1. A static spacetime has line element
\[ ds^2 = -e^{2\phi/c^2}c^2dt^2 + g_{ij}dx^idx^j , \]
where \( \phi \) and \( g_{ij} \) are independent of \( x^0 = ct \), and \( i, j = 1, 2, 3 \). Show that
\[ \Gamma^0_{\alpha\beta} = \frac{1}{c^2} \left( n_\alpha \frac{\partial \phi}{\partial x^\beta} + n_\beta \frac{\partial \phi}{\partial x^\alpha} \right) \quad \text{and} \quad \Gamma^i_{00} = g^{ij} \frac{1}{c^2} \frac{\partial \phi}{\partial x^j} e^{2\phi/c^2} , \]
where \( n_\alpha = (1, 0, 0, 0) \).

Let \( u^\alpha \) be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that \( u^i = 0 \) and \( u^0u_0 = -c^2 \)). Show that
\[ \nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi \quad \text{and so} \quad \nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha . \]
Show further that
\[ g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = R^{\alpha\beta} u_\alpha u_\beta \]
and hence that
\[ g^{ij} \nabla_i \nabla_j \phi + \frac{1}{c^2} g^{ij} \nabla_i \phi \nabla_j \phi = R^{\alpha\beta} u_\alpha u_\beta . \]
[ Hint: you may find it helpful to start from the Ricci identity \( u_\alpha;\beta\gamma - u_\alpha;\gamma\beta = R^\delta_{\alpha\beta\gamma} u_\delta \). ]

What does the last equation reduce to in the Newtonian limit (weak gravity) with \( T_{\alpha\beta} = \rho u_\alpha u_\beta \)?

2. A perfect fluid has 4-velocity \( u^\alpha \) which is tangent to the fluid flow lines (the integral curves of \( u^\alpha \)) and which satisfies \( u^\alpha u_\alpha = -c^2 \). If the fluid has particle number density \( n \), density \( \rho \) and pressure \( p \), then the particle flux density \( N_\alpha \) and energy-momentum tensor \( T_{\alpha\beta} \) are given by
\[ N_\alpha = nu_\alpha , \quad T^{\alpha\beta} = (\rho + p/c^2)u^\alpha u^\beta + pg^{\alpha\beta} , \]
and both are conserved: \( \nabla_\alpha N^\alpha = \nabla_\beta T^{\alpha\beta} = 0 \).

(i) If the fluid has zero pressure, show that \( \nabla_\alpha (\rho u^\alpha) = 0 \) and that the fluid flow lines are geodesics. Show also that \( \rho/n \) is constant on each such geodesic.

(ii) If the fluid has pressure, find an expression for \( \nabla_\alpha (\rho u^\alpha) \) and show that
\[ \left( \rho + \frac{1}{c^2} p \right) u^\beta \nabla_\beta u_\alpha + \nabla_\alpha p + \frac{1}{c^2} u^\alpha u^\beta \nabla_\beta p = 0 . \]

3. Consider a perfect fluid, with definitions and notation as in question 2, and a static, weak-field metric of the form given in question 1, but with \( g_{ij} = \delta_{ij} \). In the Newtonian limit, \( \varphi/c^2 \sim v^2/c^2 \ll 1 \), where \( v \) is a typical speed, so that \( u^\alpha \approx (c, u) \). Show that, to lowest order,
\[ \frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0 , \]
where \( \nabla \) is the usual vector operator in 3-dimensional flat space. What is the corresponding equation for \( \rho \)? Show that, in the Newtonian limit, \( \rho u^\beta u_{i,\beta} = -p, (i = 1, 2, 3) \) and hence that
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla \varphi - \frac{1}{\rho} \nabla p . \]
4. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric with \( c = 1 \) is given by:

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)
\end{align*}
\]

and

\[
\begin{align*}
G_{tt} &= \frac{3(\ddot{a}^2 + k)}{a^2}, \quad G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + k}{1 - kr^2}.
\end{align*}
\]

For a dust universe with \( T_{tt} = \rho \), show that \( \rho a^3 = \rho_0 \), where \( \rho_0 \) is a constant.

(i) In the case \( k = 0 \), show that \( a\ddot{a} = A^2 \), where \( A \) is a constant, and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case \( k < 0 \).

(ii) In the case \( k > 0 \), we define a new time coordinate \( \eta \) by \( \frac{d\eta}{dt} = \frac{1}{Ra} \), where \( R^2 = k^{-1} \). Derive the equations

\[
\begin{align*}
a(\eta) &= B(1 - \cos \eta), \quad t(\eta) = BR(\eta - \sin \eta),
\end{align*}
\]

where \( B \) is a constant, and hence show that the universe recollapses within a finite time.

(iii) For the solution in (ii), set \( r = R\sin \chi \) in the line element and use the formula for the 3-space volume element

\[
dV = (g_{\chi\chi} g_{\theta\theta} g_{\phi\phi})^{1/2} d\chi d\theta d\phi
\]

to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to \( \pi \) for \( \chi \) and \( \theta \), and from 0 to 2\( \pi \) for \( \phi \)). Hence find the maximum volume in terms of \( MG \), where \( M \) is the total mass of the universe, and use dimensional analysis to restore the dependence of the result on \( c \).

5. Obtain the geodesic equations for the closed \((k = 1)\) FLRW dust universe, using \( \eta, \chi, \theta, \phi \) coordinates and show that there are null geodesics with \( \theta = \chi = \frac{1}{4} \pi \). How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

6. Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field \( T^{\alpha\beta} = F^{\alpha\gamma} F^\beta_\gamma - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta} \)) can be written

\[
R_{\alpha\beta} = \kappa \left( F_{\alpha\gamma} F^\gamma_\beta - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} F^{\gamma\delta} \right).
\]

For a line element of the form

\[
\text{ds}^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

the only non-zero components of the Ricci tensor are given by

\[
R_{tt}/(c^2 f) = -f R_{rr} = \frac{1}{2} f'' + f'/r, \quad R_{\theta\theta} = R_{\phi\phi}/ \sin^2 \theta = 1 - r f'/f .
\]

In the case

\[
F_{tr} = -F_{rt} = \frac{Q}{r^2} \quad \text{and} \quad F_{\alpha\beta} = 0 \quad \text{otherwise},
\]

show that a solution can be found that reduces to the Schwarzschild solution when \( Q = 0 \).

Find an analogous solution in the case \( R_{\alpha\beta} = \Lambda g_{\alpha\beta} \).
7. For the Schwarzschild metric, a retarded time coordinate $u$ is defined by $u = ct - r^*$, where $dr/dr^* = F(r) = 1 - 2M/r$. Show that, with this definition, the line element can be written
\[ ds^2 = -F \, du^2 - 2du \, dr + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \]

Consider a spacecraft that is freely falling radially into a Schwarzschild black hole, with 4-velocity $V^\alpha$ and proper time $\tau$. The spacecraft emits monochromatic radio signals, of wave length $\lambda_e$, which propagate radially outwards and are received, with wavelength $\lambda_o$, by a distant observer who is at rest with respect to the Schwarzschild coordinates.

Show that
\[ \lambda_o \lambda_e = \Delta t_o \Delta \tau = \Delta u_o c \Delta \tau \approx V_u c \Delta \tau \]
where, for example, $\Delta t_o$ is the proper time interval during which the observer receives one cycle of the signal and $\Delta \tau$ is the time for the spacecraft to emit one cycle.

Now show that $V_u = -K$, where $K$ is a constant, and that
\[ V^u = \frac{K + \sqrt{K^2 - Fc^2}}{F}, \quad V^r = -\sqrt{K^2 - Fc^2}. \]

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(4M)}$.

Conclude that, just as the spacecraft is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(4M))$.

8. Show that, for an observer with proper time $\tau$ moving in the Schwarzschild spacetime,
\[ c^2 = Fc^2 \dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2), \]
where $\dot{t} = dt/dr$ etc., and $F = 1 - 2M/r$. Show, that for an observer within the Schwarzschild horizon, $\dot{r}^2 \geq -c^2 F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\pi M/c$.

9. Let $M$ be the torus ($S^1 \times S^1$) and define the metric $g_{\alpha\beta}$ on $M$ by
\[ ds^2 = \sin \theta \,(d\phi^2 - d\theta^2) + 2\cos \theta \, d\theta \, d\phi, \]
where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,
\[ \dot{\phi}^2 + 2 \dot{\phi} \dot{\theta} \cot \theta - \dot{\theta}^2 = 0, \]
where dot is differentiation with respect to an affine parameter, and deduce that the curves given by $\phi = -2 \ln \sin(\theta/2) + \phi_0$ and $\phi = -2 \ln \cos(\theta/2) + \phi_0$ are null geodesics. Use another first integral of the Euler-Lagrange equations to show that in both cases $\theta = p\lambda$, where $\lambda$ is an affine parameter and $p$ is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite range of the affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?
10. (i) A weak gravitational field has the spacetime metric \( g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2) \), where \( \eta_{\alpha\beta} \) is the Minkowski metric and \( \epsilon \) is small. Show that
\[
R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon (h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) + O(\epsilon^2).
\]
Let \( h = h^{\gamma\gamma} \) and define \( \bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta h_{\alpha\beta} \). Check that \( h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta h_{\alpha\beta} \) where \( \eta = \bar{h}^{\gamma\gamma} \), and show that
\[
R_{\alpha\beta} = \frac{1}{2} \epsilon (-\Box \bar{h}_{\alpha\beta} + \bar{h}_{\alpha}^{\gamma,\beta} + \bar{h}_{\beta}^{\gamma,\alpha} + \frac{1}{2} \eta h_{\alpha\beta} \Box \bar{h}) + O(\epsilon^2),
\]
where \( \Box = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \). What is the linearised vacuum Einstein equation for \( \bar{h}_{\alpha\beta} \)?

(ii) An infinitesimal coordinate transformation, which is also known as a gauge transformation, is given by \( x^\alpha \to x^\alpha - \epsilon f^\alpha(x) \). Show that \( h_{\alpha\beta} \to h_{\alpha\beta} + f_{\alpha,\beta} + f_{\beta,\alpha} + O(\epsilon) \), but that the curvature tensors are unchanged to leading order in \( \epsilon \). Deduce that if \( f^\alpha \) is chosen to satisfy \( \Box f^\alpha = -h_{\alpha\beta,\beta} \), then in the new coordinates the gauge condition \( \bar{h}_{\alpha}^{\alpha\beta} = 0 \) holds. Conclude that, with this choice, the linearised vacuum Einstein equation for weak fields is the wave equation:
\[
\Box \bar{h}_{\alpha\beta} = 0.
\]

(iii) Consider a gravitational wave solution \( h_{\alpha\beta} = H_{\alpha\beta} \epsilon^{ik_\alpha x^\beta} \) with \( H_{\alpha\beta,\gamma} = 0 \) (note: this is an ansatz for \( h_{\alpha\beta} \), not \( \bar{h}_{\alpha\beta} \)). Show that, in order to satisfy both the linearised Einstein equation and the gauge condition in (ii), \( k^\alpha \) must be a null vector and \( H_{\alpha\beta} k^\beta = \frac{1}{2} k^\alpha H_{\beta}^{\beta} \) must hold.

(iv) Corresponding to the remaining freedom to make gauge transformations, show that there is an arbitrariness in the solution given by \( H_{\alpha\beta} \to H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta \) for any \( v^\alpha \). How many degrees of freedom are there for a gravitational wave propagating in a given direction \( k^\alpha \)? If \( k^\alpha = k(1, 0, 0, 1) \), show that we may take the independent components of \( H_{\alpha\beta} \) to be \( H_{11} = -H_{22} \) and \( H_{12} = H_{21} \).

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