Mathematical Tripos Part II **Lent Term 2025**
General Relativity **Dr. J.M. Evans**

Example Sheet 4

1. A static spacetime has line element

$$ds^2 = -e^{2\phi/c^2} c^2 dt^2 + g_{ij}\, dx^i dx^j \,,$$

where ϕ and g_{ij} are independent of $x^0 = ct$, and $i, j = 1, 2, 3$. Show that

$$\Gamma_{\alpha\ \beta}^{\ 0} = \frac{1}{c^2}\Big(n_\alpha \frac{\partial \phi}{\partial x^\beta} + n_\beta \frac{\partial \phi}{\partial x^\alpha}\Big) \quad \text{and} \quad \Gamma_{0\ 0}^{\ i} = g^{ij}\frac{1}{c^2}\frac{\partial \phi}{\partial x^j}\, e^{2\phi/c^2} \,,$$

where $n_\alpha = (1, 0, 0, 0)$.

Let u^α be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that $u^i = 0$ and $u^0\, u_0 = -c^2$). Show that

$$\nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi \quad \text{and so} \quad \nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha \,.$$

Show further that

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = R_{\alpha\beta}u^\alpha u^\beta$$

and hence that

$$g^{ij}\nabla_i\nabla_j\phi + \frac{1}{c^2} g^{ij}\nabla_i\phi\nabla_j\phi = R_{\alpha\beta}u^\alpha u^\beta \,.$$

[*Hint: you may find it helpful to start from the Ricci identity* $u_{\alpha;\beta\gamma} - u_{\alpha;\gamma\beta} = R^\delta{}_{\alpha\beta\gamma} u_\delta$.]

What does the last equation reduce to in the Newtonian limit (weak gravity) with $T_{\alpha\beta} = \rho\, u_\alpha u_\beta$?

2. A perfect fluid has 4-velocity u^α which is tangent to the fluid flow lines (the integral curves of u^α) and which satisfies $u^\alpha u_\alpha = -c^2$. If the fluid has particle number density n, density ρ and pressure p, then the particle flux density N^α and energy-momentum tensor $T^{\alpha\beta}$ are given by

$$N^\alpha = n u^\alpha, \qquad T^{\alpha\beta} = (\rho + p/c^2) u^\alpha u^\beta + p g^{\alpha\beta},$$

and both are conserved: $\nabla_\alpha N^\alpha = \nabla_\beta T^{\alpha\beta} = 0$.

(i) If the fluid has zero pressure, show that $\nabla_\alpha(\rho u^\alpha) = 0$ and that the fluid flow lines are geodesics. Show also that ρ/n is constant on each such geodesic.

(ii) If the fluid has pressure, find an expression for $\nabla_\alpha(\rho u^\alpha)$ and show that

$$\Big(\rho + \frac{1}{c^2}p\Big) u^\beta \nabla_\beta u^\alpha + \nabla^\alpha p + \frac{1}{c^2} u^\alpha u^\beta \nabla_\beta p = 0 \,.$$

3. Consider a perfect fluid, with definitions and notation as in question 2, and a static, weak-field metric of the form given in question 1, but with $g_{ij} = \delta_{ij}$. In the Newtonian limit, $\varphi/c^2 \sim v^2/c^2 \ll 1$, where v is a typical speed, so that $u^\alpha \approx (c, \mathbf{u})$. Show that, to lowest order,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{u}) = 0 \,,$$

where $\boldsymbol{\nabla}$ is the usual vector operator in 3-dimensional flat space. What is the corresponding equation for ρ? Show that, in the Newtonian limit, $\rho\, u^\beta u_{i;\,\beta} = -p_{,\,i}$ $(i = 1, 2, 3)$ and hence that

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\phi - \frac{1}{\rho}\boldsymbol{\nabla} p \,.$$

4. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric with $c = 1$ is given by:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

and

$$G_{tt} = \frac{3(\dot{a}^2 + k)}{a^2}\,, \quad G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + k}{1-kr^2}\,.$$

For a dust universe with $T_{tt} = \rho$, show that $\rho\, a^3 = \rho_0$, where ρ_0 is a constant.

(i) In the case $k = 0$, show that $a\dot{a}^2 = A^2$, where A is a constant, and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case $k < 0$.

(ii) In the case $k > 0$, we define a new time coordinate η by $\dfrac{d\eta}{dt} = \dfrac{1}{Ra}$, where $R^2 = k^{-1}$. Derive the equations

$$a(\eta) = B(1 - \cos\eta)\,, \qquad t(\eta) = BR(\eta - \sin\eta)\,,$$

where B is a constant, and hence show that the universe recollapses within a finite time.

(iii) For the solution in (ii), set $r = R\sin\chi$ in the line element and use the formula for the 3-space volume element

$$dV \;=\; (g_{\chi\chi}\, g_{\theta\theta}\, g_{\phi\phi})^{1/2} d\chi\, d\theta\, d\phi$$

to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to π for χ and θ, and from 0 to 2π for ϕ). Hence find the maximum volume in terms of MG, where M is the total mass of the universe, and use dimensional analysis to restore the dependence of the result on c.

5. Obtain the geodesic equations for the closed ($k = 1$) FLRW dust universe, using η, χ, θ, ϕ coordinates and show that there are null geodesics with $\theta = \chi = \frac{1}{2}\pi$. How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

6. Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field $T^{\alpha\beta} = F^{\alpha\gamma}F^{\beta}{}_{\gamma} - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g^{\alpha\beta}$) can be written

$$R_{\alpha\beta} = \kappa\big(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \tfrac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\big).$$

For a line element of the form

$$ds^2 = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

the only non-zero components of the Ricci tensor are given by

$$R_{tt}/(c^2 f) = -fR_{rr} = \tfrac{1}{2}f'' + f'/r\,, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2\theta = 1 - rf' - f\,.$$

In the case

$$F_{tr} = -F_{rt} = \frac{Q}{r^2} \quad \text{and} \quad F_{\alpha\beta} = 0 \;\; \text{otherwise,}$$

show that a solution can be found that reduces to the Schwarzschild solution when $Q = 0$.

Find an analogous solution in the case $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$.

7. For the Schwarzschild metric, a retarded time coordinate u is defined by $u = ct - r^*$, where $dr/dr^* = F(r) = 1 - 2M/r$. Show that, with this definition, the line element can be written

$$ds^2 = -F\,du^2 - 2du\,dr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,.$$

Consider a spacecraft that is freely falling radially into a Schwarzschild black hole, with 4-velocity V^α and proper time τ. The spacecraft emits monochromatic radio signals, of wavelength λ_e, which propagate radially outwards and are received, with wavelength λ_o, by a distant observer who is at rest with respect to the Schwarzschild coordinates.

Show that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta\tau} = \frac{\Delta u_o}{c\Delta\tau} = \frac{\Delta u_e}{c\Delta\tau} \approx \frac{V^u}{c}$$

where, for example, Δt_o is the proper time interval during which the observer receives one cycle of the signal and $\Delta\tau$ is the time for the spacecraft to emit one cycle.

Now show that $V_u = -K$, where K is a constant, and that

$$V^u = \frac{K + \sqrt{K^2 - Fc^2}}{F}\,, \qquad V^r = -\sqrt{K^2 - Fc^2}\,.$$

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(4M)}$.

Conclude that, just as the spacecraft is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(4M))$.

8. Show that, for an observer with proper time τ moving in the Schwarzschild spacetime,

$$c^2 = Fc^2\dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2),$$

where $\dot{t} = dt/d\tau$ etc., and $F = 1 - 2M/r$. Show, that for an observer within the Schwarzschild horizon, $\dot{r}^2 \geq -c^2F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\pi M/c$.

9. Let M be the torus $(S^1 \times S^1)$ and define the metric $g_{\alpha\beta}$ on M by

$$ds^2 \;=\; \sin\theta\,(d\phi^2 - d\theta^2) \;+\; 2\cos\theta\,d\theta\,d\phi\,,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,

$$\dot{\phi}^2 \;+\; 2\,\dot{\phi}\,\dot{\theta}\cot\theta - \dot{\theta}^2 = 0\,,$$

where dot is differentiation with respect to an affine parameter, and deduce that the curves given by $\phi = -2\ln\sin(\theta/2) + \phi_0$ and $\phi = -2\ln\cos(\theta/2) + \phi_0$ are null geodesics. Use another first integral of the Euler-Lagrange equations to show that in both cases $\theta = p\lambda$, where λ is an affine parameter and p is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite range of the affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?

10. (i) A weak gravitational field has the spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$, where $\eta_{\alpha\beta}$ is the Minkowski metric and ϵ is small. Show that

$$R_{\alpha\beta\gamma\delta} \;=\; \tfrac{1}{2}\epsilon(\, h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\,) \;+\; O(\epsilon^2)\,.$$

Let $h = h^\gamma{}_\gamma$ and define $\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\,\eta_{\alpha\beta}$. Check that $h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \frac{1}{2}\overline{h}\,\eta_{\alpha\beta}$ where $\overline{h} = \overline{h}^\gamma{}_\gamma$, and show that

$$R_{\alpha\beta} \;=\; \tfrac{1}{2}\epsilon(\,-\Box\,\overline{h}_{\alpha\beta} \;+\; \overline{h}_\alpha{}^\gamma{}_{,\beta\gamma} \;+\; \overline{h}_\beta{}^\gamma{}_{,\alpha\gamma} \;+\; \tfrac{1}{2}\eta_{\alpha\beta}\,\Box\,\overline{h}\,) \;+\; O(\epsilon^2)\,,$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. What is the linearised vacuum Einstein equation for $\overline{h}_{\alpha\beta}$?

(ii) An infinitesimal coordinate transformation, which is also known as a *gauge transformation*, is given by $x^\alpha \to x^\alpha - \epsilon f^\alpha(x)$. Show that

$$h_{\alpha\beta} \to h_{\alpha\beta} + f_{\alpha,\beta} + f_{\beta,\alpha} + O(\epsilon)\,,$$

but that the curvature tensors are unchanged to leading order in ϵ. Deduce that if f^α is chosen to satisfy $\Box f^\alpha = -\overline{h}^{\alpha\beta}{}_{,\beta}$, then in the new coordinates the *gauge condition* $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$ holds. Conclude that, with this choice, the linearised vacuum Einstein equation for weak fields is the wave equation:

$$\Box\,\overline{h}_{\alpha\beta} = 0\,.$$

(iii) Consider a *gravitational wave* solution $h_{\alpha\beta} = H_{\alpha\beta}\,e^{ik_\beta x^\beta}$ with $H_{\alpha\beta,\gamma} = 0$ (note: this is an ansatz for $h_{\alpha\beta}$, not $\overline{h}_{\alpha\beta}$). Show that, in order to satisfy both the linearised Einstein equation and the gauge condition in (ii), k^α must be a null vector and $H_{\alpha\beta}k^\beta = \frac{1}{2}k_\alpha H_\beta{}^\beta$ must hold.

(iv) Corresponding to the remaining freedom to make gauge transformations, show that there is an arbitrariness in the solution given by $H_{\alpha\beta} \to H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta$ for any v_α. How many degrees of freedom are there for a gravitational wave propagating in a given direction k^α? If $k^\alpha = k(1,0,0,1)$, show that we may take the independent components of $H_{\alpha\beta}$ to be $H_{11} = -H_{22}$ and $H_{12} = H_{21}$.

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