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Feel free to set $c = 1$, and reintroduce it if required using dimensional analysis.

1 Let the Ricci tensor of a space-time be given by

$$R_{\alpha\beta} = \nabla_\alpha \nabla_\beta \phi$$

where ϕ is a scalar field. Show that

$$\nabla_\alpha (\nabla_\beta \nabla^\beta \phi) = -2R_{\alpha\beta} \nabla^\beta \phi$$

and hence that $\nabla_\alpha \phi \nabla^\alpha \phi + R$ is constant.

Note: the Ricci identity is $\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R^\gamma_{\delta\alpha\beta} V^\delta$ and the contracted Bianchi identity is $\nabla^\beta R_{\alpha\beta} = \frac{1}{2} \nabla_\alpha R$.

2 Write down the Ricci identity for a vector field. Given two vector fields U^α and V^α , evaluate $\nabla_\delta \nabla_\gamma (U^\alpha V^\beta) - \nabla_\gamma \nabla_\delta (U^\alpha V^\beta)$. Deduce that, for an arbitrary tensor field $T^{\alpha\beta}$,

$$\nabla_\delta \nabla_\gamma T^{\alpha\beta} - \nabla_\gamma \nabla_\delta T^{\alpha\beta} = R^\alpha_{\mu\delta\gamma} T^{\mu\beta} + R^\beta_{\mu\delta\gamma} T^{\alpha\mu}.$$

Deduce that $\nabla_\beta \nabla_\alpha T^{\alpha\beta} = \nabla_\alpha \nabla_\beta T^{\alpha\beta}$ for any tensor field $T^{\alpha\beta}$.

3 Use local inertial coordinates to prove that

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}.$$

4 Show, by considering its symmetries, that the Riemann curvature tensor for a 2-dimensional metric has only one independent component. Show further that for such a metric

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} R (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}).$$

Verify this result using the Christoffel symbols for 2-dimensional de Sitter space-time (obtained on sheet 1).

5 Suppose that the Maxwell tensor $F_{\alpha\beta}$ for the electromagnetic field can be derived from a vector potential A_α , i.e. $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Show that $\nabla_{[\gamma} F_{\alpha\beta]} = 0$, and that conservation of the energy momentum tensor (i.e. $T^{\alpha\beta}_{;\beta} = 0$)

$$T^{\alpha\beta} = F^\alpha_{\gamma} F^{\beta\gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta},$$

implies the other Maxwell equations $\nabla_\beta F^{\alpha\beta} = 0$, provided that the matrix F^α_{β} is non-singular.

6 Write down the radial Euler-Lagrange equation for the Schwarzschild metric and show that, in the case of a circular geodesic orbit in the equatorial plane, this determines the period of the orbit in terms of Schwarzschild coordinate time. How does this relate to the Newtonian result?

Alice stays at home (home is on the surface of the Earth, which is a non-rotating sphere of radius R_E) and Bob orbits the Earth in a circular geodesic orbit of radius R .

(i) Show that Alice's proper time τ_A is related to the Schwarzschild time coordinate by $d\tau_A^2 = (1 - 2M/R_E) dt^2$.

(ii) Show that Bob's proper time τ_B is related to the Schwarzschild time coordinate by

$$d\tau_B^2 = (1 - 2M/R) dt^2 - (M/R) dt^2.$$

Deduce that Bob and Alice age at the same rate if $R = \frac{3}{2} R_E$.

7 A massive particle moves on a circular orbit of radius R with the usual parameters E and L in Schwarzschild space-time. Use the r Euler-Lagrange equation to find conditions that R must satisfy, and show that they are the same as would be obtained by considering motion in the 1-dimensional potential $V(r)$, where

$$2V(r) = \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right).$$

- (i) Express L in terms of R and M and deduce that there are circular orbits for $R > 3M$.
- (ii) Show that these orbits are stable if $R > 6M$ and unstable if $3M < R < 6M$.
- (iii) Show that the fractional binding energy (i.e. $1 - E$) of a stable circular orbit in the limit $R \rightarrow 6M$ is

$$(1 - 2\sqrt{2}/3) \simeq 0.0572.$$

8 Show that there is a circular photon orbit at $r = 3M$ in Schwarzschild and that it is unstable.

9 In an attempt to unify Special Relativity and Newtonian Gravity, the orbits in a central potential are calculated using the Euler-Lagrange equations derived from the Lagrangian

$$\mathcal{L} = -\gamma^{-1}c^2 + \frac{GM}{r}$$

where $\gamma = (1 - \mathbf{v} \cdot \mathbf{v}/c^2)^{-\frac{1}{2}}$ (in the obvious notation). [You may wish to justify the use of this Lagrangian by constructing the Hamiltonian.]

Obtain the orbital equation for motion in the equatorial plane in the form

$$u'' + u = \gamma \ell^{-1}$$

where $u = 1/r$, prime denotes differentiation with respect to the usual ϕ coordinate, $\ell^{-1} = GM/h^2$, and h is a suitable conserved quantity.

Show that $\gamma = (1 + h^2(u'^2 + u^2)/c^2)^{\frac{1}{2}}$. Using the approximation $\gamma \approx 1 + \frac{1}{2}h^2(u'^2 + u^2)/c^2$, show that the rate of advance of the perihelion is one sixth of that obtained using the Schwarzschild metric.

10 A particle flies past a spherically symmetric star of Schwarzschild radius r_s . Show, by setting $u = r^{-1}$ in the usual effective potential $V(r)$ for Schwarzschild, that

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - 1}{L^2} - u^2 + 2Mu(1/L^2 + u^2).$$

- (i) If $M = 0$ show that coordinates can be chosen such that $u = (1/b) \sin \phi$, where b is a constant. Show also that if the speed of the particle is v when $r = b$, then $L = bv/\sqrt{1 - v^2}$. Determine v and b in terms of E and L .
- (ii) Now let $r_s := 2M$, $r_s/b = \epsilon$, where $\epsilon \ll 1$, and suppose a solution of the form $u = \sum_0^\infty \epsilon^n u_n(\phi)$ exists with $u_0 = (1/b) \sin \phi$. Find a second order differential equation for $u_1(\phi)$ and hence show that the deflection is given by

$$\Delta\phi = 2\epsilon \left(1 + \frac{b^2}{2L^2}\right) + O(\epsilon^2).$$

Show further that

$$\Delta\phi \approx \frac{r_s(1 + v^2)}{bv^2} = \frac{2M(1 + v^2)}{bv^2}.$$

Comment on the limit $v \rightarrow 1$.

11 A classical test of general relativity is the time delay caused to radar signals bounced off planets or satellites. Ignoring curvature a light ray is given by $r \sin \phi = b$. By differentiating this, show that $r^2 d\phi^2 = b^2 dr^2 / (r^2 - b^2)$.

Assuming the above relation between $d\phi$ and dr obtain the approximate equation for light rays in Schwarzschild

$$dt = \pm \frac{r}{\sqrt{r^2 - b^2}} \left(1 + \frac{2M}{r} - \frac{Mb^2}{r^3} \right) dr ,$$

where $O((M/r)^2)$ terms have been neglected. Show that, to this approximation, the time taken to move from $r = b$ to $r = r_1$ is given by

$$\Delta t = \sqrt{r_1^2 - b^2} + 4M \cosh^{-1}(r_1/b) - \frac{2M}{r_1} \sqrt{r_1^2 - b^2},$$

and identify the first term on the right hand side.