1. Let $\Gamma_{abc}^d$ be the connection in coordinates $x^a$. Show that under the coordinate transformation

$$\tilde{x}^d = x^d + \frac{1}{2} \Gamma_{bc}^d x^b x^c$$

the transformed connection vanishes at the origin. The transformation law for a connection is

$$B_b^c \tilde{\Gamma}_{ef}^d = \Gamma_{abc} B_a^d - \frac{\partial B_a^d}{\partial x^b} \quad \text{where} \quad B_a^d = \frac{\partial \tilde{x}^d}{\partial x^a}.$$ 

What further coordinate transformation is needed to reduce the coordinates to Local Inertial Coordinates at the origin?

2. The Lie derivative $\mathcal{L}_k Y^a$ of a vector field $Y^a$ with respect to a vector field $k^a$ is defined by the following conditions:

(a) Let $x^a$ be any coordinate system in which $k^a = (1, 0, 0, 0)$. Then

$$\mathcal{L}_k Y^a = \frac{\partial Y^a}{\partial x^0} \equiv k^b \frac{\partial Y^a}{\partial x^b}.$$ 

(b) $\mathcal{L}_k Y^a$ transforms as a vector.

Show that, in a general coordinate system $\tilde{x}^a$,

$$\mathcal{L}_k \tilde{Y}^a = \tilde{k}^b \nabla_{\tilde{b}} \tilde{Y}^a - \tilde{Y}^b \nabla_{\tilde{b}} \tilde{k}^a$$

Hint: You need to write condition (a) in terms of tensor quantities; your solution should be no more than four lines. Work back from the answer if you can’t see how to do it.

Given that the Lie derivative $\mathcal{L}_k \phi$ of a scalar field $\phi$ with respect to a vector field $k^a$ is defined in a general coordinate system $x^a$ by

$$\mathcal{L}_k \phi = k^a \frac{\partial \phi}{\partial x^a}$$

and that the Lie derivative obeys the usual Leibniz rule when applied to the derivative of a product, find the Lie derivative $\mathcal{L}_k Z_a$ of a covector field $Z_a$ with respect to a vector field $k^a$. 
3 Show that, if a vector $S^a$ is parallely transported along an affinely parametrized geodesic $\gamma$ with tangent vector $T^a$, then $S_0 T^a$ is constant on $\gamma$.

Consider the parallel transport of a vector $S^a$ round a closed path on the unit 2-sphere consisting of the following four segments:

(i) $0 < \theta < \frac{1}{2} \pi$, $\phi \geq 0$; (ii) $\frac{1}{2} \pi \geq \theta \geq \theta_0$, $0 \leq \phi \leq \phi_0$; (iii) $0 \leq \theta \leq \frac{1}{2} \pi$, $\phi = \phi_0$; (iv) $\theta_0 \leq \theta \leq \frac{1}{2} \pi$, $\phi = \phi_0$;

where $\theta$ and $\phi$ are the normal polar coordinates. The starting point is $(\frac{1}{2} \pi, \phi_0)$ (on the equator), where $S^a = (S_\theta, S_\phi) = (1, 0)$.

(a) Sketch a picture in the case $\theta_0 = 0$ (so the path is a spherical triangle with one vertex at the North pole) using the result of the first paragraph (no further calculation required) and hence show that the angle between the initial and final vectors $S^a$ is proportional to the area enclosed by the path.

(b) Verify that for $0 < \theta_0 < \frac{1}{2} \pi$ the parallel transport equations have solutions:

- $S^a = (1, 0)$ on (i);
- $S^a = (1, 0)$ on (ii);
- $S^a = (\cos(\phi \cos \theta_0), -\sin(\phi \cos \theta_0) / \sin \theta_0)$ on (iii);
- and $S^a = (\cos(\phi_0 \cos \theta_0), -\sin(\phi_0 \cos \theta_0) / \sin \theta_0)$ on (iv).

Hence find $S^a$ at the end point. Check that, when $\theta_0 \to 0$, your answer agrees with your answer to part (a).

**Note:** $\Gamma_{\phi \phi \phi} = -\sin \theta \cos \theta$ and $\Gamma_{\phi \theta \phi} = \Gamma_{\phi \phi \theta} = \cot \theta$.

4 The metric $g_{ab}(x)$ has the property that, if each point $x^a$ is mapped to $y^a(x)$, distances are unaltered. Such a transformation $x \mapsto y$ is called an *isometry* for this metric. Show that

$$g_{ab}(y) \frac{\partial y^a}{\partial x^c} \frac{\partial y^b}{\partial x^d} = g_{cd}(x).$$

Setting $y^a(x) = x^a + \epsilon \xi^a(x)$, where $\epsilon$ is small, show that to first order in $\epsilon$

$$\xi^c \frac{\partial g_{ab}}{\partial x^c} + g_{cb} \frac{\partial \xi^c}{\partial x^a} + g_{ca} \frac{\partial \xi^c}{\partial x^b} = 0.$$

Show also, either by using the explicit form for the Christoffel symbols or by using locally inertial coordinates, that this condition can be written in tensorial form as

$$\nabla_a \xi_b + \nabla_b \xi_a = 0.$$

This is known as Killing’s equation and solutions $\xi_a$ are called *Killing covector fields*.

If $\xi^a = (1, 0)$ is a Killing vector field, show that $\frac{\partial g_{ab}}{\partial x^0} = 0$.

5 Write down the Ricci identity for a vector field. Given two vector fields $U^a$ and $V^a$, evaluate $\nabla_d \nabla_c (U^a V^b) - \nabla_c \nabla_d (U^a V^b)$. Deduce that, for an arbitrary tensor field $T^{ab}$,

$$\nabla_d \nabla_c T^{ab} - \nabla_c \nabla_d T^{ab} = R^{a}_{\ cde} T^{eb} + R^{b}_{\ edc} T^{ae}.$$

Deduce that $\nabla_b \nabla_k T^{ab} = \nabla_k \nabla_b T^{ab}$ for any tensor field $T^{ab}$.

6 Use local inertial coordinates to prove that

$$R_{abcd} = R_{cdab}.$$
7. Show, by considering its symmetries, that the Riemann curvature tensor for a 2-dimensional metric has only one independent component. Show further that
\[ R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}) \, . \]
Verify this result using the Christoffel symbols for 2-dimensional de Sitter space-time (obtained on sheet 1).

8. Let \( \xi_a \) be a Killing covector field as defined in question 4. Use the Ricci identity and \( R^b{}_{[bcd]} = 0 \) to show that
\[ \xi_{a;bc} = -R^d{}_{cab} \xi_d \, . \]
In the case of Minkowski space, integrate this equation twice to obtain the 10 independent Killing vectors.

9. The Ricci tensor of a space-time is given by
\[ R_{ab} = \nabla_a \nabla_b \phi \]
where \( \phi \) is a scalar field. Show that
\[ \nabla_a (\nabla_b \nabla^b \phi) = -2R_{ab} \nabla^b \phi \]
and hence that \( \nabla_a \phi \nabla^a \phi + R \) is constant.
**Note:** the Ricci identity is \( \nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^e{}_{dab} V^d \) and the contracted Bianchi identity is \( \nabla^b R_{ab} = \frac{1}{2} \nabla_a R \).

10. Suppose that the Maxwell tensor \( F_{ab} \) for the electromagnetic field can be derived from a vector potential \( A_a \), i.e., \( F_{ab} = \nabla_a A_b - \nabla_b A_a \). Show that \( \nabla_a F_{ab} = 0 \), and that conservation of the energy momentum tensor
\[ T^{ab} = F^a{}_c F^{bc} - \frac{1}{4} g^{ab} F_{cd} F^{cd} \, , \]
implies the other Maxwell equations \( \nabla_b F^{ab} = 0 \), provided that the matrix \( F^a{}_b \) is non-singular.