Part II General Relativity

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Contents

1 Part II: Introduction General Relativity (16 lectures)

1.1 *Pre-requisites*

Part IB Methods and Special Relativity are essential and Part II Classical Dynamics is desirable. You are particularly advised to revise Cartesian tensors, the Einstein summation convention and the practice of 'index-shuffling'.

The ensuing notes are designed to cover *almost* all the material in the course and a small amount of additional illustrative material not included in the schedules, and which will not be lectured, but which is accessible using the techniques you should have mastered by the end of it. This material is enclosed in *asterisks*. The material will be lectured in the order given in the schedule, which reads as follows.

1.2 *The Schedule*

Curved and Riemannian spaces. Special relativity and gravitation, the Pound-Rebka experiment. Introduction to general relativity: interpretation of the metric, clock hypothesis, geodesics, equivalence principles. Static spacetimes, Newtonian limit. [4]

Covariant and contravariant tensors, tensor manipulation, partial derivatives of tensors. Metric tensor, magnitudes, angles, duration of curve, geodesics. Connection, Christoffel symbols, absolute and covariant derivatives, parallel transport, autoparallels as geodesics. Curvature. Riemann and Ricci tensors, geodesic deviation. [5]

Vacuum field equations. Spherically symmetric spacetimes, the Schwarzschild solution. Rays and orbits, gravitational red-shift, light deflection, perihelion advance. Event horizon, gravitational collapse, black holes. [5]

Equivalence principles, minimal coupling, non-localisability of gravitational field energy. Bianchi identities. Field equations in the presence of matter, equations of motion. [2]

There will be three example sheets.

1.3 *Units*

In order not to clutter up formulae, for the most part, units will be used in which the velocity of light, c, and Newton's constant of Gravitation, G, are set to unity. When required they may, and will, be restored using elementary dimensional analysis.

1.4 *Signature Convention*

Beginners often find remembering various notational conventions which abound in the subject confusing. The best strategy is to cultivate the ability to switch as desired, specially as no physical statement can depend on such arbitrary conventions. Some hints to facilitate changing conventions are given below; however you are advised not to do so in the middle of a formula. The signature convention I will use is $(+++-)$ and spacetime indices (which run over 4 values) will be denoted by lower case latin letters (rather than say Greek, Cyrillic or Hebrew) usually taken from the beginning of the alphabet. Space indices will be denoted by i, j, k and will take values from 1 to 3. If you need to change the signature conventions (for example to consult a textbook which uses the opposite one), it suffices to replace the metric g_{ab} by $-g_{ab}$.

1.5 *Curvature Conventions*

The curvature and Ricci tensor conventions (whose meaning will be explained later in the course) are $\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d$, $R_{db} = R^c{}_{dcb}$. If the signature convention is switched to the opposite one, keeping the curvature and Ricci tensor conventions unchanged, then the Christoffel symbols $\left\{{}_a{}^b{}_c\right\}$, affine connection components $\Gamma_a{}^b{}_c$, curvature tensor $R^c{}_{dab}$ and Ricci tensor $R_{db} = R^c{}_{bcd}$ are unchanged. The Ricci-scalar $R = g^{ab} R_{ab}$ changes sign.

1.6 Other miscellaneous conventions

A comma followed by a subscript $_{,a}$ after a tensor means the same as ∂_a in front of the tensor and denotes partial derivative. A semi-colon followed by a subscript after a tensor $_{;a}$ or ∇_a in front of a tensor denotes covariant derivative. The symbol $\pm(a \leftrightarrow b)$ after a tensorial expression containing the index pair ab means add or subtract the same expression with a and b interchanged. Round brackets will be used to denote symmetrization and square brackets to denote anti-symmetrization, thus $S_{(ab)} = \frac{1}{2}\big(S_{ab} + S_{ab}\big)$ and $A_{[ab]} = \frac{1}{2}\big(A_{ab} - A_{ab}\big)$.

1.7 Appropriate books

The following are listed in the schedules.

C. Clarke, Elementary General Relativity. Edward Arnold 1979 (out of print)

J B Hartle, Gravity Addison Wesley

L.P. Hughston and K.P. Tod An Introduction to General Relativity. London Mathematical Society Student Texts no. 5, Cambridge University Press 1990 $(+---)(R)$

† R. d'Inverno Introducing Einstein's Relativity. Clarendon Press 1992 $(+---)(R)$

W. Rindler Relativity: Special, General and Cosmological Oxford University Press 2001

B.F. Schutz A First Course in General Relativity. Cambridge University Press 1985

H. Stephani General Relativity 2nd edition. Cambridge University Press 1990 $(+++-)(R)$.

In addition, the following more advanced books contain much useful material at about the level of the present course. They should all be available in college libraries.

C. W. Misner, K.S. Thorne and J.A. Wheller, Gravitation. W.H. Freeman $(-+++)(G)$

S. Weinberg, Gravitation and Cosmology (Wiley) $(-+++)(G)$

L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Pergamon) $(+---)(R)$

J.M. Stewart, Advanced General Relativity (Cambridge University Press) $(+---)(G)$

R.M. Wald, General Relativity (Chicago University Press)$(-+++)(R)$

The third covers both Electrodynamics at a level suitable for the Part II course and then develops General Relativity. The first book contains a useful summary of the various conventions used in some of the better known textbooks. (R) means spactime indices are Roman, (G) means that they are Greek.

Recently an outstanding new textbook book came out which I strongly recommend for the physical side of GR. It is designed as an undergraduate text for American physics students but it is completely up-to-date and carries the mathematics quite far, almost as far as is done in the course. However it carries the applications much further. It is

J. B. Hartle, Gravity : An Introduction to Einstein's General Relativity (Addison Wesley) £35.99

2 Scope and Validity of the Theory

General Relativity results from 'unifying', or making compatible, Newtonian Gravity and Special Relativity. In particular it must give a fully consistent account of the motion of light moving in a gravitational field. Since we know from Quantum Mechanics that light, and indeed all matter, has both particle and wave aspects, a successful theory should allow a description of light both as particles and as waves.

Newton's Laws of Gravity are expressed using Newtons' constant of Gravitation G and of course Special Relativity introduces the velocity of light c. In general, formulae in General Relativity involve both.

• Gravity is important if the typical velocities v, induced by a mass M inside a radius R satisfy

$$v^2 \approx \frac{GM}{R}. \tag{1}$$

• Relativity is important if

$$v^2 \approx c^2$$

• General Relativity is important if

$$\frac{2GM}{c^2} \approx 1\,. \tag{2}$$

In other words if

$$\boxed{R \approx R_S = \frac{2GM}{c^2},} \tag{3}$$

where R_S is called the Schwarzschild radius of the body. If the radius of a body is comparable with its Schwarzschild radius, then

$$\text{escape velocity} \approx \text{light velocity} \tag{4}$$

or

$$\text{rest mass energy} \approx \text{gravitational potential energy}. \tag{5}$$

In other words the body is close to or actually is a Black Hole, a phenomenon predicted by John Michell, on the basis of the 'Ballistic theory of Light 'in 1784 and later taken up by Laplace. The name 'Black Hole'was coined by John Wheeler in the late 1960's.

According to the Ballistic Theory, light is made up of particles whose speed in the absence of gravity is c. According to Newtonian mechanics, such particles should suffer a deflection

$$\delta = \frac{2GM}{c^2 b}, \tag{6}$$

when scattered with impact parameter b. In fact according to Einstein's theory, as we shall see later in the course, the exact answer is, for impact parameters large compared with the Schwarzshcild radius, twice as large

$$\boxed{\delta = \frac{4GM}{c^2 b} = \frac{2R_S}{b}.} \tag{7}$$

The dimensionless number

$$\frac{2GM}{c^2 R} = \frac{R_S}{R} \tag{8}$$

is thus a measure of how large general relativistic effects are.

General Relativity breaks down when relativistic quantum effects become important. This happens when, if we probe a system of size R with light for example of angular frequency ω and wavelength λ we need an amount of energy $\hbar\omega$ comparable with the rest mass energy Mc^2 we are examining. To localize the system we need

$$\frac{\lambda}{2\pi} \leq R, \tag{9}$$

and hence, by Planck's relation an energy $\hbar\omega$ at least comparable with the rest mass energy Mc^2 of the system will be needed if R is smaller than

$$\boxed{R \approx R_C = \frac{\hbar}{Mc},} \tag{10}$$

where R_C is called the Compton radius. If we try to localize a particle to better than $R \approx R_c$ we need so much energy that we run the risk of creating more particles. At this point we have to use quantum field theory which is designed to describe systems with an indefinite number of particles because our body will behave more like an 'elementary particle 'than a macroscopic body. Thus the realm of classical Newtonian gravity is bounded by $R > R_S$ and non-relativistic quantum mechanics by $R > R_C$. These two realms intersect at the Planck scale which is the domain of Quantum Gravity. Since very little is known about what happens there we shall say no more about it in this course except to point out that associated with it are an *absolute or fundamental system of physical units of mass length and time*, independent of any man-made conventions, called *Planck units*, characterizing the relevant scale. They work out to be

$$\text{Planck Mass} \qquad M_P = \big(\frac{c\hbar}{G}\big)^{\frac{1}{2}} \approx 2 \times 10^{-5} g \approx 10^{19} GeV \tag{11}$$

$$\text{Planck Length} \qquad L_P = \big(\frac{G\hbar}{c^3}\big)^{\frac{1}{2}} \approx 1.6 \times 10^{-33} cm \tag{12}$$

$$\text{Planck Time} \qquad T_P = \big(\frac{G\hbar}{c^5}\big)^{\frac{1}{2}} \approx 4 \times 10^{-44} s. \tag{13}$$

2.1 Example

Long before Planck, Johnstone Stoney, the first person to recognize that nature admits a fundamental unit of electric charge and the man who invented the name 'electron'constructed an absolute or fundamental system of units using G, and c but not using $\hbar$. How did he do it? How are his units related to Planck units?

3 Review of Newtonian Theory

It will prove useful to review Newtonian theory in a form which we can make contact with in later work. A freely falling particle has the equation of motion, in an inertial coordinate system, [1]

$$m_i \frac{d^2\mathbf{x}}{dt^2} = m_p \mathbf{g}(\mathbf{x}, t), \tag{14}$$

where $\mathbf{g}$ is the "gravitational field, m_i the inertial mass and m_p the passive gravitational mass. According to experiments of Galileo, Newton and Eötvös we have *Equality of Inertial and Passive Gravitational Mass*, i.e.

$$\boxed{m_i = m_p} \tag{15}$$

and thus

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{g}(\mathbf{x}, t), \tag{16}$$

[1] sometimes called an inertial reference frame.

which implies the *Universality of Free Fall*, i.e. all particles fall with the same acceleration in an external gravitational field. This allows us to pass to a new (non-inertial) coordinate system

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{b}(t), \tag{17}$$

in which

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2} = \tilde{\mathbf{g}}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}, t) - \ddot{\mathbf{b}}(t). \tag{18}$$

We observe that (cf. *Einstein's Lift*)

i) By choosing (t) suitably we can set $\tilde{\mathbf{g}} = 0$ along the path of a single particle. Thus a uniform gravitational field (i.e. one for which $\mathbf{g}(\mathbf{x}, t)$ is independent $\mathbf{x}$) is unobservable: it can always be eliminated by passing to a suitable frame.

ii) The gravitational field $\mathbf{g}$ (i.e. the local value of the acceleration due to gravity) is not a physical variable because Newton's equations of motion admit a larger symmetry group (in fact infinite dimensional) than just the Galilei group.

Following Dicke's refinement of Einstein's original analysis it is customary to describe this situation in terms of the

Weak Equivalence Principle: All freely falling bodies with negligible gravitational self interactions follow the same path, if they have the same initial velocity.

This idea is very old, and goes back at least to John Philoponus, a passionate critic of Aristotle, and who wrote around 500 A.D.

> For if you take two weights differing from each other by a very wide measure, and drop them from the same height, you will see that the ratio of the times of their motion does not correspond with the ratio of their weights, but the difference between the times is much less. Thus if the weights did not differ by a wide measure, but if one were, say double, and the other half, the times will not differ at all from each other, or if they do, it will be by an imperceptible amount, although the weights did not have that kind of difference between them, but differed in the ratio two to one.

Galileo checked this by timing a ball rolling down an inclined plane and, by repute, dropping balls from the Leaning Tower of Pisa. Newton made a more qualitative check by showing that the periods of two simple pendula whose bobs are made from different materials are equal to better than one part in a million. Baron Eötvös showed the sun does not exert a periodic torque on the arm of a torsion balance from which are suspended two weights of different materials. Experiments by Dicke and others have used this method to test the weak equivalence principle by showing that everything on earth falls towards the sun with the same acceleration with a precision of one part in a million million (10^{12}). There are currently plans by NASA and ESA to fly a drag-free satellite (one satellite inside a larger evacuated satellite which has sensors and rockets to ensure that the inner satellite is in free fall.) using this the proposers plan to check Philoponus' claim to one part in 10^{17}. *If* his claim fails at this level, a possible explanation would be that in addition to the four forces we are familiar with (electro-magnetic, weak nuclear, strong nuclear and gravitational) there may be an additional and so far purely hypothetical long range field responsible for a *fifth force*. Since there is no evidence for such a force we shall, in this course, assume the unrestricted validity of the weak equivalence principle.

By contrast non-uniform gravitational fields *are* observable. To see how, look at the motion of two neighbouring particles with positions $\mathbf{x}$ and $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{N}$.

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{g}(\mathbf{x}, t), \tag{19}$$

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2} = \mathbf{g}(\mathbf{x} + \mathbf{N}) \tag{20}$$

and so

$$\frac{d^2\mathbf{N}}{dt^2} = (\mathbf{N}.\text{grad})\mathbf{g} + (\mathcal{O})(\mathbf{N^2}) \tag{21}$$

or to the lowest order

$$\frac{d^2N_i}{dt^2} = (\partial_j g_i)N_j. \tag{22}$$

We write this as

$$\boxed{\frac{d^2N_i}{dt^2} + E_{ij}N_j = 0. \qquad \textit{Geodesic Deviation}} \tag{23}$$

$$\boxed{E_{ij} = -\partial_j g_i. \qquad \textit{Tidal Tensor}} \tag{24}$$

Now the gravitational field is conservative, $\text{curl}\,\mathbf{g} = 0$, and so we may introduce the *Newtonian Potential* by

$$\boxed{\mathbf{g} = -\text{grad}U \qquad g_i = -\partial_i U} \tag{25}$$

whence the tidal tensor is seen to be the Hessian of the Newtonian potential

$$\boxed{E_{ij} = \partial_i\partial_j U = E_{ij}.} \tag{26}$$

Now *Poisson's equation* or Gauss's Law relates the gravitational field to the local density of active gravitational mass matter ρ_a:

$$\text{div}\mathbf{g} = -4\pi G\rho_a, \tag{27}$$

thus

$$\nabla^2 U = 4\pi G\rho_a.^2 \tag{28}$$

[2] You should check that you understand the signs in (27) and (28) and how and why they differ from those in the analogous equations in electro-statics.

Now experiment reveals the remarkable fact that the gravitational field generated by a body depends *only* on its total inertial mass which in turn equals, as we have seen, its passive gravitational mass. This fact is sometimes known as the principal of *Identity of Active and Passive Gravitational Mass*, or sometimes the *Strong Equivalence Principle*, since it implies that the Weak Equivalence Principle will hold even for bodies with significant gravitational self-interactions. In Newtonian theory this follows from Newton's Third Law that action and reaction should be equal and opposite The force $\mathbf{F}_{(21)}$ exerted by body 1 on body 2 is

$$\mathbf{F}_{(21)} = G m_p^{(2)} m_a^{(1)} \frac{\mathbf{r}_{(1)} - \mathbf{r}_{(2)}}{|\mathbf{r}_{(1)} - \mathbf{r}_{(2)}|^3}, \tag{29}$$

where $m_p^{(2)}$ is the passive gravitational mass of body 2 and $m_a^{(1)}$ is the active gravitational mass of body 2. Now Newton's Third Law, $\mathbf{F}_{(21)} = -\mathbf{F}_{(12)}$ requires

$$\frac{m_a^{(1)}}{m_p^{(1)}} = \frac{m_a^{(2)}}{m_p^{(2)}}. \tag{30}$$

If we require that this equation is true for all possible pairs of bodies, we see that, by choosing out units sensibly, that active and passive masses must be equal.

In Newtonian theory it then follows that the law of conservation of momentum, angular momentum and of the existence of a potential function such that energy is conserved will all hold. To some extent the converse holds, if the Third Law did not hold these conservation laws would not necessarily hold.

One simple way to check the Strong Equivalence Principle is by looking at the motion of the moon. The last astronauts to visit left behind some corner reflectors, i.e. three plane mirrors meeting mutually at right angles. A laser pulse sent from earth to the moon and into one of these corners is reflected back in *precisely* the opposite direction[3] and by timing how long the pulse takes to get back the orbit the earth moon distance is known to better than a centimetre or so. If the Strong Equivalence principle did not hold one would expect the centre of mass of the earth moon system to oscillate with the lunar period. No such effect is seen.

Given the identity of inertial, active gravitational and passive gravitational mass we can drop the subscript and write Poissons's equation as

$$\boxed{E_{ii} = 4\pi G\rho \qquad \textit{Field Equation.}} \tag{31}$$

Finally, because $E_{ij} = -\partial_j g_i$ we have $\partial_k E_{ij} = \partial_i E_{kj}$ or

$$\boxed{E_{i[j,k]} = 0.} \tag{32}$$

One should look upon (32) as an *integrability condition* for the existence of the gravitational field vector g_i. Similarly one should regard the symmetry condition

[3] You should be able to prove this

(26) as an integrability condition for the existence of a Newtonian potential U. Such integrability conditions are often called *Bianchi Identities*.

We are now in a position to summarize the basic equations and structure of Newtonian Gravity

$$1) \qquad \frac{d^2 N_i}{dt^2} + E_{ij} N_j = 0 \qquad \textit{Geodesic Deviation}$$

$$2) \qquad E_{ij} = E_{ji} \qquad \textit{Bianchi Identity}$$

$$3) \qquad E_{j[i,k]} = 0 \qquad \textit{Bianchi Identity}$$

$$4) \qquad E_{ii} = 4\pi G\rho \qquad \textit{Field Equation.}$$

One has that 3) $\Rightarrow$ $E_{jk} = -\partial g_j$, 3) and 4) $\Rightarrow$ $g_i = -\partial_i U \Rightarrow E_{ij} = \partial_i \partial_j U$ and 4) $\Rightarrow \nabla^2 U = 4\pi G\rho$.

3.1 Example

Calculate the tidal tensor due to a spherically symmetric star. The sun and the moon subtend approximately the same angle (about half a degree) in the sky. They also raise approximately the same tide on earth. Given that tides are produced by gravity gradients, what can you say about the mean densities of the moon and the sun?

4 Review of Special Relativity

I will usually adopt $x^a = (x^i, x^4)$, $i = 1, 2, 3$, $a = 1, 2, 3, 4$ as inertial spacetime coordinates, but sometimes I will call x^4 x^0 and make the attendent changes of conventions without further comment. Note that from now indices on coordinates will always be "upstairs". The *interval* between neighbouring spacetime points is

$$ds^2 = d\mathbf{x}^2 - dt^2 = \eta_{ab} dx^a dx^b, \tag{33}$$

with $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$. In other words we use the "mainly plus" signature convention. The interval is invariant under Lorentz transformations

$$x^a \rightarrow \tilde{x}^a = \Lambda^a{}_b x^b, \tag{34}$$

where

$$\eta_{ab} \Lambda^a{}_c \Lambda^b{}_d = \eta_{cd} \tag{35}$$

or in matrix notation

$$\Lambda^t \eta \Lambda = \eta, \tag{36}$$

where ^{t} denotes matrix transpose. The index positions on the matrices may look unfamiliar, but are consistent with the usual conventions. The first, upper, index labels rows and the second lower index labels columns, and they should be thought of as acting on column vectors x^a, where the index labels rows. Strictly speaking, Λ is an endomorphism while η is a quadratic form, and for that reason, both of its indices are lowered. For this reason it makes basis independent sense to say that it is symmetric, $\eta_{ab} = \eta_{ba}$.

The *Clock Postulate* states that a clock moving along a world line $x^a = x^a(\lambda)$ with λ some parameter along the curve measures elapsed proper time

$$\tau = \int \sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}d\lambda. \tag{37}$$

Note that τ is independent of the choice of parameter because if we replace λ by $\tilde{\lambda} = f(\lambda)$, $\frac{dx^a}{d\lambda} = f'\frac{dx^a}{d\tilde{\lambda}}$, with $f' = \frac{d\tilde{\lambda}}{d\lambda}$ but

$$\sqrt{-\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}}d\lambda = \sqrt{-\eta_{ab}\frac{dx^a}{d\tilde{\lambda}}\frac{dx^b}{d\tilde{\lambda}}}d\tilde{\lambda} \tag{38}$$

We say that τ is *reparametrization invariant.*

The Geodesic Postulate states that free particles move on straight lines

$$\frac{d^2x^a}{d\lambda^2} = 0 \Leftrightarrow x^a = x^a(0) + \lambda u^a, \tag{39}$$

where u^a is a constant vector. If the world line is *timelike* we can normalize u^a by choosing λ to be proper time

$$\lambda = \tau \Rightarrow \frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}\eta_{ab} = \eta_{ab}u^a u^b = -1. \tag{40}$$

For *light rays* this cannot be done and λ is arbitrary up to an affine transformation

$$\lambda \to a\lambda + b \qquad a, b \in R. \tag{41}$$

We call λ an *affine parameter.*

Free motion can be described using a *Variational Principle* in at least two ways:

Method I works only for timelike (or spacelike) curves. We vary the action functional

$$S[x^a(\lambda)] = -m\int\sqrt{-\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{\lambda}}d\lambda = -m\tau = -m\int\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b}d\lambda. \tag{42}$$

Note that

i) the action functional S is reparametrization invariant,

ii) Choosing $\lambda = t = x^4$, we get

$$S = -m\int dt\sqrt{1 - (\frac{d\mathbf{x}}{dt})^2} \tag{43}$$

which coincides at small velocities, up to an irrelevant additive piece [4], to the standard non-relativistic expression. We use the associated Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{44}$$

with $L = -\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b} \Rightarrow \frac{\partial L}{\partial x^a} = 0$. Now

$$\frac{\partial L}{\partial \dot{x}^a} = -\frac{1}{L}\eta_{ab}\frac{dx^b}{d\lambda} = -\eta_{ab}\frac{dx^b}{d\tau} \tag{45}$$

The equation of motion becomes

$$\frac{d}{d\lambda}\left(\eta_{ab}\frac{dx^b}{d\tau}\right) = 0 \Rightarrow \frac{d^2x^a}{d\tau^2} = 0. \tag{46}$$

Method II is rather quicker. We take as action functional

$$S = \int \eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}d\lambda, \tag{47}$$

$L = \eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}$. Note that this action is *not* reparametrization invariant. The Euler-Lagrange equations are

$$\frac{d}{d\lambda}\left(\eta_{ab}\frac{dx^b}{d\lambda}\right) = 0. \tag{48}$$

Now

$$\frac{\partial L}{\partial \lambda} = 0, \tag{49}$$

and so by Noether's theorem [5]

$$\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = \text{constant}. \tag{50}$$

If the constant is negative we choose it to be -1 and find that we can choose the additive constant in τ so that $\lambda = \tau$. In this way we recover our previous result for timelike curves. If the constant is zero, we obtain equations which are valid for light rays or other massless particles.

It turns out that both Method I and Method II can readily be extended to curved spacetimes. In practice, Method II is usually more convenient.

It may seem strange that there is no unique action principle for the motion of a particle, but this becomes less so if one reflects that from an abstract point

[4] You should be able to explain why it is irrelevant.

[5] Check this. You should also check that you understand that there are two cases of Noether's theorem, one when the Lagrangian does not depend on the independent variable which we are using here and the other when the Lagrangian does not depend upon a dependent variable, which we will be using shortly. For clarity they should perhaps be called Noether's first and second theorem respectively, but this terminology is not universal.

of view we are characterizing the motion as stationary point of a functional, i.e of a function of infinitely many variables. We are familiar with the fact that, in finite dimensions, a given point may be a stationary point of many different functions. To rub home the point you may like to verify that we get the same equations of motion if we take

$$L = f(\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}), \tag{51}$$

where $f()$ is almost *any* function of its argument.

5 Curved Spacetime

Because of the *Universality of Free Fall* the motion is independent of mass, and so it is an attractive idea to ascribe the curvature of the paths of freely falling particles to the curvature of spacetime. We assume that the constant spacetime *metric* η_{ab} is replaced by a general space and time dependent curved metric $g_{ab}(x)$ such that the interval is given by

$$\boxed{ds^2 = g_{ab}(x)dx^a dx^b, \qquad g_{ab} = g_{ba}.} \tag{52}$$

The *Clock and Geodesic Postulates* now read as before but with η_{ab} replaced by g_{ab}. Thus

$$\tau = \int \sqrt{-g_{ab}\dot{x}^a\dot{x}^b}d\lambda. \tag{53}$$

Using **Method I** we have set $L = \sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$, and the Euler Lagrange equations are

$$-\frac{d}{d\lambda}(\frac{1}{L}\frac{g_{ab}dx^b}{d\lambda}) = -\frac{1}{2L}(\frac{\partial g_{cd}}{\partial x^a})\frac{dx^c}{d\lambda}\frac{dx^d}{d\lambda}, \tag{54}$$

or, since $L = 1$ if $\lambda = \tau$,

$$\frac{d}{d\tau}(\frac{g_{ab}dx^b}{d\tau}) = \frac{1}{2}(\frac{\partial g_{cd}}{\partial x^a})\frac{dx^c}{d\tau}\frac{dx^d}{d\tau}. \tag{55}$$

This maybe re-written

$$g_{ab}\frac{d^2x^b}{d\tau^2} + \frac{\partial g_{ab}}{\partial x^c}\frac{dx^c}{d\tau}\frac{dx^b}{d\tau} - \frac{1}{2}\frac{\partial g_{cd}}{\partial x^a}\frac{dx^c}{d\tau}\frac{dx^d}{d\tau} = 0, \tag{56}$$

or, relabelling dummy indices,

$$g_{ab}\frac{d^2x^b}{d\tau^2} + \frac{1}{2}\Big(\frac{\partial g_{ad}}{\partial x^c} + \frac{\partial g_{ac}}{\partial x^d} - \frac{\partial g_{cd}}{\partial x^a}\Big)\frac{dx^c}{d\tau}\frac{dx^d}{d\tau} = 0. \tag{57}$$

We define the *inverse metric* by

$$g^{ab} = (g^{-1})^{ab} = g^{ba} \tag{58}$$

so that

$$g^{ac} g_{cb} = \delta^a_b, \tag{59}$$

where δ^a_b is the Kronecker delta and equal to 1 if $a = b$ and zero otherwise. Contraction of (57) g^{ea} and relabelling $e \to a$ now yields

$$\boxed{\frac{d^2 x^a}{d\tau^2} + \left\{ {}_c{}^a{}_d \right\} \frac{dx^c}{d\tau} \frac{dx^d}{d\tau} = 0,} \tag{60}$$

with

$$\boxed{\left\{ {}_c{}^e{}_d \right\} = \tfrac{1}{2} g^{ea} \Big(\tfrac{\partial g_{ad}}{\partial x^c} + \tfrac{\partial g_{ac}}{\partial x^d} - \tfrac{\partial g_{cd}}{\partial x^a} \Big).} \tag{61}$$

The rather strange collection of objects $\left\{ {}_c{}^a{}_d \right\}$ are called *Christoffel symbols*. We shall explore their mathematical properties shortly. For the time being, they should just be thought of as an array of functions, and in fact, to anticipate what follows, they are *not* the components of a tensor field.

We could have proceeded using **Method II**. A slightly shorter analogous calculation using $L = g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}$ yields

$$\frac{d^2 x^a}{d\lambda^2} + \left\{ {}_c{}^a{}_d \right\} \frac{dx^c}{d\lambda} \frac{dx^d}{d\lambda} = 0. \tag{62}$$

Again we have $\frac{\partial L}{\partial \lambda} = 0$ and Noether's theorem yields

$$g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = \text{constant}. \tag{63}$$

For massive particles, the constant is negative and we choose it to be -1 and thus get $\lambda = \tau$. We now obtain our previous equations. However, as in flat spacetime, Method II also works for massless particles.

In the next section we shall apply the equations we have developed to some examples.

6 Static and stationary metrics, Pound-Rebka Experiment

A metric is called *stationary* if it is independent of time. This means that one may introduce a privileged time coordinate $t = x^4$ such $\frac{\partial}{\partial t}(g_{ab}) = 0$. Thus

$$ds^2 = g_{44}(\mathbf{x}) dt^2 + g_{ij}(\mathbf{x}) dx^i dx^j + 2 g_{4i}(\mathbf{x}) dt dx^i \, . \tag{64}$$

A metric is called *static* if it is stationary and in addition invariant under *time-reversal*, i.e. invariant under an involution $\mathcal{T} : t \to -t$. This implies that $g_{4i} = 0$ and that the metric may be cast in the form

$$ds^2 = h_{ij}(\mathbf{x}) dx^i dx^j - e^{2U(\mathbf{x})} dt^2. \tag{65}$$

As we shall see shortly, $U(\mathbf{x})$ plays the role of the Newtonian potential.

6.1 The Gravitational Redshift

Suppose n pulses are sent from an emitter at $\mathbf{x}_e$ to an observer at $\mathbf{x}_o$ in coordinate time Δt. The emitted frequency will, by the clock postulate, be obatined using the proper time, so

$$\nu_e = \frac{n}{\Delta t \sqrt{-g_{44}(\mathbf{x}_e)}} \tag{66}$$

and similarly for the observed frequency

$$\nu_o = \frac{n}{\Delta t \sqrt{-g_{44}(\mathbf{x}_o)}} \tag{67}$$

and hence the ratio

$$\frac{\nu_e}{\nu_o} = \sqrt{\frac{g_{44}(\mathbf{x}_0)}{g_{44}(\mathbf{x}_e)}} = \exp\big(U(\mathbf{x}_0) - U(\mathbf{x}_e)\big). \tag{68}$$

Evidently if the emitter is at a lower value of the gravitational potential, $U(\mathbf{x}_e) < U(\mathbf{x}_o)$, the received frequency will be lower than the emitted frequency. This is called a gravitational redshift and is quantified in terms of a quantity z given by $1 + z = \frac{\nu_e}{\nu_o}$.

We remark that

i) A heuristic derivation of the gravitational redshift can also be given using Planck's relation $E = h\nu$ and Einstein's formula $E = mc^2$. One sets up a cyclic process in which a photon is sent from a lower to a higher potential, is absorbed, the absorber slowly lowered to the starting point and the photon is then re-emitted. Since energy can be obtained during the lowering process, because of the extra weight due to the energy of the absorbed photon, unless the photon loses precisely the predicted amount of energy climbing up the gravitational well, one would be able to construct a perpetual motion machine of the second kind which is impossible.

ii) The gravitational redshift is *universal*, the redshift experienced is the *same* for all massless particles. Again this could also be proved using the impossibility of perpetual motion machines along the lines given above.

iii) The gravitational redshift shows that as measured by physical clocks, spacetime really is curved. This statement is sometimes referred to as the *Schild argument*.

iv) The gravitational redshift shows that time (i.e. as mesured by clocks) runs at different rates at different places.

v) It is interesting to analyze the problem using the Ballistic Theory according to which energy is also conserved. The speed of the 'light particles'which have to climb up the gravitational potential well is reduced. Thus according to the Ballistic Theory, light coming from different sources will have different speeds.

In fact in 1784 John Michell predicted precisely this would happen and suggested an experiment with a prism to check it. But his prediction contradicts the observed fact (which we use when setting up Special Relativity) that the speed of light received here on earth is universal and independent of its source.

6.2 Particle motion in Static Spacetimes

We have

$$L = e^{2U}\dot{t}^2 - h_{ij}\dot{x}^i\dot{x}^j. \tag{69}$$

In the timelike case we may choose $\lambda = \tau$. The x^4 equation of motion is

$$\frac{d}{d\tau}\big(e^{2U}\frac{dt}{d\tau}\big) = 0. \tag{70}$$

The x^i equation of motion is

$$\frac{d}{d\tau}\big(h_{ij}\frac{dx^j}{d\tau}\big) + e^{2U}\partial_i U\big(\frac{dt}{d\tau}\big)^2 = \frac{1}{2}(\partial_i h_{jk})\dot{x}^j\dot{x}^k. \tag{71}$$

Thus, comparing with 60

$$\left\{{}_4{}^4{}_i\right\} = \partial_i U \tag{72}$$

$$\left\{{}_4{}^i{}_4\right\} = -h^{ij}\partial_j e^{2U} \tag{73}$$

$$\left\{{}_j{}^i{}_k\right\} = \frac{1}{2}h^{is}\Big(\frac{\partial h_{sk}}{\partial x^j} + \frac{\partial h_{sj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^s}\Big). \tag{74}$$

Independence of t gives, from Noether's theorem,

$$e^{2U}\frac{dt}{d\tau} = E, \tag{75}$$

where the constant E is the energy. Now $g_{ab}\dot{x}^a\dot{x}^b = -1$ (with $\lambda = \tau$) gives

$$e^{2U}\big(\frac{dt}{d\tau}\big)^2 - h_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 1 = E^2e^{-2U} - h_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}. \tag{76}$$

6.3 The Newtonian Limit

In the above we have set $c = 1$. To understand this approximation we should restore units and so $t \to ct$. The quantity U is replaced by $\frac{U}{c^2}$.

Now we can now expand the metric in inverse powers of c:

$$g_{44} = -c^2 + \mathcal{O}(1) \tag{77}$$

$$h_{ij} = \delta_{ij} + \mathcal{O}(\frac{1}{c^2})\,. \tag{78}$$

Now

$$e^{\frac{2U}{c^2}} = 1 + \frac{2U}{c^2} + \dots\,. \tag{79}$$

Thus, to the lowest non-trivial order, we put

$$\boxed{ds^2 \approx -c^2 dt^2 (1 + \frac{2U}{c^2}) + d\mathbf{x}^2 \bigl(1 + \mathcal{O}(\frac{1}{c^2})\bigr).} \tag{80}$$

If we, in addition, assume that the particle is moving slowly, we may set $\tau \approx t$, $E = 1 + \mathcal{E}$. We find from (76)

$$\boxed{\frac{1}{2}\mathbf{v}^2 + U = \mathcal{E}.} \tag{81}$$

Clearly, this is the equation of energy conservation for a non-relativistic particle of energy per unit mass $\mathcal{E}$ moving in a Newtonian gravitational potential U. This justifies our identification of the quantity $\frac{1}{2}\ln(-g_{44})$ as the Newtonian potential.

In fact, using the Einstein field equations, which we have not yet met, it is possible to improve (80) so that it is accurate to order $\frac{1}{c^4}$

$$ds^2 \approx -c^2 dt^2 (1 + \frac{2U}{c^2}) + \bigl(1 - \frac{2U}{c^2}\bigr) d\mathbf{x}^2. \tag{82}$$

We can now give the gravitational redshift suffered by a photon in the field of a body of mass M in the Newtonian approximation

$$\frac{\nu_o}{\nu_e} = GM(\frac{1}{r_e} - \frac{1}{r_o}). \tag{83}$$

For the Pound-Rebka experiment $r_o = r_e + h$, where h is the height of the tower, we obtain

$$\frac{\delta\nu}{\nu} = \frac{gh}{c^2}. \tag{84}$$

6.4 Motion of Light rays

Noether's theorem gives

$$e^{2U}\frac{dt}{d\lambda} = E, \tag{85}$$

and the fact that $g_{ab}\dot{x}^a\dot{x}^b = 0$ gives (cf. 76)

$$E^2 = e^{2U} h_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}, \tag{86}$$

$$\frac{d^2x^i}{d\lambda^2} + \Bigl[\Bigl\{ {}_j{}^i{}_k \Bigr\} + h_{jk}h^{is}\partial_s U\Bigr]\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0. \tag{87}$$

Because the affine parameter λ is defined only up to an overall multiple, $\lambda \to a\lambda$, so too is the 'Energy'E. The equations are invariant under the rescaling $\lambda \to a\lambda$, $E \to \frac{E}{a}$. This means that, in a purely particle theory of light, only the ratio of energies is well defined.

The interpretation of (87) will be given later in the course.

6.5 Further Examples

6.5.1 The Schwarzschild Metric

You should now be in a good position to study the motion of particles moving around a spherically symmetric static black hole or a star. Provided the orbiting particle has a negligible effect on the spacetime geometry, which will be true if it is very much less massive than the star or black hole, the metric is

$$ds^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2M}{r}\right) dt^2. \tag{88}$$

We will take a more detailed look at this later in the course.

6.5.2 The $k = 0$ Robertson-Walker Metric

Our universe is not static, but rather it is expanding. To a good approximation the metric of our universe is given by [6]

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2. \tag{89}$$

The function $a(t)$ is called the scale factor. For a particle of mass m, Noether's theorem implies *momentum conservation*

$$ma^2(t)\frac{d\mathbf{x}}{d\lambda} = \mathbf{p}, \tag{90}$$

where $\mathbf{p}$ is a constant vector, but because of the time-dependence, energy

$$E = m\frac{dt}{d\tau} \tag{91}$$

is *not* conserved. In fact using the normalization of the 4-velocity one gets

$$E = \sqrt{m^2 + \frac{\mathbf{p}^2}{a^2(t)}}. \tag{92}$$

Thus if the universe expands and the scale factor $a(t)$ increases the energy E decreases. The locally measured (so-called 'peculiar') velocity

$$\mathbf{v} = a(t)\frac{d\mathbf{x}}{dt} \tag{93}$$

given by

$$\mathbf{v} = \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2a^2(t)}} \tag{94}$$

also decreases except in the limit of zero mass m when $\mathbf{v}$ is constant and of unit magnitude $|\mathbf{v}| = 1$. The energy of a massless particle decreases $\propto \frac{1}{a(t)}$, thus if a photon is emitted at time t_e and received at time t_o the redshift is given by

$$\frac{E(t_e)}{E(t_e)} = 1 + z = \frac{a(t_o)}{a(t_e)}. \tag{95}$$

[6] In cosmology one studies a more general class of Robertson-Walker metrics in which the spatial sections are curved.

6.5.3 Example: End point variations and momentum conservation

If the end points of the world line from A to X are allowed to vary, the variation of the action $S(X,A)$ is

$$\delta S(X,A) = \int_A^X \delta x^a \Big(\frac{\partial L}{\partial x^a} - \frac{dp_a}{d\lambda}\Big) d\lambda + \big[p_a \delta x^a\big]_A^X, \tag{96}$$

where the canonical momentum p_a is defined by

$$p_a = \frac{\partial L}{\partial \dot{x}^a}. \tag{97}$$

Now consider a 2-particle collison at X in which particle 1 with mass m_1 starts from A, particle two with mass m_2 from B and after the collision particle 3 arrrives at C with mass m_3 and particle 4 arrives at D with mass m_4. We vary the total action and demand that it vanish:

$$\delta S(C,X) + \delta S(D,X) + \delta S(X,A) + \delta S(X,B) = 0, \tag{98}$$

where the points A, B, C, D are held fixed but X is allowed to vary. As well as learning that AX, BX, XC and XD must be geodesics, we also discover from the variation at X that momentum is conserved at the collision:

$$p^1_a + p^2_a = p^3_a + p^4_a, \tag{99}$$

where

$$p^i_a = m_i g_{ab} \frac{dx^b}{d\tau}. \tag{100}$$

7 Lengths, Angles and Conformal Rescalings

An important aspect of General Relativity is that locally the laws of Special Relativity should hold. Thus if we work in a small neigbourhood of a point x^a in spacetime, we can use the metric g_{ab} to define an inner product

$$g_{ab} V^a U^b \tag{101}$$

on infinitesimal vector displacements V^a, U^a and define the length of a vector V^a by

$$|V^a| = \sqrt{|g_{ab} V^a V^b|}. \tag{102}$$

If the two vectors are *spacelike*, the *angle* θ between them is given by

$$\cos\theta = \frac{g_{ab} U^a V^b}{|U^a||V^b|}. \tag{103}$$

If the two vectors are *timelike*, and both are future directed, the *rapidity* θ between them is given by

$$\cosh\theta = -\frac{g_{ab} U^a V^b}{|U^a||V^b|}. \tag{104}$$

If we change the metric by multiplying by a positive function (a process called *Weyl-rescaling*):

$$g_{ab} \to \Omega^2(x) g_{ab} = \tilde{g}_{ab} \tag{105}$$

we find that all lengths rescale

$$|V^a| \to \Omega |V^a|, \tag{106}$$

but angles and rapidities are unchanged

$$\theta \to \theta. \tag{107}$$

For this reason Weyl-rescalings are also called *Conformal Rescalings*. An important example is provided by the time-dependent Robertson-Walker metric of an expanding universe

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 = a^2(t)\Big(-d\eta + d\mathbf{x}^2\Big) \tag{108}$$

which is obtained by a conformal rescaling of the flat and static Minkowski metric inside the bracket. The coordinate η is defined by

$$\eta = \int \frac{dt}{a(t)}, \tag{109}$$

and $\Omega(x) = a(t)$. As we shall see later, the null geodesics of two conformally related metrics coincide. It follows that the angles made by system of light rays in this type of expanding universe are the same as they would be in flat spacetime.

For example, a galaxy emitting light of intrinsic proper size d at time t_e has size $\frac{d}{a(t_e)}$ in the conformally related Minkowski metric and thus subtends an angle (assumed very small) at time t_o given by

$$\Delta\theta = \frac{d}{a(t_e)(\eta_o - \eta_e)}, \tag{110}$$

where

$$\eta_o - \eta_e = \int_{t_e}^{t_o} \frac{dt}{a(t)}. \tag{111}$$

The formula (110) has an amusing consequence. If we consider a family of galaxies, all of the same intrinsic size, at greater and greater distances, or equivalently greater and greater redshifts, the apparent angular size at first decreases, as we should expect in flat spacetime, and then *increases*. For example, if $a(t) \propto t^p$, with $0 < p < 1$ a short calculation, which you should check, shows that the apparent angular size is least at a redshift

$$1 + z = \big(\frac{1}{p}\big)^{\frac{p}{1-p}}. \tag{112}$$

Which metric is the 'correct'metric depends upon our measuring instruments. If we use conventional measuring instruments, built say of ordinary atoms, we find that they measure lengths as given by the Roberston-Walker metric, and relative to them the universe expands. Of course it is always possible to maintain, rather as one imagines Alice would have, that the universe is not expanding but it is we who are getting smaller, but if she had she would have also have had to agree that the atoms of which she is made are also getting smaller. If one really wishes to maintain that the flat Minkowski metric is the 'correct'one, one should provide a set of instruments which measure it.

It is perhaps striking that philosophical speculations about what would happen if the universe doubled in size overnight and whether we would notice were quite frequent towards the end of the nineteenth century, long before Einstein formulated General Relativity. Only later in the 1920's with Hubble's discovery of the expansion of the universe did they become relevant for physics.

8 Tensor Analysis

One can only get so far using just geodesics. To make further progress and to be able to write down the analogue of Poisson's equation, i.e. Einstein's field equations, we need to develop some more geometry. In this introductory course we shall proceed at what is mathematically a relatively unsophisticated level. Much deeper accounts of differentiable manifolds can of course be given, but for practical purposes they are much less relevant than a good understanding of what has become a standard part of Mathematical Physics.

Our analysis of the Weak Equivalence Principle shows that privileged global inertial coordinates cease to exist in a general curved spacetime. Our formalism must therefore allow the use of arbitrary coordinate systems. This desire is formalized in the

Principle of General Covariance which states that one should be able to write down the equations of physics in a way which is valid in all coordinate systems.

Such equations are said to be *covariant* or sometimes *form-invariant* since they should take the same form in all coordinate systems. It is important to realize that the Principle of General Covariance does not preclude the use of particular coordinate systems which may be extremely useful in practice. Neither does it rule out *a priori* the possibility of privileged systems of coordinates or frames of reference. It simply requires that such spacetimes can in principle be described without reference to particular cordinate systems. Thus if spacetime really were flat, we would like to say so without introducing global inertial coordinates. Tensor Calculus or Tensor Analysis allows us to do just that.

Given one coordinate system x^a we can always pass to a new coordinate system $\tilde{x}^a = \tilde{x}^a(x^b)$ and calculate the *Jacobian matrix*

$$\Lambda^a{}_b = \frac{\partial \tilde{x}^a}{\partial x^b}. \tag{113}$$

The *chain rule* implies that if $\tilde{\tilde{x}}^a = \tilde{\tilde{x}}^a(\tilde{x}^c)$ is a third coordinate system, then

$$\boxed{\frac{\partial \tilde{\tilde{x}}^a}{\partial x^b} = \frac{\partial \tilde{\tilde{x}}^a}{\partial \tilde{x}^c}\frac{\partial \tilde{x}^c}{\partial x^b}\,.} \tag{114}$$

Moreover, if the transformation $x^a \rightarrow \tilde{x}^a$ is invertible

$$\det\Big(\frac{\partial \tilde{x}^a}{\partial x^b}\Big) \neq 0, \tag{115}$$

we have

$$\boxed{\frac{\partial x^a}{\partial \tilde{x}^b}\frac{\partial \tilde{x}^b}{\partial x^c} = \delta^a_c\,.} \tag{116}$$

For example, for a linear transformation, such as a Lorentz transformation,

$$\tilde{x}^a = \Lambda^a{}_b x^b, \qquad \frac{\partial \tilde{x}^a}{\partial x^b} = \Lambda^a{}_b, \tag{117}$$

the Jacobian matrix is a constant matrix. Note how nicely the index positions accommodate themselves to the rules of partial differentiation and matrix multiplication, and of course, the Einstein summation convention.

One sometimes encounters the use of tildes on the indices as a further aid to remembering what variables are being differentiated. In that case, the Einstein summation convention holds but repeated indices must be of the same type, i.e. either both un-tilded or both tilded. I shall not adopt that usage here because it strains the eye and tends to make printed formulae difficult to read. However for beginners, or when engaged in complicated calculations, this notation can be useful.

We now consider *Vector Fields*. To motivate the definition consider a curve $x^a = x^a(\lambda)$ in spacetime. Its tangent vector in the x^a coordinates is (by definition)

$$T^a = \frac{dx^a}{d\lambda}, \tag{118}$$

and in coordinates $\tilde{x}^a$, it is

$$\tilde{T}^a = \frac{d\tilde{x}^a}{d\lambda}\,. \tag{119}$$

Using the chain rule

$$\frac{d\tilde{x}^a}{d\lambda} = \frac{\partial \tilde{x}^a}{\partial x^b}\frac{dx^b}{d\lambda}, \tag{120}$$

and thus

$$\boxed{\tilde{T}^a = \frac{\partial \tilde{x}^a}{\partial x^b}T^b, \qquad \text{contravariant vector field.}} \tag{121}$$

This formula motivates the definition of what is called a *contravariant vector field* as set of quantities $T^a(x)$ transforming under a coordinate change as (121).

The strange epithet contravariant suggests that there is some other kind of vector field. This is true, and they are called *covariant vector fields* and roughly speaking, they transform in the opposite way. An example is the *gradient*

$$F_a = \partial_a f = \frac{\partial f}{\partial x^a} \tag{122}$$

of a function $f(x)$. The chain rule gives

$$\frac{\partial f}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a}\frac{\partial f}{\partial \tilde{x}^b}\,, \tag{123}$$

or, if $\tilde{F}_a = \frac{\partial f}{\partial \tilde{x}_a}$,

$$\boxed{F_a = \frac{\partial \tilde{x}^a}{\partial x^b}\tilde{F}_b, \qquad \text{covariant vector field}} \tag{124}$$

If the coordinate transformation $x^a \rightarrow \tilde{x}^a$ is invertible, we also have

$$\boxed{\tilde{F}_a = \frac{\partial x^b}{\partial \tilde{x}^a}F_b,,.} \tag{125}$$

Given a contravariant vector field T^a and a covariant vector field F_a, one may form the *contraction* $F_a T^a = T^a F_a$. Because

$$\tilde{F}_a \tilde{T}^a = \frac{\partial \tilde{x}^a}{\partial x^b}\frac{\partial x^e}{\partial \tilde{x}^a} T^b F_e = \delta^e_b T^b F_e = T^a F_a, \tag{126}$$

the contraction is invariant, that is, it is a *scalar field.*

The transformation properties of the *metric tensor field* follow, just as in Special Relativity, by demanding that the interval

$$ds^2 = g_{ab} dx^a dx^b = \tilde{g}_{cd} d\tilde{x}^c d\tilde{x}^d \tag{127}$$

is invariant. Now since,

$$dx^a = \frac{\partial x^a}{\partial \tilde{x}^c} d\tilde{x}^c, \tag{128}$$

we have

$$\boxed{\tilde{g}_{cd} = g_{ab}\frac{\partial x^a}{\partial \tilde{x}^c}\frac{\partial x^b}{\partial \tilde{x}^d}, \qquad \text{symmetric second rank covariant tensor field}\,.} \tag{129}$$

A general (not necessarily symmetric) second rank covariant tensor field Q_{ab} transforms as

$$\boxed{\tilde{Q}_{cd} = Q_{ab}\frac{\partial x^a}{\partial \tilde{x}^c}\frac{\partial x^b}{\partial \tilde{x}^d}, \qquad \text{symmetric second rank covariant tensor field}\,.} \tag{130}$$

We can decompose

$$Q_{cd} = Q_{(cd)} + Q_{[dc]}, \tag{131}$$

into

$$Q_{(cd)} = \frac{1}{2}\big(Q_{cd} + Q_{dc}\big) = Q_{(dc)} \qquad \text{its symmetric part} \tag{132}$$

and

$$Q_{[cd]} = \frac{1}{2}\big(Q_{cd} - Q_{dc}\big) = -Q_{[dc]} \qquad \text{its anti}-\text{symmetric part.} \tag{133}$$

This decomposition is invariant under a general coordinate transformation. Thus, for example

$$Q_{(cd)}\frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^c} = Q_{cd}\frac{\partial x^{(c}}{\partial \tilde{x}^a}\frac{\partial x^{d)}}{\partial \tilde{x}^c} = Q_{cd}\frac{\partial x^c}{\partial \tilde{x}^{(a}}\frac{\partial x^d}{\partial \tilde{x}^{b)}} = \big(\tilde{Q}\big)_{(ab)}. \tag{134}$$

An identical argument holds for the anti-symmetric part with round brackets exchanged for square brackets.

One may define contravariant second rank tensors analogously

$$\tilde{P}^{ab} = P^{cd}\frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial \tilde{x}^b}{\partial x^c}. \tag{135}$$

Again the symmetric and anti-symmetric parts transform into themselves under general coordinate transformations.

In general, one may consider arbitrary tensor fields of type $\binom{p}{q}$ with

p indices upstairs, i.e. contravariant

q indices downstairs, i.e. covariant

The transformation rule now has p factors of $\frac{\partial \tilde{x}}{\partial x}$ and q factors of $\frac{\partial x}{\partial \tilde{x}}$.

For example a $\binom{1}{1}$ tensor field transforms as

$$\tilde{M}^a{}_b = \frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial x^d}{\partial \tilde{x}^b}M^c{}_d\,. \tag{136}$$

The upstairs and downstairs indices can be contracted to yield a scalar called the *trace*

$$\tilde{M}^a{}_a = \frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial x^d}{\partial \tilde{x}^a}M^c{}_d = \delta^d_c M^c{}_d = M^d{}_d. \tag{137}$$

In general, one may always contract r contravariant indices with r covariant indices and the contraction will be a tensor field of type $\binom{p-r}{q-r}$, $r \le \min(p, q)$.

8.1 Operations on tensors preserving their tensorial properties

(i) **Addition** (of same type)

(ii) **Multiplication** by scalars

(iii) **Outer or Tensor Products**, e.g.

$$V_a W^b \qquad \text{is a } \binom{1}{1} \text{ tensor.} \tag{138}$$

(iv) **Contractions**

(v) **Index Interchange**

$$T_{ab} \,\text{is a tensor} \Leftrightarrow T_{ba} \,\text{is a tensor.} \tag{139}$$

(vi) **Symmetrization and Anti-symmetrization**. For example

$$T_{[abc]} = \frac{1}{3!}\Big(T_{abc} + T_{cab} + T_{bca} - T_{bac} - T_{cba} - T_{acb}\Big) \qquad \text{is a tensor}\,. \tag{140}$$

8.2 Quotient Theorem

For example, if $V^{ab}W_{ab}$ is a scalar for an arbitrary contravariant tensor V^{ab}, then W_{ab} is a covariant tensor. This result is sometimes called *Tensor Detection.*

8.3 Rules for Index Shuffling

For example

$$V^{(ab)}W_{ab} = V^{ab}W_{(ab)}, \tag{141}$$

and

$$V^{(ab)}W_{[ab]} = 0. \tag{142}$$

8.4 *Graphical Notation*

For some purposes, especially those involving complicated index manipulations, it may be convenient to adopt a graphical or 'chemical'notation for tensors first introduced by Clifford, Cayley and Sylvester in the nineteenth century. In this notation

(i) Each tensor is represented by a vertex,

(ii) A contravariant index is represented by attaching an edge with an outgoing arrow on it

(iii) A covariant index is represented by by an ingoing arrow.

(iv) In order to keep track of the order of the indices the arrows are attached in a definite cyclic order, for example anti-clockwise, around each node. Of course the relative order of an ingoing and outgoing node is unimportant

(v) Contractions correspond to joining an ingoing and an outgoing arrow, not necessarily attached to the same vertex.

(vi) The Kronecker delta δ^b_a is represented by a nodeless arrow.

If one has a distinguished metric or bi-linear form, then the arrows may be omitted. The obvious analogy with chemical diagrams explains why a tensor of type $\binom{p}{q}$ is sometimes said to have *valence* $p+q$. The main draw-back of this notation is the need to keep track of the cyclic order. Of course for totally symmetric or totally anti-symmetric tensors, this is not a problem since then lifting one arrow over another of the same type gives rise to the same tensor up to a sign. This can be used to give graphical proofs of tensor identities or the enumeration of all possible invariants constructed from a set of tensors. The latter problem is analogous to counting the number of *isomers* of a certain molecule.

9 Differentiating Tensors

We have seen that $\frac{\partial \phi}{\partial x^a}$ is a co-vector field but what about the Hessian?

$$\frac{\partial \phi}{\partial x^a \partial x^b} = \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial}{\partial \tilde{x}^c} \Big(\frac{\partial \tilde{x}^d}{\partial x^b} \frac{\partial \phi}{\partial \tilde{x}^d} \Big) \tag{143}$$

$$= \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial \tilde{x}^d}{\partial x^b} \frac{\partial^2 \phi}{\partial \tilde{x}^c \partial \tilde{x}^d} + \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial^2 \tilde{x}^d}{\partial \tilde{x}^c \partial x^b} \frac{\partial \phi}{\partial \tilde{x}^d} \tag{144}$$

$$= \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial \tilde{x}^d}{\partial x^b} \frac{\partial^2 \phi}{\partial \tilde{x}^c \partial \tilde{x}^d} + \frac{\partial^2 \tilde{x}^d}{\partial x^a \partial x^b} \frac{\partial \phi}{\tilde{x}^d}\,. \tag{145}$$

The first term is *good* but the second is clearly *bad*. In other words, the Hessian is *not* a second rank covariant tensor field. For similar reasons, neither are the Christoffel symbols $\left\{ {}_a{}^b{}_c \right\}$ the components of a tensor field of type $\binom{1}{2}$.

In order to construct tensor fields we must introduce a so-called *covariant derivative operator* ∇_a which maps $\binom{p}{q}$ tensors to $\binom{p}{q+1}$ tensors . Moreover we want ∇_a to have properties as close as possible to those of ∂_a.

We demand of ∇_a that

(i) $\nabla_a \phi = \partial_a \phi$,

(ii) ∇ is Leibnizian:

$$\nabla_a (UV) = (\nabla_a U)V + U(\nabla_a V), \tag{146}$$

for any pair of $\binom{p}{q}$ and $\binom{p'}{q'}$ tensor fields U and V,

(iii) ∇_a commutes with contractions,

(iv) Acting on a vector field,

$$\boxed{\nabla_a V^b = \partial_a V^b + \Gamma_a{}^b{}_c V^c.} \tag{147}$$

A covariant derivative operator is also called an *affine connection* and the $\Gamma_a{}^b{}_c$ are its components [7].

[7]For a general affine connection, there is no symmetry with respect to interchange of the the lower indices. I am using the convention that the first index is the 'differentiating'index. This convention is not in universal use, and so one should take care when consulting text-books. However, shortly we will restrict attention to symmetric affine connections and this distinction becomes unnecessary.

They are *not* the components of a $\left\{ {1 \atop 2} \right\}$ tensor field. In fact, under a coordinate transformation

$$\Gamma_a{}^b{}_c \to \tilde{\Gamma}_a{}^b{}_c = \frac{\partial \tilde{x}^b}{\partial x^e} \frac{\partial x^g}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^c} \Gamma_g{}^e{}_d + \frac{\partial \tilde{x}^b}{\partial x^e} \frac{\partial^2 x^e}{\partial \tilde{x}^a \partial \tilde{x}^c}. \tag{148}$$

Properties (i),(ii),(iii),(iv) determine the action of the covariant derivative operator ∇_a on any tensor field. For example, to see how ∇_a acts on a covector field note that

$$\nabla_a (W_b V^b) = (\nabla_a W_b) V^b + W_b (\nabla_a V^b) = \partial_a (W_b V^b) \tag{149}$$

$$= (\partial_a W_b) V^b + (\partial_a V^b) W_b = (\nabla_a W_b) V^b + W_b (\partial_a V^b + \Gamma_a{}^b{}_c V^c), \tag{150}$$

which implies that

$$\boxed{\nabla_a W_b = \partial_a W_b - \Gamma_a{}^c{}_b W_c \,.} \tag{151}$$

Note the minus sign compared with the expression (iv) for a vector field. Similarly,

$$\boxed{\nabla_a W_{cb} = \partial_a W_{cb} - \Gamma_a{}^e{}_c W_{eb} - \Gamma_a{}^e{}_b W_{ce} \,.} \tag{152}$$

It is a useful exercise to check that

$$\nabla_a \nabla_b \phi = \nabla_a \partial_b \phi \tag{153}$$

are the components of a tensor field.

9.1 Symmetric Affine Connections

Using index shuffling and the equality of mixed partials we deduce that

$$\tilde{\Gamma}_{[a}{}^b{}_{c]} = \frac{\partial \tilde{x}^b}{\partial x^e} \frac{\partial x^g}{\partial \tilde{x}^{[a}} \frac{\partial x^d}{\partial \tilde{x}^{c]}} \Gamma_g{}^e{}_d \tag{154}$$

$$= \frac{\partial \tilde{x}^b}{\partial x^e} \frac{\partial x^{[g}}{\partial \tilde{x}^a} \frac{\partial x^{d]}}{\partial \tilde{x}^c} \Gamma_g{}^e{}_d \tag{155}$$

$$= \frac{\partial \tilde{x}^b}{\partial x^e} \frac{\partial x^g}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^c} \Gamma_{[g}{}^e{}_{d]} \,. \tag{156}$$

It follows that

$$T_a{}^b{}_c = \Gamma_a{}^b{}_c - \Gamma_c{}^b{}_a = -T_c{}^b{}_a \tag{157}$$

is a tensor of type $\binom{1}{2}$ called the *torsion tensor*. In what follows, we shall always assume that the torsion vanishes, i.e the components of the connection are symmetric

$$\boxed{\Gamma_a{}^b{}_c = \Gamma_c{}^b{}_a \,.} \tag{158}$$

You should check that you understand why the work we did above shows that this is a statement which holds in all coordinate systems.

9.1.1 Example: Exterior Derivative

An anti-symmetric co-tensor is called a *p-form*.

One may construct a $(p+1)$ form, called the *exterior derivative* or *generalized curl*, using just partial differentiation. We define [8]

$$d\omega_{abc\ldots} = (p+1)\partial_{[a}\omega_{bc\ldots]}. \tag{159}$$

To prove this is a tensor field we can either write out the behaviour under a coordinate transformation and see that the bad terms cancel, or more expeditiously, show that for *any symmetric* affine connection

$$\partial_{[a}\omega_{bc\ldots]} = \nabla_{[a}\omega_{bc\ldots]}. \tag{160}$$

The right hand side is a tensor and therefore the left hand side is a tensor. The operator d so defined acting on p forms is nilpotent: $d^2 = 0$.

9.1.2 Example: The Nijenhuis Bracket

Similarly, using the same technique, one may prove that given two $\binom{1}{1}$ tensor fields $A^a{}_b$ and $B^a{}_b$, then

$$S_a{}^b{}_c = A^e{}_a\partial_e B^b{}_c - A^b{}_e\partial_a B^e{}_c - A^e{}_c\partial_e B^b{}_a + A^b{}_e\partial_c B^e{}_a + B^e{}_a\partial_e A^b{}_c - B^b{}_e\partial_a A^e{}_c - B^e{}_c\partial_e A^b{}_a + B^b{}_e\partial_c A^e{}_a \tag{161}$$

is also a $\binom{1}{2}$ tensor field which is antisymmetric in a and c.

9.2 The Levi-Civita Connection

It is a striking fact that

(i)

$$\left\{{}_b{}^a{}_c\right\} = \left\{{}_c{}^a{}_b\right\} \tag{162}$$

(ii) $\left\{{}_b{}^a{}_c\right\}$ transform *precisely* as a symmetric affine connection.

We call this connection the *Levi-Civita connection.* It has the following remarkable property

$$\boxed{\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma_a{}^e{}_b g_{ec} - \Gamma_a{}^e{}_c g_{eb} = 0.} \tag{163}$$

In fact, a stronger statement is true

The Fundamental Theorem of Differential Geometry

The Levi-Civita connection $\left\{{}_b{}^a{}_c\right\}$ *is the unique affine connection s.t.*

(i)

$$\Gamma_b{}^a{}_c = \Gamma_c{}^a{}_b, \qquad \text{i.e. is symmetric} \tag{164}$$

[8] The $p+1$ factor is conventional and makes for simplifications in the formulation of Stokes's theorem.

(ii)

$$\nabla_a g_{cb} = 0, \qquad \text{the metric is covariantly constant.} \tag{165}$$

Proof: We write out the covariant constancy condition three times, cyclically permuting the three indices and then take a suitable linear combination. The symmetry of $\Gamma_a{}^b{}_c$ leads to cancellations:

$$(i) \qquad \nabla_a g_{bc} = \partial_a g_{bc} - \Gamma_a{}^e{}_b g_{ec} - \Gamma_a{}^e{}_c g_{eb} = 0. \tag{166}$$

$$(ii) \qquad \nabla_c g_{ab} = \partial_c g_{ab} - \Gamma_c{}^e{}_a g_{eb} - \Gamma_c{}^e{}_b g_{ea} = 0. \tag{167}$$

$$(iii) \qquad \nabla_b g_{ca} = \partial_b g_{ca} - \Gamma_b{}^e{}_c g_{ea} - \Gamma_b{}^e{}_a g_{ec} = 0. \tag{168}$$

Now take $(i) - (ii) - (iii)$ and use the symmetry of the connection to get

$$2g_{ea}\Gamma_b{}^e{}_c + \partial_a g_{bc} - \partial_c g_{ab} - \partial_b g_{ac} = 0, \tag{169}$$

which gives

$$\Gamma_b{}^e{}_c = \frac{1}{2} g^{es} \big(\partial_b g_{sc} + \partial_c g_{sb} - \partial_s g_{bc} \big) . \tag{170}$$

9.2.1 Example: metric-preserving connections with torsion

Repeat the above exercise for a connection with torsion to find that the connection coefficients are now given by

$$\Gamma_b{}^e{}_c = \frac{1}{2} g^{es} \big(\partial_b g_{sc} + \partial_c g_{sb} - \partial_s g_{bc} \big) + T_b{}^e{}_c - T_{bc}{}^e - T_{cb}{}^e. \tag{171}$$

The additional term

$$K_b{}^e{}_c = T_b{}^e{}_c - T_{bc}{}^e - T_{cb}{}^e \tag{172}$$

is sometimes called the *contorsion tensor*.

10 Parallel transport

If γ is a curve given by $x^a(\lambda)$ and $T^a = \frac{dx^a}{d\lambda}$ its tangent vector we define the *absolute derivative* of a vector V^a along γ by

$$\boxed{\frac{DV^a}{D\lambda} = T^b \nabla_b V^a \, .} \tag{173}$$

We often denote $\nabla_b V^a$ by $V^a{}_{;b}$ and so

$$\frac{DV^a}{D\lambda} = V^a{}_{;b} T^b. \tag{174}$$

We say that V^a is *parallely transported along* γ if

$$\boxed{\frac{DV^a}{D\lambda} = 0.} \tag{175}$$

That is

$$\frac{dV^a}{d\lambda} + (T^b \Gamma_b{}^a{}_c) V^c = 0, \tag{176}$$

or, infinitesimally,

$$dV^a = -\Gamma_b{}^a{}_e V^e dx^b \tag{177}$$

along γ. This is a linear o.d.e. along γ and has a unique solution given the initial value of the vector, $V^a(0)$. However *parallel transport is path dependent*, it depends on γ and two curves γ and γ' joining the same two events in spacetime will have different vectors V^a at the end points.

Note that we could have demanded the apparently weaker condition

$$\frac{DV^a}{D\lambda} = f(\lambda) V^a \tag{178}$$

along γ for some function $f(\lambda)$, but if

$$V^a = gU^a, \tag{179}$$

we have

$$\dot{g} U^a + g \frac{DU^a}{D\lambda} = fgU^a, \tag{180}$$

and so, by setting $\frac{\dot{g}}{g} = f$, we get

$$\frac{DU^a}{D\lambda} = 0. \tag{181}$$

10.1 Autoparallel curves

These are curves along which the tangent vector T^a is parallely transported [9]

$$\boxed{\frac{DT^a}{D\lambda} = 0} \tag{182}$$

or

$$\boxed{\frac{d^2 x^a}{d\lambda^2} + \Gamma_b{}^a{}_c \frac{dx^c}{d\lambda} \frac{dx^b}{d\lambda} = 0\,.} \tag{183}$$

For the Levi-Civita connection, $\Gamma_a{}^b{}_c = \left\{{}_a{}^b{}_c\right\}$ we recover our old definition of a *geodesic*.

[9] Although we won't use this fact in this course, the notions of parallel transport and auto-parallel curve also make sense for affine connections which are not necessarily symmetric.

Note that we could have demanded the apparently weaker condition that

$$\frac{DT^a}{D\lambda} = f(\lambda)T^a, \tag{184}$$

for some $f(\lambda)$. However if we change parameter

$$\lambda \rightarrow \tilde{\lambda} = \tilde{\lambda}(\lambda), \tag{185}$$

we find

$$T^a = g\tilde{T}^a, \tag{186}$$

where

$$g = \frac{d\tilde{\lambda}}{d\lambda}, \qquad \tilde{T}^a = \frac{dx^a}{d\tilde{\lambda}}, \tag{187}$$

and we can now use our previous remark to set $\frac{\dot{g}}{g} = f$ and find a new parameter $\tilde{\lambda}$ such that

$$\frac{D\tilde{T}^a}{D\tilde{\lambda}} = 0, \tag{188}$$

and we get back to our previous condition. Such a choice of parameter is called an *affine parameter* and it is unique up to an affine transformation

$$\tilde{\lambda} \rightarrow a\tilde{\lambda} + b, \qquad a, b \in R. \tag{189}$$

10.2 Acceleration and Force

Let τ be proper time along a timelike curve γ, then the *4-acceleration* is defined by

$$\boxed{a^a = \frac{DU^a}{D\tau},} \tag{190}$$

where $U^a = \frac{dx^a}{d\tau}$ is the *4-velocity* and is normalized so that $U^a U_a = g_{ab}U^a U^b = -1$.

Differentiating $g_{ab}U^a U_b = 1$ along γ and remembering that the metric is covariantly constant, $\frac{Dg_{ab}}{D\lambda} = 0$, we find that

$$2U^a a_a = 0. \tag{191}$$

Thus the 4-velocity and acceleration vector are orthogonal, in particular the acceleration vector is spacelike. The *4-force* F^a is defined by

$$\frac{d(mU^a)}{d\tau} = F^a, \tag{192}$$

where m is the rest mass of the particle, which we will assume is a constant. In this case Newton's law holds in the form $F^a = ma^a$. Any expression for the force must be orthogonal to the 4-velocity

$$F^a g_{ab} U^b = 0. \tag{193}$$

10.2.1 Example: The acceleration of a particle at rest

A particle at rest in a static metric has $U^a = \frac{1}{\sqrt{|g_{44}|}}\delta^a_4$ and a 4-acceleration

$$\dot{U}_a = -(\frac{\partial_i g_{44}}{2g_{44}}, 0). \tag{194}$$

Thus the 4-acceleration a particle at rest is directed radially *outward* and its magnitude squared is given by

$$\dot{U}^a \dot{U}_a = \frac{1}{4(g_{44})^2} h^{ij} \partial_i g_{44} \partial_j g_{44}. \tag{195}$$

In the Schwarzschild metric, this equals $\frac{\frac{M^2}{r^4}}{1-\frac{2M}{r}}$. The acceleration, and hence the force needed to keep the particle from falling radially inwards, diverge at the horizon $r = 2M$.

To check that this is reasonable, it is worth working through this example in the case of 2-dimensional Minkowski space $E^{1,1}$ in accelerating coordinates which is a good local model for the behaviour of the metric near a Killing horizon as we shall see later. If

$$x^0 = \rho \sinh t, \qquad x^1 = \rho \cosh t, \tag{196}$$

the metric in the *Rindler wedge* $x^1 > |x^0|$ is

$$ds^2 = -\rho^2 dt^2 + d\rho^2. \tag{197}$$

Curves $\rho =$ constant have acceleration $\frac{1}{\rho}$ and this is clearly in the positive x^1 direction.

10.2.2 Example: Charged Particles

For a particle of charge e moving in an electromagnetic field we have the *Lorentz equation*

$$m\frac{DU^a}{D\lambda} = eF^a{}_b U^b, \tag{198}$$

where

$$F_{ab} = g_{ac} F^c{}_b = -F_{ba} = F_{[ab]}, \tag{199}$$

is the second rank antisymmetric covariant *Faraday tensor*, the Lorentz 4-force is indeed orthogonal to the 4-velocity.

10.2.3 Example: Relativistic Rockets

Relativistic rockets have variable rest-mass, $m = m(\tau)$. Their equation of motion is

$$\frac{DmU^a}{D\tau} = J^a, \tag{200}$$

where J^a is the rate of emission of 4-momentum of the ejecta. Physically J^a must be timelike, $J_a J^a < 0$, which leads to the inequality

$$\frac{\dot{m}}{m} > |a^a|. \tag{201}$$

Thus to obtain a certain acceleration, as in the Twin-Paradox set-up, over a certain proper time requires a lower bound on the total mass of the fuel used

$$\ln(\frac{m_{\text{final}}}{m_{\text{initial}}}) < \int |a^a| d\tau. \tag{202}$$

In two dimensional Minkowski spacetime $E^{1,1}$

$$U^a = (\cosh\theta, \sinh\theta) \Rightarrow |a^a| = \frac{d\theta}{d\tau}, \tag{203}$$

where θ is the *rapidity*. We find

$$\frac{m_{\text{final}}}{m_{\text{initial}}} < \sqrt{\frac{1+v_{\text{initial}}}{1-v_{\text{initial}}}}\sqrt{\frac{1-v_{\text{final}}}{1+v_{\text{final}}}} = \frac{1}{1+z}\,. \tag{204}$$

Consider two observers, one of whom is at rest and engaged in checking Goldbach's conjecture that every even number is the sum of two primes using a computer. The second observer, initially at rest with respect to the first observer $v_{\text{initial}} = 0$, decides to use time dilation to find out faster by accelerating *towards* the stationary observer thus acquiring a velocity v_{final} and a *blue shift* factor $(1+z)$. The increase in the rate of gain of information is bounded by the energy or mass of the fuel expended.

10.3 The Levi-Civita connections of conformally related metrics

Supose we have two conformally related metrics such that

$$g'_{ab} = \Omega^2 g_{ab}. \tag{205}$$

A short calculation reveals that

$$\left\{{}_b{}^a{}_c\right\}' = \left\{{}_b{}^a{}_c\right\} + \delta^a_b \frac{\partial_c\Omega}{\Omega} + \delta^a_c \frac{\partial_b\Omega}{\Omega} - g^{as}\frac{\partial_s\Omega}{\Omega} g_{bc} \tag{206}$$

Now, given a curve γ, $x^a(\lambda)$ we have

$$\frac{d^2x^a}{d\lambda^2} + \left\{{}_b{}^a{}_c\right\}' \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = \frac{d^2x^a}{d\lambda^2} + \left\{{}_b{}^a{}_c\right\} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} + 2\frac{dx^a}{d\lambda}\frac{\dot{\Omega}}{\Omega} - g^{as}\frac{\partial_s\Omega}{\Omega} g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}. \tag{207}$$

Now, in general, if $x^a(\lambda)$ is a geodesic of the metric g_{ab} it will *not* be a geodesic of the metric g'_{ab}. However there is an exception. Supposing that γ is a null geodesic of the metric g_{ab} then we have

$$g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 0 \Rightarrow g'_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 0, \tag{208}$$

and so γ certainly has a null tangent vector with respect to the confomally related metric g'_{ab}. Now

$$\frac{d^2x^a}{d\lambda^2} + \left\{{}_b{}^a{}_c\right\}' \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = \frac{d^2x^a}{d\lambda^2} + \left\{{}_b{}^a{}_c\right\} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} + 2\frac{dx^a}{d\lambda}\frac{\dot{\Omega}}{\Omega}, \tag{209}$$

and so if $x^a(\lambda)$ is a null geodesic of the metric g_{ab} with λ an affine parameter, then the left hand side vanishes and we deduce from the vanishing of the right hand side that it is also a null geodesic of the conformally related metric g'_{ab}. But now λ *is no longer an affine parameter*. Thus while being a null geodesic is a conformally invariant statement, being affinely parametrized is not.

Physically two conformally related metrics cannot be distinguished by means of measurements made solely with light rays.

10.4 Static metrics and Fermat's Principle

If

$$\boxed{ds^2 = -e^{2U}dt^2 + h_{ij}dx^i dx^j} \tag{210}$$

we found that the equation of motion of a photon is

$$\frac{d^2x^i}{d\lambda^2} + \Big[\left\{{}_j{}^i{}_k\right\} + h^{is}h_{jk}\partial_s U\Big]\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0\,. \tag{211}$$

Now introduce a new spatial metric

$$h'_{ij} = e^{-2U}h_{ij}\,. \tag{212}$$

One has

$$\left\{{}_j{}^i{}_k\right\}' = \left\{{}_j{}^i{}_k\right\} - \delta^i_j\partial_k U - \delta^i_k\partial_j U + h^{is}\partial_s U h_{jk}. \tag{213}$$

Thus

$$\frac{d^2x^i}{d\lambda^2} + \left\{{}_j{}^i{}_k\right\}' \frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 2\frac{dx^i}{d\lambda}\partial_k U\frac{dx^k}{d\lambda}\,. \tag{214}$$

It follows that x^λ is a non-affinely parmetrized geodesic of the *optical or Fermat 3-metric*

$$ds_o^2 = e^{-2U}h_{ij}dx^i dx^j = h'_{ij}dx^i dx^j. \tag{215}$$

Since

$$ds_4^2 = e^{2U}\Big[-dt^2 + h'_{ij}dx^i dx^j\Big], \tag{216}$$

this is, in fact, a special case of our previous example with $\Omega = e^U$.

10.4.1 Isotropic coordinates and optical metric for the Schwarzschild solution: Shapiro time delay

Any spherically symmetric 3-metric is conformally flat[10]. In fact, if one introduces an *isotropic* coordinate ρ by

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2, \tag{217}$$

which maps $\infty > \rho > \frac{M}{2}$ to $\infty > r > 2M$ in a 1-1 way, then the Schwarzschild metric becomes

$$ds^2 = -\Big(\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\Big)^2 dt^2 + \Big(1+\frac{M}{2\rho}\Big)^4 \Big\{d\rho^2 + \rho^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\Big\} \tag{218}$$

The 3-metric inside the brace is flat. Thus the optical metric is of the form

$$ds_o^2 = n^2(\mathbf{x})d\mathbf{x}^2, \tag{219}$$

and $n(\mathbf{x})$ is an effective space-dependent refractive index given by

$$n(\mathbf{x}) = \frac{\big(1+\frac{M}{2|\mathbf{x}|}\big)^3}{\big(1-\frac{M}{2|\mathbf{x}|}\big)}. \tag{220}$$

The effective refractive index $n(\mathbf{x})$ increases from unity at infinity to infinity as one approaches the horizon at $r = 2M, \rho = \frac{M}{2}$. Light or radio waves passing near a star or black hole are thus slowed down compared with radio waves or light which keep away. This effect is called the *Shapiro time delay effect* and has been verifed recently (using the Cassini satellite when it was in opposition to the earth) to one part in a hundred thousand [11]. A radio wave is sent from earth to the satellite and back. To the relevant accuracy, the time delay compared with the flat space result may be obtained from the line integral $2\int_\gamma ndl$ where γ is a straight line in isotropic coordinates joining the earth and the satellite. If these are at radii r_1 and r_2 respectively, and b is the distance of nearest approach, the relevant time delay is

$$\frac{4GM}{c^3}\ln\left(\frac{4r_1r_2}{b^2}\right). \tag{221}$$

10.4.2 Example: Circular null geodesics

The circumference of a circle of coordinate radius ρ centred on the origin is

$$2\pi n(\rho)\rho. \tag{222}$$

[10]You should be able to prove this when you have understood the proof of Birkhoff's Theorem later in the course.

[11]Those interested can look up the article on the web (B Bertotti, L Iess and P Torora, A test of general relativity using radio links with the Cassini spacecraft, *Nature* **425** (2003) 374-376.

In the case of the Schwarzschild metric the circumference has a minimum value at $\rho = M(1+\sqrt{\frac{3}{4}})$, i.e. $r = 3M$ This corresponds to a circular geodesic of the optical metric, called in this context a circular null geodesic, although the null geodesic itself is of course not a closed curve in spacetime.

10.4.3 *Example: Stereographic projection, null geodesics in closed Friedman-Lemaitre universes and Maxwell's fish eye lens*

An n-sphere S^n of unit radius may be defined by its embedding into E^{n+1}, $n+1$ dimensional Euclidean space whose coordinates $X^0, X^1, X^2, \ldots, X^n$ are constrained to satisfy

$$(X^0)^2 + (X^i)^2 = 1, \tag{223}$$

$i = 1, 2, \ldots, n$. We may set

$$X^0 = \cos\chi, \qquad X^i = n^i \sin\chi, \tag{224}$$

with $n^i n^i = 1$, i.e. $n^i \in S^{n-1}$ and $0 \le \chi \le \pi$ .The metric induced from the embeddings, call it $d\Omega_n^2$, is given by

$$d\Omega_n^2 = (dX^0)^2 + (dX^i)^2, \tag{225}$$

which works out to be given by

$$d\Omega_n^2 = d\chi^2 + \sin^2\chi d\Omega_{n-1}^2, \tag{226}$$

where $d\Omega_{n-1}^2$ is the metric on S^{n-1}. If we set $\rho = \tan\frac{\chi}{2}$, we find that

$$ds^2 = \frac{4}{(1+\rho^2)^2}\left\{d\rho^2 + \rho^2 d\Omega_{n-2}^2\right\}. \tag{227}$$

The metric inside the brace is flat and the map taking χ, n^i to ρ, n^i is called *stereographic projection* and is clearly conformal. In other words the metric on S^n is *conformally flat*. The problem of finding the geodesics on S^n is thus the same as finding the light rays in a medium of refractive index

$$n(\mathbf{x}) = \frac{2}{1+\mathbf{x}^2}. \tag{228}$$

Note that there is a *circular geodesic* of length 2π at $|\mathbf{x}| = 1$, and hence from the high degree of symmetry $SO(n+1)$, *every* geodesic is closed with length 2π. In fact, every geodesic can be thought of as the intersection of the sphere with a 2-plane through the origin of E^{n+1}. Every distinct pair of 2-planes intersect on a line through the origin which cuts S^n at antipodal points X^α and $-X^\alpha$.

If $n = 3$, an optical device of this sort is called a *Maxwell Fish Eye Lens*. It has the remarkable property (obvious from the description in terms of the sphere) that for every 'object point' $\mathbf{x}_e$ there is a single 'image point' $\mathbf{x}_o$ through which every ray starting from $\mathbf{x}_e$ passes.

A related type of lens, used for some radar systems, is the *Luneburg lens*. In its idealized form, we set $n = 1$ outside the hemisphere $x^3 > 0$, $|\mathbf{x}| = 1$ and use formula (228) inside the hemisphere. The planar face of the lens is at $x^3 = 0$ and all rays incident orthogonally on this planar surface will be focussed onto the axis at $x^3 = 1$.

Now consider the $k = 1$ Friedman-Lemaitre metric

$$ds^2 = -dt^2 + a^2(t)d\Omega_3^2, \tag{229}$$

where $a(t)$ is the *scale factor*. This metric is conformal to the Einstein Static Universe

$$ds^2 = a^2(t)\{-d\eta^2 + d\Omega_3^2\}, \tag{230}$$

where

$$d\eta = \frac{dt}{a(t)} \tag{231}$$

is called *conformal time*. Thus from our work above we see that light rays in the universe behave precisely as they would in a Maxwell Fish Eye Lens.

10.5 *Projective Equivalence*

This concept is similar to that of conformal equivalence except that we focus on auto-parallels. We say two linear affine connections $\Gamma_b{}^c{}_d$ and $\tilde{\Gamma}_b{}^c{}_d$ are *projectiveley equivalent* if they share the same autoparallel paths[12], not necessarily with the same affine parameter. It follows that

$$\Gamma_{(b}{}^c{}_{d)} = \tilde{\Gamma}_{(b}{}^c{}_{d)} + \delta_b^c A_d + \delta_d^c A_b, \tag{232}$$

for some co-vector field A_b.

A *projective transformation* or *collineation* takes auto-parallel paths to auto-parallel paths.

The basic example is constructed from straight lines in R^n.

Globally one should consider RP^n. To describe this, introduce homogeneous coordinates X^α, $\alpha = 0, 1, \dots, n$. We can think of R^n as the hyperplane Π given by $X^0 = 1$. Straight lines correspond to the intersections of 2-planes through the origin with the hyperplane Π. Acting with $SL(n + 1, R)$ in the obvious way, will take straight lines to straight lines. However, some $SL(n + 1, R)$ transformations will take straight lines to straight lines at 'infinity', i.e. to 2-planes which do not intersect the hyperplane Π. Moreover while almost all straight lines intersect once, some which are pararallel do not intersect at all. To obtain a more symmetrical picture, one adds extra points at infinity to Π. One defines RP^n as the set of lines through the origin in R^{n+1}, i.e. $(n + 1)$-tuples X^α identified such that $X^\alpha \equiv \lambda X^\alpha$, $\lambda \neq 0$. Clearly $GL(n + 1, R)$ takes all straight lines to all straight lines but we need to factor by the action of $R \setminus 0$ to

[12]It is convenient to define a path as the image of a curve, i.e. to throw away the information about the parametrization.

get an effective action of $PSL(n+1,R)$. One may also check that every pair of distinct straight lines intesect once and only once.

Clearly, $S^n \subset R^{n+1}$ may be mapped onto the set of directions through the origin. However, anti-podal points on S^n must be identifed since $X^\alpha \equiv -X^\alpha$. Under this $2-1$ mapping great circles, i.e. geodesics on S^n map to straight lines in RP^n. As we have seen, by considering Maxwell's lens, every distinct pair of geodesics on S^n intersect twice. On RP^n these two intersections are identified.

10.5.1 *Example: metric preserving connections having the same geodesics*

A metric preserving connection with torsion $T_a{}^b{}_c$ is projectively equivalent to the Levi-Civita connection if and only if the torsion is totally antisymmetric

$$T_{abc} = T_{[abc]}. \tag{233}$$

11 Curvature

In general, parallel transport is non-commutative; it depends upon the path one takes. The extent to which it fails to do so is measured by the *Riemann Curvature Tensor*, which may be defined by

$$\boxed{(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c = R^c{}_{dab}V^d,} \tag{234}$$

a formula sometimes known as the *Ricci identity* .

We need to check that $R^c{}_{dab}$ is a $\binom{1}{3}$ tensor. Whatever it is, it is obviously anti-symmetric in a and b:

$$\boxed{R^c{}_{dab} = -R^c{}_{dba}\,.} \tag{235}$$

The point is the righthand side of the Ricci identity is a tensor field by construction. Now using the fact that

$$\nabla_a V^c = \partial_a V^c + \Gamma_a{}^c{}_e V^e, \tag{236}$$

we find that

$$\boxed{R^c{}_{dab} = \partial_a \Gamma_b{}^c{}_d + \Gamma_a{}^c{}_e \Gamma_b{}^e{}_d - (a \leftrightarrow b)\,.} \tag{237}$$

We see that $R^c{}_{dab}$ depends only on $\Gamma_a{}^b{}_c$ and its first derivatives, but not on derivatives of V^a. It is linear in V^d and therefore we can use the Quotient Theorem to show that $R^c{}_{dab}$ is indeed a tensor.

11.0.2 The Ricci Identity for co-vectors

Using the expresion for the curvature tensor and the definition of the covariant derivative for co-vectors one may check that *if the connection is symmetric* then

$$\boxed{(\nabla_a\nabla_b - \nabla_b\nabla_a)W_c = -R^d{}_{cab}W_d\,.} \tag{238}$$

There is an analogous expression for an arbitrary tensor of type $\binom{p}{q}$: there are p terms with positive sign and q terms with negative sign. In particular, for a second rank covariant tensor Q_{cd}, one has

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) Q_{cd} = -R^e{}_{cab} Q_{ed} - R^e{}_{dab} Q_{ce} \,. \tag{239}$$

11.0.3 The Ricci tensor

By contraction we can define the *Ricci tensor*

$$\boxed{R_{db} = R^a{}_{dab} = -R^a{}_{dba} \,.} \tag{240}$$

The definitions given above work for any affine connection, symmetric or not. In this course we are mainly interested in the Levi-Civita connection which satisfies

$$\boxed{\nabla_a g_{bc} = 0.} \tag{241}$$

This imposes extra symmetries which are most easily seen by introducing local inertial coordinates.

11.1 Local Inertial Coordinates

These are such that at any chosen point in spacetime, $x^a = 0$ say,

(i)

$$g_{ab}(0) = \eta_{ab} = \mathrm{diag}(+1, +1, +1, -1) \tag{242}$$

(ii)

$$\frac{\partial g_{ab}}{\partial x^c}\Big|_{x=0} = 0, \tag{243}$$

or, equivalently,

$$\Gamma_a{}^c{}_b(0) = 0. \tag{244}$$

Local inertial coordinates are also called *Riemann normal coordinates.*

11.1.1 Existence of Local Inertial Coordinates

We start off in a coordinate system x^a which is not inertial and solve for the necessary coordinate transformations taking us to $\tilde{x}^a$ which *are* inertial. We shall find we need to fix only the first and second derivatives of $\tilde{\mathbf{x}}$ with respect to x or, equivalently, of x with respect to $\tilde{x}$ at the origin 0.

Under a coordinate transformation we have

$$\tilde{g}_{ab} = g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} \tag{245}$$

and

$$\tilde{\Gamma}_a{}^b{}_c = \frac{\partial \tilde{x}^b}{\partial x^e} \Big[\frac{\partial x^g}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^c} \Gamma_g{}^e{}_d + \frac{\partial^2 x^e}{\partial \tilde{x}^a \partial \tilde{x}^c} \Big] \tag{246}$$

Now restrict to $x = 0$. We have

$$\tilde{g}_{ab}(0) = g_{cd}(0) \frac{\partial x^c}{\partial \tilde{x}^a}\Big|_{x=0} \frac{\partial x^d}{\partial \tilde{x}^b}\Big|_{x=0} \tag{247}$$

and

$$\tilde{\Gamma}_a{}^b{}_c(0) = \frac{\partial \tilde{x}^b}{\partial x^e}\Big|_{x=0} \Big[\frac{\partial x^g}{\partial \tilde{x}^a}\Big|_{x=0} \frac{\partial x^d}{\partial \tilde{x}^c}\Big|_{x=0} \Gamma_g{}^e{}_d(0) + \frac{\partial^2 x^e}{\partial \tilde{x}^a \partial \tilde{x}^c}\Big|_{x=0} \Big] \tag{248}$$

Now we can pick $\frac{\partial x^d}{\partial \tilde{x}^b}\Big|_{x=0}$ and $\frac{\partial^2 x^e}{\partial \tilde{x}^a \partial \tilde{x}^c}\Big|_{x=0}$ as we wish. We do so to make $\tilde{g}_{ab}(0) = \eta_{ab}$ and $\tilde{\Gamma}_a{}^b{}_c(0) = 0$. The first condition involves diagonalizing $g_{ab}(0)$ which can always be done provided the metric g_{ab} has the correct signature. The second then involves solving a linear equation for the second derivatives.

11.2 Physical significance of local inertial coordinates, justification of the clock postulate

The physical significance of local inertial coordinates is that they allow one to 'abolish'the local effect of gravity by passing to a freely falling frame. In local inertial coordinates, the geodesic equations becomes

$$\frac{d^2 \tilde{x}^a}{d\tau^2} + \left\{ {}_b{}^a{}_c \right\} \frac{d\tilde{x}^b}{d\tau} \frac{d\tilde{x}^c}{d\tau t} = 0 = \frac{d^2 \tilde{x}^a}{d\tau^2} + \dots \tag{249}$$

and thus

$$\tilde{x}^a(\tau) = \tilde{x}^a(0) + U^a(0)\tau + \dots \tag{250}$$

and so geodesics are straight lines to lowest order, just as in Minkowski spacetime.

In fact, Fermi showed that one may introduce local inertial coordinates along any timelike geodesic.

Note that local physics in local inertial coordinates is the same as in Minkowski spacetime. Thus if one writes down the equations of quantum mechanics describing an atomic clock at rest, then the elapse of x^4 is what the clock measures. But this coincides with proper time along a timelike geodesic with x^i = constant.

11.2.1 Consequences: The Cyclic Identity

In local inertial coordinates at a point we have

$$(i) \qquad R^d{}_{cab} \;''='' \; \partial_a \Gamma_b{}^d{}_c - \partial_b \Gamma_a{}^d{}_c \tag{251}$$

$$(ii) \qquad R^d{}_{bca} \;''='' \; \partial_c \Gamma_a{}^d{}_b - \partial_a \Gamma_c{}^d{}_b \tag{252}$$

$$(iii) \qquad R^d{}_{abc} \;''='' \; \partial_b \Gamma_c{}^d{}_a - \partial_c \Gamma_b{}^d{}_a \tag{253}$$

We have introduced the notation $''=''$ to indicate that an equation is true only in inertial coordinates. Using $\Gamma_a{}^b{}_c = \Gamma_c{}^b{}_a$ we deduce that in inertial coordinates

$$\boxed{R^d{}_{cab} + R^d{}_{bca} + R^d{}_{abc} = 0 = R^d{}_{[abc]} \, .} \tag{254}$$

Now we can drop the $'' ='' $ because the equation is a tensorial relation, and hence, if it is true in one coordinate system, it is true in all coordinate systems.

11.2.2 Consequences: The Bianchi Identity

In local inertial coordinates at a point we also have

$$(i) \qquad R^d{}_{cab;e} '' ='' \partial_e \partial_a \Gamma_b{}^d{}_c - \partial_e \partial_b \Gamma_a{}^d{}_c \tag{255}$$

$$(ii) \qquad R^d{}_{cea;b} '' ='' \partial_b \partial_e \Gamma_a{}^d{}_c - \partial_b \partial_a \Gamma_e{}^d{}_c \tag{256}$$

$$(iiii) \qquad R^d{}_{cbe;a} '' ='' \partial_a \partial_b \Gamma_e{}^d{}_c - \partial_a \partial_e \Gamma_b{}^d{}_c \tag{257}$$

Now using the equality of mixed partials we get

$$\boxed{R^d{}_{cab;e} + R^d{}_{cea;b} + R^d{}_{cbe;a} = 0 = R^d{}_{c[ab;e]} .} \tag{258}$$

11.2.3 *Example: The other possible contraction*

For a general symmetric connection the Ricci tensor has no particular symmetries. Let $S_{ab} = -S_{ab} = R^e{}_{eab}$ be the other possible contraction. The cyclic identity tells us that this is not independent:

$$S_{bd} = R_{bd} - R_{db}. \tag{259}$$

The second Bianchi identity gives

$$S_{[bd;c]} = 0, \tag{260}$$

thus, locally,

$$S_{bd} = \partial_b S_d - \partial_d S_b, \tag{261}$$

for some co-vector S_b.

11.2.4 Example: *Weyl Connections*

These preserve angles under parallel transport but not the metric. Thus

$$\nabla_a g_{bc} = A_a g_{bc}, \tag{262}$$

for some one-form A_a. Thus

$$\Gamma_b{}^c{}_d = \left\{ {}_b{}^c{}_d \right\} - \frac{1}{2}\big(\delta^c_b A_d + \delta^c_d A_b - g_{db} g^{ce} A_e\big). \tag{263}$$

Note that, if $g_{ab} = \Omega^2 \tilde{g}_{ab}$, then $A_a = 2\frac{\partial_a \Omega}{\Omega}$. Conversely, if $A_a = 2\frac{\partial_a \Omega}{\Omega}$, then we can find a conformally related metric which is invariant under parallel transport. A Weyl connection of that type is rather trivial because it can be eliminated by a conformal 'gauge'transformation. A gauge-invariant measure of triviality in this sense is the second contraction

$$S_{db} = R^a{}_{adb} = n\big(\partial_d A_b - \partial_b A_d\big). \tag{264}$$

Weyl suggested that one might generalize Einstein's theory by endowing spacetime with what we now call a Weyl connection in which measuring rods moved from A to B along a path γ suffer parallel transport. Einstein pointed out that if it were non-trivial, then the measuring rods would not return to their original size on returning to A by some other curve unless the curvature $R^a{}_{abd} = S_{bd}$ vanishes. This contradicts the observed fact that measuring rods are made from atoms and atoms have quite definite sizes.

In fact, atomic units of length and time are constructed from the Bohr radius

$$R_B = \frac{4\pi\epsilon_0 \,\tilde{}^{2}}{m_e e^2}. \tag{265}$$

Even if the charge on the electron e or the mass of the electron m_e varied with position, under parallel transport the Bohr radius, and hence the metric would undergo a trivial conformal change which could be eliminated by a conformal rescaling.

Although Weyl rapidily abandoned his theory that the equations of physics should be gauge-invariant in the sense of independent of units of length, the idea of gauge-invariance resurfaced soon after when quantum mechanics was discovered and it was realized that in the presence of an electromagnetic field, one should replace $\partial_a \Psi$ by $\left(\partial_a - i\frac{e}{\tilde{}} A_a\right)\Psi$ in the Schrödinger equation, where now A_b is the electro-magnetic vector potential. Parallel transport of an electron now results in a change of the phase of its wave function Ψ. It is now believed that the equations of physics are exactly gauge-invariant in this sense.

11.2.5 Example:* Projective Curvature*

In general two projectively equivalent connections have different Riemann and Ricci tensors. However, using normal coordinates one may show that the following tensor

$$W^a{}_{bcd} = R^a{}_{bcd} + \frac{2}{(n+1)(n-1)}\delta^a_{[d}R_{c]b} + \frac{2n}{(n+1)(n-1)}\delta^a_{[d}R_{|b|c]} - \frac{2}{n+1}\delta^a_b R_{[cd]}, \tag{266}$$

where $|b|$ means that the index b is omitted from the anti-symmetrization,is the same for both connections.

In the case of the Levi-Civita connection, the Ricci tensor is symmetric and the projective curvature simplifies to

$$W^a{}_{bcd} = R^a{}_{bcd} + \frac{2}{n-1}\delta^a_{[d}R_{c]b}. \tag{267}$$

Thus a space of constant curvature with $R^a{}_{bcd} = K\,\delta^a_{[d}g_{c]b}$ has $W^a{}_{bcd} = 0$.This is consistent with our discussion of S^n and RP^n.

11.3 Consequences of the metric Preserving Property

The cyclic identity and the Bianchi identity hold for *any* symmetric affine connection. To make further progress we must use the metric preserving condition.

The quickest way of proceeding is to note that

$$\boxed{R_{abcd} \text{''}=\text{''} \ \frac{1}{2}\Big(\partial_b\partial_c g_{ad} + \partial_a\partial_d g_{bc} - \partial_b\partial_d g_{ac} - \partial_a\partial_c g_{bd}\Big)} \tag{268}$$

We deduce that

$$\boxed{R_{abcd} = R_{[ab]cd}} \tag{269}$$

and

$$\boxed{R_{abcd} = R_{cdab}} \tag{270}$$

It follows that the Ricci tensor R_{ab} is symmetric

$$\boxed{R_{ab} = R_{ba}} \tag{271}$$

In local inertal coordinates the first derivatives of the metric vanish and the second derivatives determine and are determined by the curvature tensor. One has

$$g_{ab} \text{''}=\text{''} \ \eta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + \mathcal{O}(x^3), \tag{272}$$

whence

$$\partial_e\partial_f g_{ab} = -\frac{1}{3}\Big(R_{aebf} + R_{afbe}\Big). \tag{273}$$

It is a useful exercise to substitute (273) into the left hand side of (268) and check that it works.

If one does not wish to use inertial coordinates one may procede as follows. One applies the the Ricci identity for co-tensors (239) to the metric

$$\nabla_e\nabla_f g_{ab} - \nabla_f\nabla_e g_{ab} = -R^g{}_{aef} g_{gb} - R^g{}_{bef} g_{ag}. \tag{274}$$

The right hand side vanishes and hence

$$R_{abfe} + R_{bafe} = 0. \tag{275}$$

To establish that R_{abfe} is symmetric with respect to swapping the index pairs ab and ef one may take four copies of the cyclic identity

$$(i) \qquad R_{abcd} + R_{adbc} + R_{acdb} = 0 \tag{276}$$

$$(ii) \qquad R_{badc} + R_{bdca} + R_{bcad} = 0 \tag{277}$$

$$(iii) \qquad R_{cbda} + R_{cabd} + R_{cdab} = 0 \tag{278}$$

$$(iv) \qquad R_{dbac} + R_{dacb} + R_{dcba} = 0 \tag{279}$$

One now takes $(i) + (ii) - (iii) - (iv)$ and the skew-symmetry on the first and on the second pair of indices to obtain the result.

11.3.1 *Example: The curvature of the sphere*

Stereographic coordinates on the unit n-sphere S^n introduced earlier, in which the metric is

$$ds^2 = 4\frac{dx_i dx_i}{(1 + x_k x_k)^2} \tag{280}$$

are Riemann normal coordinates centred on the origin.

From this one deduces that at the origin

$$R_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{281}$$

But, by $SO(n+1)$ invariance, the origin is not a privileged point on the sphere and hence (281) must be true everywhere on S^n with unit radius. If the radius is a then

$$R_{abcd} = \frac{1}{a^2}\big(g_{ac}g_{bd} - g_{ad}g_{bc}\big). \tag{282}$$

.

11.4 Contracted Bianchi Identities

Contracting the Bianchi identity

$$R^d{}_{cab;e} + R^d{}_{cea;b} + R^d{}_{cbe;a} = 0 \tag{283}$$

on d and a gives

$$R_{cb;e} - R_{ce;b} + R^d{}_{cbe;d} = 0 \tag{284}$$

Now contract on with g^{ce} to get

$$R^c{}_{b;c} - R_{;b} + R^d{}_{b;d} = 0 \tag{285}$$

where we define the *Ricci scalar* by

$$R = g^{ab}R_{ab} = g^{ab}R^e{}_{aeb} = g^{ab}g^{cd}R_{acbd} \tag{286}$$

and have used the symmetries of the Riemann tensor to give

$$R^a{}_b = R^a{}_{cbe}g^{ce}. \tag{287}$$

We can tidy up this expression to give

$$\boxed{\Big(R^{ab} - \frac{1}{2}g^{ab}R\Big)_{;b} = 0} \tag{288}$$

or defining the Einstein tensor by

$$G^{ab} = R^{ab} - \frac{1}{2}g^{ab}R, \tag{289}$$

$$\boxed{G^{ab}{}_{;b} = 0.} \tag{290}$$

11.5 Summary of the Properties of the Riemann Tensor

$$(i) \qquad R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab} \tag{291}$$

$$(ii) \qquad R_{abcd} + R_{acdb} + R_{adcb} = 0. \tag{292}$$

$$(iii) \qquad R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0. \tag{293}$$

$$\text{in addition}\,(iii) \Rightarrow \Bigl(R^{ab} - \frac{1}{2}g^{ab}R\Bigr)_{;b} = 0. \tag{294}$$

12 The Einstein Equations

We have now developed sufficient tensor analysis to obtain the Einstein equations. In fact *in vacuo*

$$R_{ab} = 0, \tag{295}$$

or possibly

$$\boxed{R_{ab} = \Lambda g_{ab}} \tag{296}$$

with Λ the *Cosmological Constant*. Why is Λ a constant? We know that

$$R_{ab}{}^{;b} = \frac{1}{2}\partial_a R, \tag{297}$$

but since $g^{ab}g_{ab} = \delta^a_a = 4$, contraction of (296) with g^{ab} gives $R = 4\Lambda$.

Thus

$$(\Lambda g_{ab})^{;b} = \Lambda_{;a} = \frac{1}{2}(4\Lambda)_{;a} = 2\Lambda_{;a} \tag{298}$$

which implies that

$$\Lambda_{;a} = \partial_a \Lambda = 0. \tag{299}$$

Note that if $\Lambda \neq 0$, we cannot put $g_{ab} = \eta_{ab}$, since then $R_{ab} = 0$. That is if $\Lambda \neq 0$, then spacetime cannot be flat.

Remarkably very recent observations of the recession of distant galaxies (which seem to be accelerating away from us) and of the cosmic microwave background indicate that Λ is very small but *positive*. The dimensions of the cosmological constant Λ are length^{-2} and roughly

$$\Lambda \approx (10^{27}\text{cm})^{-2}. \tag{300}$$

12.1 Uniqueness of the Einstein equations: Lovelock's theorem

Following a programme initiated by Weyl and Cartan, Lovelock proved the following

Theorem *In four spacetime dimensions, the only tensor V^{ab} constructed solely from the metric g_{ab} and its first and second partial derivatives $\partial_a g_{bc}$ and $\partial_a \partial_b g_{cd}$ which is conserved, ${V^{ab}}_{;b} = 0$, is a linear combination of the Einstein tensor and the metric with constant coefficients. In particular V^{ab} is symmetric.*

In higher dimensions there exist tensors V^{ab} which contain higher than first powers of second derivatives of the metric $\partial_a \partial_b g_{cd}$. In four spacetime dimensions this cannot happen. If we restrict to tensors V^{ab} which are no more than linear in $\partial_a \partial_b g_{cd}$, then only linear combinations of the Einstein tensor and metric with constant coefficients are allowed.

12.2 *Dilatation and Conformal symmetry*

In the absence of a cosmological term, Einstein's vacuum equations contain no dimensionful quantity, no length for example. As a consequence for every solution with metric g_{ab} there is another solution with metric $g'_{ab} = \lambda^2 g_{ab}$, where λ is an arbitrary positive constant. This is because under such a scaling we have $R'_{ab} = R_{ab}$. All lengths, times and masses in the metric g'_{ab} are rescaled by a factor λ. To see this in detail, consider the Schwarzschild metric. One easily sees that mutiplying the metric by a factorλ^2 is equivalent to rescaling the time t and radial coordinates $t \rightarrow \lambda t$, $r \rightarrow \lambda r$ provided one rescales the mass $M \rightarrow \lambda M$. The symmetry thus generalizes the dilatation symmetry of Minkowski spacetime, hence its name. Of course the symmetry is broken by the cosmological term.

Note that in general, even if $\Lambda = 0$, the Einstein equations do not admit general conformal symmetry: one *cannot* allow λ to depend upon position.

12.2.1 *The Weyl Conformal Curvature Tensor*

Let $\Omega = e^{\Phi}$. If $\tilde{g}_{ab} = \Omega^2 g_{ab}$, then the Weyl tensor

$$
{C^{ab}}_{cd} = {R^{ab}}_{cd} - \frac{1}{n-1}\left(R^a_c \delta^b_d - R^a_d \delta^b_c - R^b_c \delta^a_d + R^b_d \delta^a_c\right) + \frac{1}{(n-1)(n-2)} R\left(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c\right) \tag{301}
$$

is the same for the two metrics. Moreover, all of its contractions vanish.

12.3 Geodesic Deviation

To motivate the Einstein equations we consider the *Geodesic deviation equation* in General Relativity and compare it with Newtonian theory for which, as we have seen

$$
\frac{d^2 N_i}{dt^2} + E_{ij} N_j = 0. \tag{302}
$$

We begin by considering a *family of geodesics* such that $x^a(\lambda, \sigma)$ is a geodesic for fixed σ.

$$T^a = \frac{\partial x^a}{\partial \lambda}\Big|_\sigma \tag{303}$$

is the tangent vector to the geodesic labelled by σ. We call

$$N^a = \frac{\partial x^a}{\partial \sigma}\Big|_\lambda \tag{304}$$

a *connecting vector* . Thus

$$\frac{\partial T^a}{\partial \sigma} = \frac{\partial N^a}{\partial \lambda} = \frac{\partial^2 x^a}{\partial \sigma \partial \lambda} = \frac{\partial^2 x^a}{\partial \lambda \partial \sigma}. \tag{305}$$

We may think of σ and λ as a space and time coordinate. If we introduce two more coordinates x^1, x^2 say we have, in the coordinate system $(x^1, x^2, \sigma, \lambda)$, (labeling the last two coordinates by σ and λ)

$$T^a = \delta^a_4, \qquad N^a = \delta^a_3. \tag{306}$$

Thus

$$T^a{}_{,b} N^b - N^a{}_{,b} T^b = 0. \tag{307}$$

Now

$$T^a{}_{;b} N^b = T^a_{,b} N^b + \Gamma_b{}^a{}_c T^c N^b, \tag{308}$$

and

$$N^a{}_{;b} T^b = N^a_{,b} T^b + \Gamma_b{}^a{}_c N^c T^b. \tag{309}$$

Thus *for a symmetric connection* we have

$$\boxed{T^a{}_{;b} N^b - N^a{}_{;b} T^b = 0.}. \tag{310}$$

In fact for any two vector fields, it is the case that

$$\Big[T, N\Big]^a = T^a \partial_a N^b - N^a \partial_a T^b, \tag{311}$$

is also a vector field,called the *commutator* or *bracket*. As you may check, the bad terms in the coordinate transformation cancel. In our case, we say that the tangent vector T^a and the connecting vector N^a *commute*.

We now claim that if

$$(i) \qquad T^a{}_{;b} T^b = 0, \tag{312}$$

(i.e. T^a is the tangent of a geodesic) and

$$(ii) \qquad T^a{}_{;b} N^b - N^a{}_{;b} T^b = 0, \tag{313}$$

(i.e. T^a and N^a) commute, then the general relativistic version of the equation of geodesic deviation, sometimes called the Jacobi equation, is

$$\boxed{\frac{D^2N^a}{D\tau^2} + E^a{}_b N^b = 0 \qquad \text{Geodesic Deviation}} \tag{314}$$

where

$$\boxed{E^a{}_b = R^a{}_{dbc} T^d T^c.} \tag{315}$$

Proof The Ricci identity, i.e. the definition of curvature is

$$T^a{}_{;b;c} - T^a{}_{;c;b} = -R^a{}_{dbc} T^d. \tag{316}$$

Contracting with $T^a N^b$ gives

$$\star \qquad N^b \big(T^a{}_{;b;c} T^c - T^a{}_{;c;b} T^c \big) = -E^a{}_b N^b \tag{317}$$

But

$$\frac{D^2N^a}{D\tau^2} = \big(N^a{}_{;b} T^b \big)_{;c} T^c = \big(T^a{}_{;b} N^b \big)_{;c} T^c \tag{318}$$

(by commutivity, (ii)). Expanding we have

$$\frac{D^2N^a}{D\tau^2} = T^a{}_{;b;c} N^b T^c + T^a{}_{;b} N^b{}_{;c} T^c. \tag{319}$$

Now using ($\star$) we get

$$\frac{D^2N^a}{D\tau^2} + E^a{}_b N^b = T^a{}_{;b} N^b{}_{;c} T^c + T^a{}_{;c;b} T^c N^b \tag{320}$$

$$= T^a{}_{;b} N^b{}_{;c} T^c + (T^a{}_{;c} T^c)_{;b} N^b - T^a{}_{;c} T^c{}_{;b} N^b \tag{321}$$

$$= T^a{}_{;b} N^b{}_{;c} T^c - T^a{}_{;c} T^c{}_{;b} N^b \tag{322}$$

(using the geodesic equation (ii))

$$= T^a{}_{;b} N^b{}_{;c} T^c - T^a{}_{;c} N^c{}_{;b} T^b = 0. \tag{323}$$

(by the commutivity (ii)).

12.4 A convenient choice of connecting vector

The calculations given above work if the geodesic γ with parameter λ is spacelike, timelike or null. It is often convenenient to choose N^a to be orthogonal to T^a,

$$T^a N_a = 0. \tag{324}$$

This is possible because of the following

Lemma *If $T^a N_a = 0$ at one point of γ, then $T^a N_a = 0$ on all points of γ.*

Proof

$$\big(T^a N_a\big)_{;c} T^c = \frac{D}{D\lambda}\big(T^a N_a\big) = T^a N_{a;c} T^c \tag{325}$$

(because T^a is tangent to a geodesic)

$$= T^a T_{a;c} N^c \tag{326}$$

(by commutivity)

$$= \frac{1}{2}(g_{ab} T^a T^b)_{;c} N^c = 0. \tag{327}$$

The last step uses the fact the the metric is covariantly constant and that $g_{ab}T^aT^b =$ constant along a geodesic.

Thus

$$\frac{D}{D\lambda}\big(N_a T^a\big) = 0. \tag{328}$$

If it is zero for one value of λ it will be zero for all values of λ.

12.5 Example

A spacetime of constant sectional curvature has by definition

$$R_{abcd} = K\big(g_{bd}g_{ac} - g_{ad}g_{bc}\big) \tag{329}$$

You should check that this expression has all the required symmetries.

In a general spacetime, if W^a and V^a span a two-plane Π, the sectional curvature of that plane is defined by

$$K(V,W) = R_{abcd} V^a W^b V^c W^d. \tag{330}$$

Note that, by the symmetries of the Riemann tensor, if $V^a \to aV^a + bW^a$, $K \to a^2 K$, If the 2-plane Π is *timelike* we may choose V^a and W^a such that $V^aV_a = -1, \quad W^aW_a = +1, \quad V^cW_c = 0$. This fixes the scale of K. If the 2-plane is spacelike we may choose $V^aV_a = 1, \quad W^aW_a = +1, \quad V^cW_c = 0$. A space of constant curvature thus has $K =$ constant on all 2-planes. Moreover

Schur's Lemma *In a space of constant sectional curvature K is independent of position.*

Proof

By contraction with g^{ac} one finds

$$R_{bd} = 3K g_{bd}, \tag{331}$$

$$3K = \Lambda = \text{constant}, \tag{332}$$

by the contracted Bianchi identity.

In a space of constant sectional curvature, the equation of geodesic deviation becomes

$$\frac{D^2 N^a}{D\lambda^2} + K\big(T_c T^c N^a - T_c N^c T^a\big) = 0\,. \tag{333}$$

We set $T^a N_a = 0$ and obtain

$$\frac{D^2 N^a}{D\lambda^2} + K T_c T^c N^a = 0 \tag{334}$$

In the timelike case we may set $T^a T_a = -1$ and if $\Lambda > 0$ (which corresponds to de-Sitter spacetime) then timelike geodesics separate exponentially as $\sqrt{\frac{\Lambda}{3}}\tau$, where τ is propertime. If $\Lambda < 0$ (which corresponds to Anti-de-Siter spactime) then they oscillate with period $2\pi\sqrt{\frac{3}{-\Lambda}}$.

13 The Einstein Field Equations with Matter

Comparing the general relativistic geodesic equation with that in Newtonian theory we have recal Poisson's equation

$$E_{ii} = 4\pi G\rho \tag{335}$$

Now in the Newtonian limit $E_{ii} = E^i{}_i \approx R^i{}_{4i4} = R_{44}$, that is

$$R_{ab} T^a T^b \approx 4\pi G\rho. \tag{336}$$

This suggests that the relativistic analogue shold be something like

$$R_{ab} \propto T_{ab}, \tag{337}$$

where T_{ab} is a suitable tensor determined by the matter distribution. This is *roughly* correct, but to see what the correct answer is we need to introduce

13.1 The Energy Momentum Tensor

Consider a theory in flat spacetime. We can define an energy density $\mathcal{H}$ say with units energy per unit spatial volume and an *energy flux vector* s_i say with units energy per unit area per unit time. Conservation of energy now reads[13]

$$\boxed{(i) \qquad \frac{\partial \mathcal{H}}{\partial t} + \partial_i s_i = 0} \tag{338}$$

One may also define a *momentum density* π_i with units of momentum per unit 3-volume such that such that the momentum π in a time-independent domain D in E^3 is

$$p_i = \int_D \pi_i d^3x\,. \tag{339}$$

Now we postulate that the only forces act through the boundary ∂D of the domain D

$$\frac{dp_i}{dt} = \int_D \frac{\pi_i}{\partial t} d^3x = -\int_{\partial D} T_{ij} d\sigma_j\,, \tag{340}$$

[13]For the next few equations we revert to Cartesian tensor notation and place all indices downstairs, just as one does in PartIB. Their positions will be adjusted shortly.

where the force exterted by the material inside the domain F_i on a surface element $d\sigma_i$ is given by

$$F_i = T_{ij} d\sigma_j. \tag{341}$$

If this equation is true for all domains we must have

$$\boxed{(ii) \qquad \frac{\partial \pi_i}{\partial t} + \partial_j T_{ij} = 0.} \tag{342}$$

Now conservation of angular momentum imposes the symmetry condition

$$T_{ij} = T_{ji}. \tag{343}$$

Proof

$$\frac{\partial}{\partial t}\big(x_k\pi_i - x_k\pi_i\big) = -x_k\partial_j T_{ij} + x_i \partial_k T_{kj} \tag{344}$$

$$= -\partial_j\big(x_k T_{ij} - x_i T_{kj}\big) + T_{ik} - T_{ki} \tag{345}$$

i.e.

$$\frac{\partial}{\partial t}\big(x_k\pi_i - x_i\pi_k\big) + \partial_j\big(x_k T_{ij} - x_i T_{kj}\big) = T_{ki} - T_{ik} \tag{346}$$

Integrating we get

$$\frac{d}{dt}\int_D \big(x_k\pi_i - x_i\pi_k\big) d^3x = -\int_{\partial D}\big(x_k T_{ij} - x_i T_{kj}\big) d\sigma_j + \int_D \big(T_{ik} - T_{ki}\big) d^3x. \tag{347}$$

The left hand side is the rate of change of $\epsilon_{kij} L_j$, where L_j is the total angular momentum inside the domain D. The first term on the righthand side $\epsilon_{kij} G_k$, where G_i is is the external torque on the system. If these are to be equal for all domains D then T_{ij} must be symmetric.

Our two fundamental equations may be combined in a single equation

$$\boxed{\partial_a T^{ab} = 0,} \tag{348}$$

provided we set

$$T^{44} = \mathcal{H} \qquad T^{i0} = \frac{s_i}{c}, \qquad T^{0i} = c\pi_i, \qquad T^{ij} = T_{ij},. \tag{349}$$

that is

$$T^{ab} = \begin{pmatrix} \mathcal{H} & c\pi_i \\ \frac{s_i}{c} & T_{ij} \end{pmatrix} \tag{350}$$

and $\partial_a = (\partial_i, \partial_4)$ with $x^4 = ct$.

You should check that all components of T^{ab} have the dimensions $\frac{\text{energy}}{3-\text{volume}}$. In order to render the conservation of angular momentum condition $T_{ij} = T_{ji}$ invariant under Lorentz transformations must extend the symmetry property $T_{ij} = T_{ij}$ to the relativistically invariant condition

$$\boxed{T^{ab} = T^{ba}}. \tag{351}$$

This implies a relation between energy flux s_i and momentum density π

$$c\pi_i = \frac{s_i}{c} \tag{352}$$

13.2 Generalization to curved spacetime

We make what is often referred to as a *minimal coupling* assumption, that is when we see an equation in special relativity containing a comma we simply replace the comma by a semi-colon. In this way the equation will certainly be covariant and hopefully it will also be consistent. Thus we assume that

$$\boxed{T^{ab}{}_{;b} = \partial_b T^{ab} + \Gamma_c{}^b{}_b T^{ac} + \Gamma_c{}^a{}_d T^{cd} = 0.} \tag{353}$$

In fact

$$\Gamma_a{}^b{}_b = \frac{1}{\sqrt{-g}} \partial_a \big(\sqrt{-g}\big), \tag{354}$$

where

$$g = \det g_{ab}. \tag{355}$$

Proof

Recall that

$$\Gamma_b{}^a{}_c = \frac{1}{2} g^{ad} \Big(g_{bd,c} + g_{cd,b} - g_{bc,d} \Big), \tag{356}$$

so that

$$\Gamma_a{}^b{}_b = \frac{1}{2} g^{cd} \partial_a g_{cd}. \tag{357}$$

Now for any one parameter family of symmetric matrices $M(t)$ with determinant $\Delta(t)$ one has (by diagonalization)

$$\dot{\Delta} = \Delta \, \mathrm{Trace} \, M^{-1} \dot{M}, \tag{358}$$

and the result follows. The conservation law may now be re-written

$$T^{ab}{}_{;b} = \frac{1}{\sqrt{-g}} \partial_a \big(\sqrt{-g} T^{ab} \big) + \Gamma_c{}^a{}_d T^{cd} = 0. \tag{359}$$

In the Newtonian limit when T^{ab} is dominated by T^{44} the second term $\Gamma_4{}^i{}_4 \rho$ may be interpreted as a Newtonian force since $\Gamma_4{}^i{}_4 \approx \partial_i U$.

14 The Einstein field equations

The obvious candidate equations, which are tensorial and constructed from no higher than second derivatives of the metric, are

$$R^{ab} = AT^{ab} + Bg^{ab} + CRg^{ab}, \tag{360}$$

where A, B, C are constants. Acting with ∇_a shows that $C = \frac{1}{2}$, and comparison with Poissons equation now shows that$A = \frac{8\pi G}{c^4}$.

Thus

$$R^{ab} - \frac{1}{2} g^{ab} R = \frac{8\pi G}{c^4} T^{ab} - \Lambda g^{ab}. \tag{361}$$

14.1 Example and derivation of the geodesic postulate

A *perfect fluid*, i.e.an inviscid fluid has

$$T^{ab} = \rho U^a U^b + P\big(g^{ab} + \frac{1}{c^2} U^a U^b\big),) \tag{362}$$

where the *Eulerian 4-velocity* of the fluid U^a satisfies $U^a U_a = -c^2$ and ρ is the mass density and P the pressure. This is consistent with the fact that in the local rest frame of the fluid for which $U^a = (c, \mathbf{0})$,

$$T^{44} = \rho c^2, \qquad T^{ij} = P\delta^{ij}. \tag{363}$$

An *equation of state* is a relation $P = P(\rho)$. For example for a *radiation gas*

$$\frac{1}{c^2} P = \frac{1}{3}\rho, \tag{364}$$

which implies that T^{ab} is trace-free $T = T^{ab} g_{ab} = 0$.

For *dark energy* or a cosmological constant we have

$$T^{ab} = -\frac{\Lambda c^4}{8\pi G} g^{ab}, \tag{365}$$

thus

$$\rho = +\frac{\Lambda c^2}{8\pi G} \tag{366}$$

and

$$\frac{P}{c^2} = -\rho. \tag{367}$$

Thus if the cosmological constant is positive then the pressure is negative, in other words, the medium is in a state of tension,

A pressure-free fluid is often called *dust*. It has $P = 0$ and hence $T^{ab} = \rho U^a U^b$. We have

$$T^{ab}{}_{;b} = \big(\rho U^a U^b\big)_{;b} = \rho \dot{U}^a + U^a \big(\rho U^b\big)_{;b} = 0, \tag{368}$$

where $\dot{U}^a = U^a{}_{;b} U^b$ is the acceleration of the fluid, i.e. of the worldlines of the fluid, whose tangent vectors are parallel to to U^a and which satisfy

$$\frac{dx^a}{d\tau} = U^a. \tag{369}$$

The acceleration is orthogonal to the 4-velocity $\dot{U}^a U_a = 0$. Thus taking the dot product of (368) with U_a gives

$$\dot{U}^a = 0. \tag{370}$$

This equation means that the world lines are geodesics. Physically this is becaues the vanishing pressure implies that any pressure gradient which would act as a force must vanish. We now deduce that

$$\big(\rho U^b\big)_{;b} = 0. \tag{371}$$

This is a law of conservation of rest mass.

$$\big(\rho U^b\big)_{;b} = \partial_b\big(\rho U^b\big) + \Gamma_b{}^c{}_c U^b = 0. \tag{372}$$

But using the fact that

$$\Gamma_b{}^c{}_c = \frac{1}{\sqrt{-g}}\partial_b\big(\sqrt{-g}\big), \tag{373}$$

we can convert this to a *true conservation law*

$$\partial_b\big(\sqrt{-g}\rho U^b\big) = 0. \tag{374}$$

The name true conservation law arises because we have

$$\frac{\partial}{\partial t}\big(\sqrt{-g}\frac{U^4}{c}\rho\big) + \partial_i\big(\sqrt{-g}U^i\rho\big) = 0. \tag{375}$$

We can now integrate over a spatial domain D to get

$$\frac{d}{dt}\int_D \sqrt{-g}\frac{U^4}{c}\rho d^3x = -\int_{\partial D}\rho\sqrt{-g}U^i d\sigma_i \tag{376}$$

This equation expresses the conservation of rest mass.

14.1.1 Example: *perfect fluid with pressure*

If $T_{ab} = (\rho + P)U_aU_b + Pg_{ab}$ the conservation law gives

$$\Big((\rho + P)U_a\Big)^{;a}U_b + (\rho + P)\dot{U}_b + \partial_a P = 0. \tag{377}$$

Introduce a tensor $h^a_b = U^aU_b + \delta^a_b$. It has the property that $h^a_b h^b_c = h^a_c$, $h_{ab} = h_{ba}$ and $h^a_b U^b = 0$, but if $V^aU_a = 0$ then $h^a_b V^b = V^a$, in other words h^a_b is a *projection operator* which projects vectors into the spacelike 3-plane orthogonal to the timelike 4-velocity U^a.

Taking the inner product with U^b gives

$$\Big((\rho + P)U_a\Big)^{;a} - \partial_a P U^a = 0 \tag{378}$$

Thus

$$(\rho + P)U^a{}_{;a} + U^a\partial_a\rho = 0. \tag{379}$$

Projecting orthogonal to U_b gives (remembering that $h^a_b\dot{U}^b = \dot{U}^a$)

$$\dot{U}_a = -\frac{h^b_a\partial_b P}{\rho + P}. \tag{380}$$

To proceed further we need some thermodynamics. A homogeneous 'perfect fluid 'has the rather unusual property that there is only one independent thermodynamic variable which we may take to be the energy density ρ , pressure P

, entropy density s or temperature T as we wish. They are linked by the first law which reads

$$TdS = dU + PdV \tag{381}$$

Now for a homogeneous substance, if $U = V\rho$, $S = sV$, this gives

$$Tds = d\rho \tag{382}$$

and

$$Ts = \rho + P. \tag{383}$$

The latter formula is sometimes called the *Gibbs-Duhem relation.* It follows that

$$\frac{ds}{s} = \frac{d\rho}{\rho + P}. \tag{384}$$

Thus we may re-write (379) as the law of entropy conservation

$$\left(sU^a\right)_{;a} = 0. \tag{385}$$

To understand the other equation we note that

$$\frac{dP}{\rho + P} = \frac{dT}{T}. \tag{386}$$

Thus

$$\dot{U}_a = -\frac{h^b_a \partial_b T}{T}. \tag{387}$$

For a fluid at rest in a static metric,

$$\dot{U}_a = \left(0, \partial_i \ln(\sqrt{-g_{44}})\right) \tag{388}$$

and hence

$$\partial_i \ln(\sqrt{-g_{44}}) = -\partial_i \ln T \tag{389}$$

thus

$$T\sqrt{-g_{44}} = \text{constant}, \tag{390}$$

which is called *Tolman's redshift law.* A gas in equilibrium in a static gravitational potential must have a temperature which increases as the gravitational potential decreases by precisely the redshift factor.If this were not so one could construct a perpetual motion machine by converting the local energy at high potential into thermal photons and sending them to a location of lower potential. The photons would gain energy and have the redshifted temperature. If this were not equal to the local value, the temperature difference could be used to do work.

14.2 Example: Einstein's Greatest Mistake

After constructing his theory of gravity, Einstein applied it to cosmology. He believed that the universe should be static. He was unable to construct a static model with his original theory which did not have a cosmological constant term. However thinking along the lines we followed earlier, he realized that he had left out a posible integration constant, which is of course Λ. Using it he managed to construct a static universe called the *Einstein Static Universe.* The metric is

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{391}$$

where g_{ij} is a metric of constant curvature $K = \frac{1}{a^2}$ on S^3. The 3-sphere may be envisaged as embedded in four flat dimensions

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = a^2 \tag{392}$$

It is not difficult to persuade one of the following **fact**

$$R_{ikjl} = \frac{1}{a^2}\big(g_{ij}g_{kl} - g_{il}g_{kj}\big) \tag{393}$$

It follows that $R_{ij} = \frac{2}{a^2}g_{ij}$ and $R^i{}_i = \frac{6}{a^2}$.
Thus

$$R_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{a^2}g_{ij} \end{pmatrix} \qquad R^a{}_a = \frac{6}{a^2}. \tag{394}$$

and

$$\frac{1}{2}Rg_{ab} = \begin{pmatrix} -\frac{3}{a^2} & 0 \\ 0 & \frac{3}{a^2}g_{ij} \end{pmatrix} \tag{395}$$

Now if the fluid is at rest $U^a = \delta^a_4$ and

$$T_{ab} = \rho U_a U_b = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \tag{396}$$

Substitution into the Einstein equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G\rho U_a U_b - \Lambda g_{ab} \tag{397}$$

gives

$$4\pi G\rho = \frac{1}{a^2} = \Lambda. \tag{398}$$

14.3 Example Friedman-Lemaitre-Robertson-Walker metrics

About 5 years after Einstein's paper, Friedman pointed out that one could have time dependent cosmological solutions with or without Λ, a fact also realized by Lemaitre. Shortly after Lemaitre's paper Hubble announced that the universe is not static, distant galaxies were receding from us. Einstein realized that he had failed to make this a prediction of his theory, and alledgedly refered to his introduction of Λ as the greatest blunder of his life. Blunder or not, it has no re-surfaced as vacuum energy, and seems to dominate the expansion of the universe. To see how we consider the metric

$$ds^2 = -dt^2 + a^2(t) g_{ij}(x^k) dx^i dx^j \tag{399}$$

where now the spatial metric is taken to be a 3-sphere of radius $\sqrt{k}$, $k = 0, 1$. The case $k = -1$ is also allowed.

A straightforward but tiresome calculation shows that the Einstein equations reduce to

$$(i) \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} \tag{400}$$

$$(ii) \qquad \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{401}$$

$$(iii) \qquad \dot{\rho} + \frac{3\dot{a}}{a}\rho = 0. \tag{402}$$

15 Spherically Symmetric vacuum metrics

We start by establishing

Birkhoff's Theorem *The unique spherically symmetric vacuum metric is (locally) the Static Schwarzschild metric*

$$ds^2 = -\big(1 - \frac{2GM}{r}\big)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2\big(d\theta^2 \sin^2\theta d\phi^2\big) \tag{403}$$

This generalizes Newton's theorem and greatly simplifies the study of gravi- ational collapse. Any spherically symmetric source, e.g. a pulsating, exploding, or collapsing star has an exterior metric exactly given by the Schwarzschild metric. Physically this can be understood in part from the fact that gravitational waves are transverse and polarizable. Any polarization would define a direction field on S^2 but by the Hairy Ball Theorem it cannot therefore be spherically symmetric.

Proof
Step 1: The metric The meric may be cast in the form

$$ds^2 = A(r,t)dt^2 + 2B(r,t)drdt + C(r,t)dr^2 + R^2(r,t)\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{404}$$

There can be no $g_{a\theta}$ or $g_{a\phi}$ terms with $a = r, t$ by spherical symmetry. We are still allowed to change the coordinates r, t

$$r \to \tilde{r} = \tilde{r}(r,t), \qquad t \to \tilde{t} = \tilde{t}(r,t). \tag{405}$$

Using this freedom we can define

$$\tilde{r} = R(r,t)^{14}. \tag{406}$$

The remaining coordinate freedom may be used to eliminate g_{rt}. Dropping the tilde's we have shown that the most general metric that we need to consider is

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{407}$$

Step 2: The field equations Routine calculations give

$$(i) \qquad G_{tt} \propto \frac{e^{-\lambda}}{r^2}\big(-1 + e^{\lambda} + r\lambda'\big) = 0 \tag{408}$$

$$(ii) \qquad G_{rt} \propto e^{-\frac{(\nu+\lambda)}{2}}\frac{\dot{\lambda}}{r} = 0 \tag{409}$$

$$(iii) \qquad G_{rr} \propto \frac{e^{-\lambda}}{r^2}\big(1 - e^{-\lambda} + r\nu'\big) = 0 \tag{410}$$

$$(iv) \qquad G_{\theta\theta} = \frac{1}{\sin^2\theta}G_{\phi\phi} \propto \frac{e^{-\lambda}}{4}\big(2\nu''+(\nu')^2+\frac{2}{r}(\nu'-\lambda')-\nu'\lambda'\big)-\frac{e^{-\nu}}{4}\big(2\ddot{\lambda}+(\dot{\lambda})^2-\dot{\lambda}\dot{\nu}\big) = 0 \tag{411}$$

We are using the notation $\dot{f} \equiv \partial_t f$, $f' \equiv \partial_r f$.

Step 3: Proof of staticity

$$(ii) \Rightarrow \dot{\lambda} = 0 \Rightarrow \lambda = \lambda(r). \tag{412}$$

$$(i) + (iii) \Rightarrow \lambda' + \nu' = 0 \Rightarrow \nu = -\lambda(r) + f(t), \tag{413}$$

where $f(t)$ is an arbitrary function of integration. Thus the metric is

$$ds^2 = e^{-\lambda(r)}e^{f(t)}dt^2 + e^{\lambda(r)}dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{414}$$

We still have the freedom to change the time coordinate. Define a new time coordianate $\tilde{t}$ by

$$d\tilde{t} = e^{\frac{1}{2}f(t)}dt. \tag{415}$$

The metric now bcomes manifestly static

$$\boxed{ds^2 = e^{-\lambda(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\big(d\theta^2 \sin^2\theta d\phi^2\big),} \tag{416}$$

where we have dropped the tilde on t.

Step 4: Solution of the Static field equation We now need to solve the first order non-linear o.d.e. (i)

$$-1 + e^{\lambda} + r\lambda' = 0. \tag{417}$$

[14]Strictly speaking we need to know that $R(r,t)$is not actually a constant. A separate discussion rules out this possibility

This can be done by making the substitution

$$e^{-\lambda} = 1 + D(r), \tag{418}$$

to get

$$\frac{D'}{D} = -\frac{1}{r}. \tag{419}$$

The solution is

$$D = -\frac{2M}{r}, \tag{420}$$

where M is an arbitrary integration constant.

16 Geodesics in the Schwarzschild metric

Without loss of generality we can set $\theta = \frac{\pi}{2}$. The Lagrangian becomes

$$L = -\big(1 - \frac{2M}{r}\big)\Big(\frac{dt}{d\lambda}\Big)^2 + \frac{1}{1 - \frac{2M}{r}}\Big(\frac{dr}{d\lambda}\Big)^2 + r^2\Big(\frac{d\phi}{d\lambda}\Big)^2. \tag{421}$$

Since L is independent of λ, Noether's theorem gives

$$(i) \qquad \big(1 - \frac{2M}{r}\big)\Big(\frac{dt}{d\lambda}\Big)^2 - \frac{1}{1 - \frac{2M}{r}}\Big(\frac{dr}{d\lambda}\Big)^2 - r^2\Big(\frac{d\phi}{d\lambda}\Big)^2 = k, \tag{422}$$

where $k = +1, 0, -1$ for timelike, lightlike or spacelike geodesics respectively. Since L is independent of t Noether's theorem gives

$$(ii) \qquad \big(1 - \frac{2M}{r}\big)\Big(\frac{dt}{d\lambda}\Big) = E, \tag{423}$$

where E is the energy (per unit mass in the timelike case). Since L is independent of ϕ Noether's theorem gives

$$(iii) \qquad r^2\Big(\frac{d\phi}{d\lambda}\Big) = h, \tag{424}$$

where h is the angular momentum (per unit mass in the timelike case). Substitution of (ii) and (iii) in (i) gives

$$\Big(\frac{dr}{d\lambda}\Big)^2 = E^2 - \big(1 - \frac{2M}{r}\big)\big(k + \frac{h^2}{r^2}\big). \tag{425}$$

16.1 The shape of the orbit

Using (iii) we get

$$\frac{h^2}{r^2}\Big(\frac{dr}{d\phi}\Big)^2 = E^2 - \big(1 - \frac{2M}{r}\big)\big(k + \frac{h^2}{r^2}\big). \tag{426}$$

Now set

$$u = \frac{1}{r} \tag{427}$$

to get

$$\frac{1}{2}\Big(\frac{du}{d\phi}\Big)^2 + \frac{1}{2}u^2 = \frac{E^2}{2h^2} - \frac{k}{2h^2}(1 - 2MU) + Mu^3. \tag{428}$$

We recognize this as the first integral of the second order o.d.e.

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{Mk}{h^2} + 3Mu^2}. \tag{429}$$

16.1.1 Application 1: Newtonian limit

If we drop the last term on the right hand side we get

$$\frac{d^2u}{d\phi^2} + u = \frac{Mk}{h^2}. \tag{430}$$

which is the standard Newtonian result provided we identify M as the mass of the gravitating body. The ratio of the term neglected to that retained on the left hand side is $\mathcal{O}(h^2u^2)$ which is easily seen to be $\mathcal{O}(\frac{v^2}{c^2})$.

16.1.2 Application 2: light bending

We set $k = 0$ to get

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2. \tag{431}$$

This equation can be reduced to quadrature and the answer expressed in terms of elliptic functions, however perhaps the simplest method, vaild for large impact parameter b is to work perturbatively

To order zero

$$\frac{d^2u}{d\phi^2} + u = 0\,.\ \Rightarrow u = \frac{\sin\phi}{b}, \tag{432}$$

that is

$$r\sin\phi = b, \tag{433}$$

which is the equation of a straight line whose distance of nearest approach to the origin is b,

Next Order Substituting the zero order solution in the right hand side of (431) gives

$$\frac{d^2u}{d\phi^2} + u = \frac{3M}{b^2}\sin^2\phi = \frac{3M}{2b^2}(1 - \cos 2\phi) \tag{434}$$

This has as solution

$$\boxed{u = \frac{1}{b}\big(\sin\phi + \frac{M}{2b}(3 + \cos 2\phi)\big)}, \tag{435}$$

where we have chosen the integration constant to make the u symmetric about $\phi = \frac{\pi}{2}$.

To find the deflection angle δ we note that initial asymptote of the orbit corresponds to vanishing u and occurs for small $\phi = \phi_\infty$. We expand

$$u \approx \frac{1}{b}\big(\phi + \frac{2M}{b} + \mathcal{O}(\phi^2)\big), \tag{436}$$

so the asymptote is at $\phi_\infty = -\frac{2M}{b}$.The total deflection is $\delta = -2\phi_\infty$ and hence, restoring dimensions

$$\boxed{\delta = \frac{4GM}{c^2 b}}. \tag{437}$$

Numerically, if b is a solar radius and M a solar mass then a radio wave grazing the sun is deflected by about 1.75 seconds of arc.

16.1.3 Application 3: Precession of the perihelion

We now set $k = 1$.We assume that the orbit is nearly circular.
To order zero we have

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} \tag{438}$$

which has solution

$$u = \frac{1}{l}(1 + e\cos\phi), \qquad l = \frac{h^2}{M}. \tag{439}$$

This is an ellipse of semi-latus rectum l ,eccentricity $0 \le e < 1$ and semi major axis $a = \frac{l}{1-e^2}$. The aphelion (furthest) distance is $\frac{l}{1-e}$ and the perihelion (nearest) distance is $\frac{l}{1+e}$.

We shall assume that the eccentricity is small and so the orbit is nearly circular.

Next order

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2. \tag{440}$$

We set $u = \frac{M}{h^2} + w$ and expand to lowest (linear) order in w ignoring quadratic terms.

$$\frac{d^2w}{d\phi^2} + w(1 - \frac{6M^2}{h^2}) = \frac{3M^3}{h^4}. \tag{441}$$

This is a forced simple harmonic motion with frequency $\omega = \sqrt{1 - \frac{6M}{h^2}} \approx 1 - \frac{3M}{h^2}$. Thus to next order

$$u = \frac{1}{l'}(1 + e'\cos\omega\phi), \tag{442}$$

where $l' \approx l$ and $e' \approx e$ are the corrected values of l and e which can easily be calculated but which are not relevant for us. This is a precessing ellipse in which the furthest and nearest points, the aphelion and perihelion respectively, overshoot or advance by an amount per revolution

$$\frac{2\pi}{\omega} - 2\pi \approx \frac{6\pi M^2}{h^2} = \frac{6\pi M}{l} \,. \tag{443}$$

Thus the perihelion advance is

$$\frac{6\pi GM}{c^2 a(1-e^2)}. \tag{444}$$

For Mercury we get about 43 seconds of arc per century. For the binary pulsar we get about 4 degrees per year.