1 A static space-time has line element
\[ds^2 = -e^{2\phi/c^2}c^2dt^2 + h_{ij}dx^i dx^j \quad (i, j = 1, 2, 3)\]
where \(\phi\) and \(h_{ij}\) are independent of \(t\). Show that
\[\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{c^2} \left( V_\alpha \frac{\partial \phi}{\partial x^\beta} + V_\beta \frac{\partial \phi}{\partial x^\alpha} \right) \quad \text{and} \quad \Gamma^t_{00} = h^{ij} \frac{\partial \phi}{\partial x^j} e^{2\phi/c^2}\]
where \(V_\alpha = (1, 0, 0, 0)\).

Let \(u^\alpha\) be the 4-velocity of a co-moving observer (i.e., an observer at rest in these coordinates, so that \(u^t = 0\) and \(u^0 u_0 = -c^2\)). Show that
\[\nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi\]
and deduce that \(\nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha\). Show further that
\[g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi - R^{\alpha\beta} u_\alpha u_\beta = 0\]
and hence that
\[h^{ij} \nabla_i \nabla_j \phi + \frac{1}{c^2} h^{ij} \nabla_i \phi \nabla_j \phi = R^{\alpha\beta} u_\alpha u_\beta\]
[The Ricci identity is \(u^{\alpha\beta\gamma} - u^{\alpha\gamma\beta} = R^{\delta\alpha\beta\gamma} u_\delta\).] What does this reduce to in the Newtonian limit with \(T^{\alpha\beta} = \rho u_\alpha u_\beta\)?

2 A perfect fluid has 4-velocity \(u^\alpha\) and particle number density \(n\), density \(\rho\) and pressure \(p\). The particle flux density \(N^\alpha\) and energy-momentum tensor \(T^{\alpha\beta}\) are given by
\[N^\alpha = nu^\alpha, \quad T^{\alpha\beta} = (\rho + p/c^2)u^\alpha u^\beta + pg^{\alpha\beta},\]
and both are conserved: \(N^\alpha;_\alpha = T^{\alpha\beta;_\beta}_\alpha = 0\).

(i) Suppose first that the fluid has zero pressure. Show that the fluid flow lines (integral curves of \(u^\alpha\)) are geodesics and that \(\rho\) is proportional to \(n\) on each such geodesic.

(ii) Now consider a general perfect fluid and the weak-field metric
\[ds^2 = -e^{2\varphi/v^2}c^2dt^2 + dx^2 + dy^2 + dz^2\]
with \(\varphi/c^2 \lesssim v^2/c^2 \ll 1\), where \(v\) is a typical speed, so that \(u^\alpha \approx (1, u)\). Show that, to lowest order,
\[\frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0,\]
where \(\nabla\) is the usual 3-dimensional flat space derivative. What is the corresponding equation for \(\rho\)? [Recall (sheet 2) that \(\Gamma^{\beta}_{\alpha\gamma} = \frac{1}{2} (\log(-g))_{\alpha\gamma}\).]

Show that
\[\rho + p/c^2)u_\alpha;_\beta u^\beta + p,\alpha + c^{-2} p,\beta u^\beta u_\alpha = 0\]
and hence that, in the Newtonian limit, \(\rho u_\alpha;_\beta u^\beta = -p,\alpha\) \((i = 1, 2, 3)\) i.e.
\[\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla \phi - \frac{1}{\rho} \nabla p.\]
The Friedman-Lemaître-Robertson-Walker (FLRW) metric is given by

\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \]

and

\[ G_{tt} = \frac{3(\dot{a}^2 + kc^2)}{a^2}, \quad c^2 G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{1 - kr^2}. \]

For a dust universe \( (T_{tt} = \rho c^4) \), show that \( \rho a^3 = \rho_0 \), where \( \rho_0 \) is a constant.

(i) In the case \( k = 0 \), show that \( \frac{\ddot{a}}{a^2} = \frac{A^2}{a^2} \), where \( A \) is a constant and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case \( k < 0 \).

(ii) In the case \( k > 0 \), we define a new coordinate \( \eta \) by \( \frac{d\eta}{dt} = cRa \), where \( R^2 = k - 1 \). Derive the equations

\[ a(\eta) = B(1 - \cos \eta), \quad c t(\eta) = BR(\eta - \sin \eta), \]

where \( B \) is a constant. Hence show that the universe recollapses within a finite time. Now set \( r = R \sin \chi \) in the line element and use the formula for the 3-space volume element

\[ dV = \sqrt{g_{\chi\chi} g_{\theta\theta} g_{\phi\phi}} \, d\chi \, d\theta \, d\phi \]

to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to \( \pi \) for \( \chi \) and \( \theta \), and from 0 to \( 2\pi \) for \( \phi \)). Hence find the maximum volume in terms of \( MG \), where \( M \) is the total mass of the universe, and \( c \).

Obtain the geodesic equations for the closed \((k = 1)\) FLRW dust universe, using \( \eta, \chi, \theta, \phi \) coordinates and show that there are null geodesics with \( \theta = \chi = \frac{1}{2} \pi \). How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field \( T^{\alpha\beta} = F^{\alpha\gamma} F_{\beta\gamma} \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} \)) can be written

\[ R_{\alpha\beta} = \kappa \left( F_{\alpha\gamma} F_{\beta}^\gamma - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta} \right). \]

You are given that, for a line element of the form

\[ ds^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

the only non-zero components of the Ricci tensor are

\[ c^{-2} R_{tt}/f = -f R_{rr} = \frac{1}{2} f'' + f'/r, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2 \theta = 1 - rf'/f. \]

In the case

\[ F_{tr} = -F_{rt} = \frac{Q}{r^2}, \quad \text{with} \quad F_{\alpha\beta} = 0 \quad \text{otherwise}, \]

show that a solution can be found that reduces to the Schwarzschild solution for \( Q = 0 \). Find an analogous solution in the case \( R_{\alpha\beta} = \Lambda g_{\alpha\beta} \).
A spacecraft is freely falling radially into a Schwarzschild black hole. It has 4-velocity \( V^\alpha \) and proper time \( \tau \). It emits monochromatic radio of wavelength \( \lambda_o \). Its signals propagate radially outwards and are received, with wavelength \( \lambda_o \), by a distant observer who is at rest with respect to the Schwarzschild coordinates.

A retarded time coordinate \( u \) is defined by \( u = ct - r^* \) where \( dr/dr^* = F(r) \) and \( F(r) = 1 - 2M/r \). Show that

\[
ds^2 = -F \, du^2 - 2du \, dr + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

Show that

\[
\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{c \Delta \tau} = \frac{\Delta u_o}{c \Delta \tau} \approx \frac{V^u}{c}.
\]

where, for example, \( \Delta t_o \) is the proper time interval during which the observer receives one cycle of the signal and \( \Delta \tau \) is the time for the spacecraft to emit one cycle.

Show next that \( V^u = -K \), where \( K \) is a constant, and that

\[
V^u = \frac{K + \sqrt{K^2 - Fc^2}}{F}, \quad V^r = -\sqrt{K^2 - Fc^2}.
\]

Deduce that on the world line of the spacecraft near the horizon \( du/dr \approx -2/F \), and that \( u \approx -2r^* \) and \( F \sim e^{-u/(4M)} \).

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence \( \propto \exp(-ct/(4M)) \).

Show that, for an observer with proper time \( \tau \) moving in the Schwarzschild space-time,

\[
c^2 = Fc^2 \dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2),
\]

where \( \dot{t} = dt/d\tau \) etc., and \( F = 1 - 2M/r \). Show, that for an observer within the Schwarzschild horizon, \( \dot{s}^2 \geq -c^2F \) however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach \( r = 0 \) within a proper time \( \pi M/c \).

Let \( M \) be the torus \( (S^1 \times S^1) \) and define the metric \( g_{\alpha\beta} \) on \( M \) by

\[
ds^2 = \sin \theta (d\phi^2 - d\theta^2) + 2 \cos \theta d\theta d\phi,
\]

where \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq 2\pi \). Show that, for a null geodesic,

\[
\dot{\phi}^2 + 2\dot{\phi} \cot \theta - \dot{\theta}^2 = 0,
\]

where dot is differentiation with respect to an affine parameter, and deduce that the curves given by \( \phi = -2 \ln \sin(\theta/2) + \phi_0 \) and \( \phi = -2 \ln \cos(\theta/2) + \phi_0 \) are null geodesics. Use another first integral of Lagrange’s equations to show that in both cases \( \theta = p\lambda \), where \( \lambda \) an affine parameter and \( p \) is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite affine parameter, never reaching the null curve \( \theta = 2\pi \), and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?
A weak gravitational field has the spacetime metric \( g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2) \), where \( \eta_{\alpha\beta} \) is the Minkowski metric and \( \epsilon \) small. Show that

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon [h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}] + O(\epsilon^2).
\]

Let \( h = h^{\gamma}{}_{\gamma} \) and define \( \overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} \). Show that \( h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \frac{1}{2} \overline{h} \eta_{\alpha\beta} \). Show also that

\[
R_{\alpha\beta} = \frac{1}{2} \epsilon [-\square \overline{h}_{\alpha\beta} + \overline{h}_{\alpha}\gamma,\beta\gamma + \overline{h}_{\beta}\gamma,\alpha\gamma + \frac{1}{2} \eta_{\alpha\beta} \square \overline{h}] + O(\epsilon^2).
\]

where \( \square = \eta^{\alpha\beta} \nabla_\alpha \nabla_\beta \). What are the linear vacuum equations for \( \overline{h}_{\alpha\beta} \)?

An infinitesimal coordinate transformation (which may be called a gauge transformation) is given by \( x^\alpha \rightarrow x^\alpha + \epsilon f^\alpha(x) \). Show that

\[
h_{\alpha\beta} \rightarrow h_{\alpha\beta} - f_{\alpha,\beta} - f_{\beta,\alpha} + O(\epsilon),
\]

but that the curvature tensors are unchanged (to leading order in \( \epsilon \)). Deduce that if \( f^\alpha \) is chosen to satisfy \( \square f^\alpha = \overline{h}^{\alpha\beta}{}_{,\beta} \), then in the new coordinates \( \overline{h}^{\alpha\beta}{}_{,\beta} = 0 \). Conclude that the linearised Einstein equation for weak fields in vacuum is the wave equation

\[
\square \overline{h}_{\alpha\beta} = 0.
\]

Consider a gravitational wave \( h_{\alpha\beta} = H_{\alpha\beta} e^{ik^\beta x^\beta} \) in the above gauge, where \( H_{a,b,c} = 0 \). (Note: we really mean \( h_{\alpha\beta} \propto H_{\alpha\beta} \) here unlike in the lecture where we started setting \( \overline{h}_{\alpha\beta} \propto H_{\alpha\beta} \).) Show that \( H_{\alpha\beta} k^\beta = \frac{1}{2} k_a H_{\beta}{}^{\beta} \) and that \( k^\alpha \) is null. Show also that through remaining gauge freedom there is arbitrariness in \( H_{\alpha\beta} \rightarrow H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta \) for any \( v_\alpha \). How many degrees of freedom are there for a gravitational wave propagating in a given direction?

Show that \( R_{\alpha\beta\gamma\delta} k^\delta = 0 \) to lowest order in \( \epsilon \).

If \( k^\alpha = k(1, 0, 0, 1) \), show that we may take the independent components to be \( H_{11} = -H_{22}, H_{12} = H_{21} \).