Mathematical Tripos Part II **Lent Term 2025**
General Relativity **Dr. J.M. Evans**

Example Sheet 1

1. Consider the motion of light in a laboratory on board an accelerating spacecraft in deep space. By using the Equivalence Principle (in a form which you should state), obtain an expression for the deflection of light moving approximately horizontally through a laboratory on the surface of the Earth. Give your answer in the form

$$\frac{\Delta\theta}{\Delta\phi} = \frac{kGM}{c^2 R} \,,$$

where $\Delta\theta$ is the angle through which light is deflected as it traverses the laboratory, $\Delta\phi$ is the angle the laboratory subtends at the centre of the Earth, M and R are the mass and radius of the Earth, and k is a numerical constant, to be determined.

2. An astronaut, Alice, moves on a circular orbit of radius R around the Earth, while her twin, Bob, stays at home on the Earth's surface. They each measure time by counting radio pulses from a distant, stationary source. By applying the formulas for gravitational redshift and for time dilation (from Special Relativity), show that the rate at which Bob ages, compared to the rate at which Alice ages, is given by

$$\Big(1 - \frac{GM}{c^2 R_E}\Big)\Big/\Big(1 - \frac{GM}{c^2 R}\Big)^{3/2},$$

where R_E is the Earth's radius and M is its mass. (You may disregard effects due to the Earth's rotation.)

Deduce that Alice and Bob age at approximately the same rate if $R = \frac{3}{2} R_E$.

3. Write down the line element of (flat) Euclidean 3-space in cylindrical polar coordinates (r, ϕ, z). Show that the 2-dimensional line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - r_s/r} + r^2 \mathrm{d}\phi^2,$$

where r_s is a constant, can be regarded as the line element on a surface of revolution $z = f(r)$ in Euclidean 3-space (where the function $f(r)$ is to be found). Sketch the surface, and comment on the behaviour as r is reduced towards r_s.

4. Starting with the line element for flat spacetime in cylindrical polar coordinates (t, r, ϕ, z), obtain the *Langevin* line element for a rotating observer

$$ds^2 = -dt^2(c^2 - r^2\omega^2) + dr^2 + 2r^2\omega d\theta dt + r^2 d\theta^2 + dz^2,$$

where $\theta = \phi - \omega t$ and ω is constant.

A perfect fibre-optic cable is laid round the Earth's equator ($r = R$, $z = 0$) and two photons travel around the cable in opposite directions at the speed of light (so that $ds^2 = 0$ on their trajectories), starting from the same point. Show that one photon will arrive back at its starting point (for the first time) a time Δt before the other, where

$$\Delta t = \frac{4\pi\omega R^2}{c^2 - \omega^2 R^2} \,.$$

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5. A *geodesic* can be defined as a curve $x^\alpha(\mu)$ that satisfies

$$\ddot{x}^\alpha + \Gamma_{\beta}{}^{\alpha}{}_{\gamma}\, \dot{x}^\beta \dot{x}^\gamma = f(\mu)\, \dot{x}^\alpha ,$$

where $\Gamma_{\beta}{}^{\alpha}{}_{\gamma}$ is the (Levi-Civita) connection, a dot denotes differentiation with respect to the parameter μ, and $f(\mu)$ is some function. If, in addition, $f(\mu) = 0$, then the geodesic is said to be *affinely parametrized.*

Show that, by changing to a new parameter $\lambda(\mu)$, any geodesic can be affinely parametrized. If λ is an affine parameter, show that any other affine parameter can be written in the form $A\lambda + B$, where A and B are constants (and $A \neq 0$).

6. Suppose that a system with coordinates $q^r(\lambda)$ is governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ with no explicit dependence on λ, and that L is homogeneous of degree k in the 'velocities' $\dot{q}^{\,r}(\lambda)$, i.e.,

$$\dot{q}^{\,r} \frac{\partial L}{\partial \dot{q}^{\,r}} = kL \, .$$

Show that if $k > 1$ then $\mathrm{d}L/\mathrm{d}\lambda = 0$ on the extremal curves, or solutions of the Euler-Lagrange equations.

Does the same conclusion hold for a Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$ that is homogeneous of degree $k = 1$ in velocities? Given such a Lagrangian, let $L = \mathcal{L}^2$. By relating the Euler-Lagrange equations, show that any extremal curve for L on which $L > 0$ is also an extremal curve for $\mathcal{L}$. Is the converse true?

7. Obtain the geodesic equations for the Langevin metric (in question 4) in the form of four first integrals. What is their physical significance?

Show that for a particle moving initially with speed v in the radial direction the initial accelerations in the radial and θ directions are $r\omega^2$ and $-2\omega v$ (with velocities and accelerations measured using coordinate time t). How should this be interpreted?

8. Two metrics $g_{\alpha\beta}$ and $\widehat{g}_{\alpha\beta}$ are *conformally related* if $\widehat{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ for some scalar function Ω. Show that their Christoffel symbols $\left\{ {\alpha \atop \beta\,\gamma} \right\}$ and $\widehat{\left\{ {\alpha \atop \beta\,\gamma} \right\}}$ are related by

$$\widehat{\left\{ {\alpha \atop \beta\,\gamma} \right\}} = \left\{ {\alpha \atop \beta\,\gamma} \right\} + \Omega^{-1}(\delta^\alpha{}_\beta \Omega_{,\gamma} + \delta^\alpha{}_\gamma \Omega_{,\beta} - g^{\alpha\delta} g_{\beta\gamma} \Omega_{,\delta}).$$

In Nordstrøm's theory of gravity the metric is given by $g_{\alpha\beta} = e^{2\varphi}\eta_{\alpha\beta}$, where φ is a scalar function of position and $\eta_{\alpha\beta}$ is the Minkowski metric. Compute the equation of a geodesic in Nordstrøm's theory and use your result to show that $T^\alpha T^\beta g_{\alpha\beta}$, where T^α is the tangent vector corresponding to an affine parameter, is constant on the geodesic.

Using the results of question 5, show that a null geodesic is also a null geodesic in Minkowski spacetime, and hence deduce that light rays are not subject to gravitational deflection.

Show that for any time-like geodesic, with suitably chosen parameter μ,

$$\frac{d^2 x^\alpha}{d\mu^2} = -\eta^{\alpha\gamma}\psi_{,\gamma}$$

for some function ψ.

[*A comma denotes differentiation with respect to a coordinate, e.g.* $\varphi_{,\alpha} = \partial_\alpha \varphi$, *and the Christoffel symbols are defined by* $\left\{ {\alpha \atop \beta\,\gamma} \right\} = \frac{1}{2} g^{\alpha\mu}(\, g_{\beta\mu,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \, .$]

9. 2-dimensional *de Sitter space-time* has the line element

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \cosh^2 u \, \mathrm{d}\phi^2 \,,$$

where $-\infty < u < \infty$ and $0 \le \phi < 2\pi$. Compute the Christoffel symbols and hence the geodesic equations. Verify that the equations of an affinely parametrized geodesic $x^\alpha = x^\alpha(\lambda)$ can be derived from the variational principle

$$\delta \int (\dot{u}^2 - \cosh^2 u \, \dot{\phi}^2) \, \mathrm{d}\lambda = 0,$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}\lambda$ etc. Verify from the variational principle that there are two first integrals

$$\begin{aligned} \cosh^2 u \, \dot{\phi} \; &= K \,, \\ \cosh^2 u \, \dot{u}^2 \; &= K^2 + L \cosh^2 u \,, \end{aligned}$$

along the geodesic, where K and L are constants.

Show that if $K = 0$ the geodesics are $\phi =$ const. If $K \neq 0$, show that λ may be eliminated in favour of ϕ as a parameter along the geodesic and obtain the equation

$$v'^{\,2} = M^2 - v^2,$$

where $v = \tanh u$, $v' = \mathrm{d}v/\mathrm{d}\phi$ and M is a constant depending on L and K. Hence show that the $K \neq 0$ geodesics are given by

$$\tanh u = M \sin(\phi - \phi_0) \,,$$

where ϕ_0 is a constant. Show also that $M^2 > 1$ for timelike geodesics, $M^2 = 1$ for null geodesics and $M^2 < 1$ for spacelike geodesics. Regarding u and ϕ as cartesian coordinates, sketch the set of geodesics starting from $(0, 0)$.

Show from your diagram that no two such timelike geodesics will meet again, but that spacelike geodesics may recross each other. Demonstrate also that there are pairs of points which cannot be joined by a geodesic. Which, if any, of these statements would be valid in Minkowski spacetime?

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