MATHEMATICAL TRIPOS
PART II

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General Relativity - Examples III

The starred question is intended as an extra; do it if you have time, but not at the expense of unstarred questions.

1. Consider the intrinsic metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2\mathrm{d}\phi^2$$

of the equatorial plane $\theta = \pi/2$ of Schwarzschild space-time, on a surface of fixed time coordinate t. Show that this can be regarded as the metric on a surface of revolution $z = z(r)$ in Euclidean 3-space, using cylindrical polar coordinates (z, r, ϕ). Sketch the surface, and comment on the behaviour as r is reduced towards $2M$.

2. The radial motion $r(\tau)$ of a massive particle with radial Schwarzschild coordinate r, regarded as a function of the proper time τ along the particle's path in the Schwarzschild geometry, obeys

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = f(r)\,,$$

where

$$f(r) = E^2 - \left(1+\frac{h^2}{r^2}\right)\left(1-\frac{2M}{r}\right)\,.$$

Here M is the mass of the Schwarzschild geometry, and the constants E, h of the motion give the velocity components for the Schwarzschild coordinates t, ϕ, for motion in the equatorial plane, as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E}{1-\frac{2M}{r}}\,, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{h}{r^2}\,.$$

Show, graphically or otherwise, that circular orbits occur where

$$f(r) = f'(r) = 0\,, \quad f''(r)\begin{cases} <0, & \text{stable;} \\ >0, & \text{unstable.}\end{cases}$$

Deduce

(a) that there are stable circular massive orbits for $6M < r < \infty$,

(b) that the fractional binding energy of the most tightly bound, stable circular orbit at $r = 6M$ is

$$1 - E = 1 - \sqrt{8/9} \simeq 0.0572\,,$$

(c) that there are unstable circular massive orbits for $3M < r < 6M$,

(d) that there is an unstable circular massless photon orbit at $r = 3M$.

3. A small particle flies past a massive object of Schwarzschild radius r_S with a small gravitational deflection $\Delta\phi$. Defining $u = 1/r$, show that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{(E^2-1)}{h^2} + \frac{\epsilon}{h^2}\,u - u^2 + \epsilon u^3,$$

where $\epsilon = r_S$, and the constants h, E should be identified.

Assume a solution of the form $\sum_0^\infty \epsilon^n u_n(\phi)$. Show that $u_0 = (1/b)\sin\phi$, where b is the distance of closest approach. By constructing $u_1(\phi)$ show that the deflection is

$$\Delta\phi = 2\epsilon b(\tfrac{1}{2}h^{-2} + b^{-2}) + O(\epsilon^2).$$

Suppose that at the position of closest approach the particle has speed v. Identify $h = bv/\sqrt{1-v^2}$ and show that

$$\Delta\phi \approx \frac{r_S(1+v^2)}{bv^2},$$

and comment on the limits $v \to 0$, $v \to 1$.

4. A classical test of general relativity is the time delay caused to radar signals bounced off planets or satellites. Ignoring curvature a light ray is given by $r\sin\phi = b$, where b is the distance of closest approach to the origin. By differentiating this, show that $r^2\mathrm{d}\phi^2 = b^2\mathrm{d}r^2/(r^2-b^2)$. For geodesic motion we require $F\mathrm{d}t^2 = \mathrm{d}r^2/F + r^2\mathrm{d}\phi^2$ for $F = 1 - r_S/r$. Assuming the above relation between $\mathrm{d}\phi$ and $\mathrm{d}r$ obtain the approximate equation

$$\mathrm{d}t = \pm\frac{r}{\sqrt{r^2-b^2}}\left(1 + \frac{r_S}{r} - \frac{b^2 r_S}{2r^3}\right)\mathrm{d}r\ ,$$

where $\mathrm{O}\left((r_S/r)^2\right)$ terms have been neglected. Show that to this approximation, the time taken to move from the position of closest approach to r is given by

$$\Delta t = \sqrt{r^2-b^2} + r_S\log\left(r/b + \sqrt{r^2/b^2-1}\right) - \frac{r_S}{2r}\sqrt{r^2-b^2},$$

and identify the first term on the right hand side. The total time taken has been verified experimentally.

Hint: $\int \mathrm{d}x/\sqrt{x^2-1} = \log(x + \sqrt{x^2-1})$.

5. Setting $F = 1 - r_S/r$, define a new radial coordinate r^* by $\mathrm{d}r^*/\mathrm{d}r = 1/F$, a retarded time coordinate $u = t - r^*$, and express the Schwarzschild line element in (u, r, θ, ϕ) coordinates

$$\mathrm{d}s^2 = -F\,\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\ .$$

A monochromatic radio transmitter e with proper time τ, 4-velocity $V^a = \mathrm{d}x^a/\mathrm{d}\tau$ is freely falling radially into a Schwarzschild black hole. Its signals propagate radially outwards and are received by a distant observer o who sits at a constant (large) value for r. Show first that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta\tau} = \frac{\Delta u_o}{\Delta\tau} = \frac{\Delta u_e}{\Delta\tau} = V^u.$$

Show next by considering the Lagrangian that determines the geodesics, that since g_{ab} is u-independent, $V_u = -K$, where K is a constant. Use the equations $g_{ua}V^a = -K$, $g_{ab}V^aV^b = -1$ to deduce that

$$V^u = \frac{K + \sqrt{K^2 - F}}{F}, \qquad V^r = -\sqrt{K^2 - F},$$

and that near the horizon $\mathrm{d}u/\mathrm{d}r \sim -2/F$, $u \sim -2r^*$ and $F \sim e^{-u/(2r_S)}$ as $u \to \infty$.

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(t/(2r_S))$.

6. Show that, for a particle with proper time τ moving in the Schwarzschild space-time:

$$1 = F\dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2),$$

where $\dot{t} = \mathrm{d}t/\mathrm{d}\tau$ etc., and $F = 1 - r_S/r$. Show that, whatever the transmitter of question 5 does once it is within the horizon (e.g., accelerating, moving non-radially), one has $\dot{r}^2 \geq -F$. Hence deduce that it reaches nemesis or nirvana at $r = 0$ within a proper time $\frac{1}{2}\pi r_S$.

7. Why can Einstein's equations for gravity coupled to the electromagnetic field be written in the form

$$R_{ab} = K\big(F_{ac}F_b{}^c - \tfrac{1}{4}g_{ab}F_{cd}F^{cd}\big)\,?$$

Assuming

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \quad F_{tr} = -F_{rt} = \frac{Q}{r^2}\,,$$

with $F_{ab} = 0$ otherwise, and $R_{tt}/f = -fR_{rr} = \frac{1}{2}f'' + f'/r$, $R_{\theta\theta} = R_{\phi\phi}/\sin^2\theta = 1 - rf' - f$ show that a solution can be found for suitable $f(r)$ which reduces to the Schwarzchild solution for $Q = 0$. Find an analogous solution of $R_{ab} = \Lambda g_{ab}$.

8. Suppose that the Maxwell tensor F_{ab} for the electromagnetic field can be derived from a vector potential A_a, i.e., $F_{ab} = A_{b;a} - A_{a;b}$. Show that $F_{[ab;c]} = 0$, and that conservation of the energy momentum tensor

$$T^{ab} = F^a{}_cF^{bc} - \tfrac{1}{4}g^{ab}F_{cd}F^{cd}\,,$$

implies the other Maxwell equations $F^{ab}{}_{;b} = 0$, provided that the matrix $F^a{}_b$ is non-singular. *Hint: no knowledge of electrodynamics is needed to do this question!*

9. A perfect fluid has 4-velocity U^a and number density n, inertia density ρ and pressure p, all three measured in the rest frame of U^a. The flux density and energy-momentum tensor

$$N^a = nU^a, \qquad T^{ab} = (\rho + p)\, U^a U^b + p g^{ab},$$

are both conserved, $N^a{}_{;a} = T^{ab}{}_{;b} = 0$.

i) Suppose that the fluid has zero pressure, so that the particles fall freely in the mean gravitational field. Show that the fluid flow lines (integral curves of U^a) are geodesics.

ii) Consider the weak-field metric

$$\mathrm{d}s^2 = -e^{2\varphi}\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\ ,$$

with $\varphi \sim v^2 \ll 1$, where v is a typical speed. Obtain the conservation laws for number, momentum and energy and compare them with their Newtonian counterparts.

10*. Let

$$H_1{}^a{}_b = \delta^a{}_{[b}\delta^e{}_c\delta^f{}_{d]}R^{cd}{}_{ef} = \delta^a{}_{[b}R^{cd}{}_{cd]} = \delta^{[a}{}_b R^{cd]}{}_{cd}\,,$$

where $[\ldots]$ denotes antisymmetrisation of indices. Show directly from the Bianchi identity for the Riemann tensor that $\nabla_a H_1{}^a{}_b = 0$ and also show that $H_1{}^a{}_b = H_{1b}{}^a$ and hence $H_1{}^{ab} = H_1{}^{ba}$. What is the relation of $H_1{}^a{}_b$ to the Einstein tensor? Suppose

$$H_2{}^a{}_b = \delta^a{}_{[b}R^{cd}{}_{cd}R^{ef}{}_{ef]}\,.$$

Show also $\nabla_a H_2{}^a{}_b = 0$ and $H_2{}^{ab} = H_2{}^{ba}$ so that $H_2{}^{ab} = KT^{ab}$ would be a possible equation for the metric tensor. Why does $H_2{}^{ab}$ vanish in four dimensions?