1 A static space-time has line element
\[ ds^2 = -e^{2\phi/c^2}c^2 dt^2 + h_{ij}dx^i dx^j \quad (i, j = 1, 2, 3) \]
where \( \phi \) and \( h_{ij} \) are independent of \( t \). Show that
\[ \Gamma^0_{\alpha\beta} = \frac{1}{c^2} \left( V_\alpha \frac{\partial \phi}{\partial x^\beta} + V_\beta \frac{\partial \phi}{\partial x^\alpha} \right) \quad \text{and} \quad \Gamma^i_{00} = h^{ij} \frac{\partial \phi}{\partial x^j} e^{2\phi/c^2} \]
where \( V_\alpha = (1, 0, 0, 0) \).

Let \( u^\alpha \) be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that \( u^i = 0 \) and \( u^0 u_0 = -c^2 \)). Show that
\[ \nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi \]
and deduce that \( \nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha \). Show further that
\[ g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = R_{\alpha\beta} u^\alpha u^\beta \]
and hence that
\[ h^{ij} \nabla_i \nabla_j \phi + \frac{1}{c^2} h^{ij} \nabla_i \phi \nabla_j \phi = R_{\alpha\beta} u^\alpha u^\beta \]
[The Ricci identity is \( u_\alpha;\beta - u_\beta;\alpha = R^{\delta \alpha \beta \gamma} u_\delta \). What does this reduce to in the Newtonian limit with \( T_{\alpha\beta} = \rho u_\alpha u_\beta \)?]

2 A perfect fluid has 4-velocity \( u^\alpha \) and particle number density \( n \), density \( \rho \) and pressure \( p \). The particle flux density \( N^\alpha \) and energy-momentum tensor \( T^{\alpha\beta} \) are given by
\[ N^\alpha = nu^\alpha, \quad T^{\alpha\beta} = (\rho + p/c^2) u^\alpha u^\beta + pg^{\alpha\beta}, \]
and both are conserved: \( N^\alpha; \alpha = T^{\alpha\beta; \beta} = 0 \).

(i) Suppose first that the fluid has zero pressure. Show that the fluid flow lines (integral curves of \( u^\alpha \)) are geodesics and that \( \rho \) is proportional to \( n \) on each such geodesic.

(ii) Now consider a general perfect fluid and the weak-field metric
\[ ds^2 = -e^{2\phi/c^2}c^2 dt^2 + dx^2 + dy^2 + dz^2, \]
with \( \varphi/c^2 \sim v^2/c^2 \ll 1 \), where \( v \) is a typical speed, so that \( u^\alpha \approx (1, u) \). Show that, to lowest order,
\[ \frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0, \]
where \( \nabla \) is the usual 3-dimensional flat space derivative. What is the corresponding equation for \( \rho \)? [Recall (sheet 2) that \( \Gamma^\beta_{\alpha\gamma} = \frac{1}{2} \left( \log (-g) \right)_{\alpha\gamma} \).]

Show that
\[ (\rho + p/c^2) u^\alpha u^\beta + p_{\alpha} + c^{-2} p_{\beta} u^\beta u_\alpha = 0 \]
and hence that, in the Newtonian limit, \( \rho u_{i,\beta} u^\beta = -p_{i,\beta} \) (i.e. \( \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla \phi - \frac{1}{\rho} \nabla p \)).
The Friedman-Lemaître-Robertson-Walker (FLRW) metric is given by
\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \]
and
\[ G_{tt} = \frac{3(\dot{a}^2 + kc^2)}{a^2}, \quad c^2 G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{1-kr^2}. \]

For a dust universe \((T_{tt} = \rho c^4)\), show that \(\rho a^3 = \rho_0\), where \(\rho_0\) is a constant.

(i) In the case \(k = 0\), show that \(\ddot{a}a^2 = A^2\), where \(A\) is a constant and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case \(k < 0\).

(ii) In the case \(k > 0\), we define a new coordinate \(\eta\) by \(d\eta/dt = cRa\), where \(R^2 = k^{-1}\). Derive the equations
\[ a(\eta) = B(1 - \cos \eta), \quad c(\eta) = BR(\eta - \sin \eta), \]
where \(B\) is a constant. Hence show that the universe recollapses within a finite time. Now set \(r = R \sin \chi\) in the line element and use the formula for the 3-space volume element
\[ dV = \sqrt{g_{\chi\chi} g_{\theta\theta} g_{\phi\phi}} \, d\chi \, d\theta \, d\phi \]
to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to \(\pi\) for \(\chi\) and \(\theta\), and from 0 to \(2\pi\) for \(\phi\)). Hence find the maximum volume in terms of \(MG\), where \(M\) is the total mass of the universe, and \(c\).

Obtain the geodesic equations for the closed \((k = 1)\) FLRW dust universe, using \(\eta, \chi, \theta, \phi\) coordinates and show that there are null geodesics with \(\theta = \chi = \frac{1}{2} \pi\). How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field \(T^{\alpha\beta} = F^{\alpha\gamma} F^{\beta}_\gamma - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta}\)) can be written
\[ R_{\alpha\beta} = \kappa(F_{\alpha\gamma} F^{\beta}_\gamma - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta}). \]
You are given that, for a line element of the form
\[ ds^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]
the only non-zero components of the Ricci tensor are
\[ c^{-2} R_{tt}/f = -f R_{rr} = \frac{1}{2} f'' + \frac{f'}{r}, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2 \theta = 1 - rf' - f. \]
In the case
\[ F_{tr} = -F_{rt} = \frac{Q}{r^2}, \quad \text{with } F_{\alpha\beta} = 0 \text{ otherwise}, \]
show that a solution can be found that reduces to the Schwarzschild solution for \(Q = 0\).

Find an analogous solution in the case \(R_{\alpha\beta} = \Lambda g_{\alpha\beta}\).
6 A spacecraft is freely falling radially into a Schwarzschild black hole. It has 4-velocity $V^\alpha$ and proper time $\tau$. It emits monochromatic radio of wavelength $\lambda_e$. Its signals propagate radially outwards and are received, with wavelength $\lambda_o$, by a distant observer who is at rest with respect to the Schwarzschild coordinates.

A retarded time coordinate $u$ is defined by $u = ct - r^*$ where $dr/dr^* = F(r)$ and $F(r) = 1 - 2M/r$. Show that

$$ds^2 = -F du^2 - 2du dr + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Show that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta \tau} = \frac{\Delta u_o}{c \Delta \tau} = \frac{\Delta u_e}{c \Delta \tau} \approx \frac{V^u}{c},$$

where, for example, $\Delta t_o$ is the proper time interval during which the observer receives one cycle of the signal and $\Delta \tau$ is the time for the spacecraft to emit one cycle.

Show next that $V^u = -K$, where $K$ is a constant, and that

$$V^u = \frac{K + \sqrt{K^2 - F c^2}}{F}, \quad V^r = -\sqrt{K^2 - F c^2}.$$

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(4M)}$.

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(4M))$.

7 Show that, for an observer with proper time $\tau$ moving in the Schwarzschild space-time,

$$c^2 = Fc^2 i^2 - i^2/F - i^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2),$$

where $i = dt/d\tau$ etc., and $F = 1 - 2M/r$. Show, that for an observer within the Schwarzschild horizon, $i^2 \geq -c^2 F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\pi M/c$.

8 Let $M$ be the torus ($S^1 \times S^1$) and define the metric $g_{\alpha\beta}$ on $M$ by

$$ds^2 = \sin \theta (d\phi^2 - d\theta^2) + 2 \cos \theta d\theta d\phi,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,

$$\dot{\phi}^2 + 2\dot{\phi} \cot \theta - \dot{\theta}^2 = 0,$$

where dot is differentiation with respect to an affine parameter, and deduce that the curves given by $\phi = -2 \ln \sin(\theta/2) + \phi_0$ and $\phi = -2 \ln \cos(\theta/2) + \phi_0$ are null geodesics. Use another first integral of Lagrange's equations to show that in both cases $\theta = p\lambda$, where $\lambda$ an affine parameter and $p$ is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?
A weak gravitational field has the spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$, where $\eta_{\alpha\beta}$ is the Minkowski metric and $\epsilon$ small. Show that

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon [h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}] + O(\epsilon^2).$$

Let $h = h^\gamma_\gamma$ and define $\tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$. Show that $h_{\alpha\beta} = \tilde{h}_{\alpha\beta} - \frac{1}{2} \tilde{h} \eta_{\alpha\beta}$. Show also that

$$R_{\alpha\beta} = \frac{1}{2} \epsilon [-\Box \tilde{h}_{\alpha\beta} + \tilde{h}_{\alpha,\gamma\beta} + \tilde{h}_{\beta,\gamma\alpha} + \frac{1}{2} \eta_{\alpha\beta} \Box \tilde{h}] + O(\epsilon^2).$$

where $\Box = \eta^{\alpha\beta} \nabla_\alpha \nabla_\beta$. What are the linear vacuum equations for $\tilde{h}_{\alpha\beta}$?

An infinitesimal coordinate transformation (which may be called a gauge transformation) is given by $x^\alpha \to x^\alpha + \epsilon f^\alpha(x)$. Show that

$$h_{\alpha\beta} \to h_{\alpha\beta} - f_{\alpha,\beta} - f_{\beta,\alpha} + O(\epsilon),$$

but that the curvature tensors are unchanged (to leading order in $\epsilon$). Deduce that if $f^\alpha$ is chosen to satisfy $\Box f^\alpha = \tilde{h}^\alpha_\beta,\beta$, then in the new coordinates $\tilde{h}^\alpha_\beta,\beta = 0$. Conclude that the linearised Einstein equation for weak fields in vacuum is the wave equation

$$\Box \tilde{h}_{\alpha\beta} = 0.$$

Consider a gravitational wave $h_{\alpha\beta} = H_{\alpha\beta} e^{i k^\alpha x_\alpha}$ in the above gauge, where $H_{\alpha\beta\gamma} = 0$. (Note: we really mean $h_{\alpha\beta} \propto H_{\alpha\beta}$ here unlike in the lecture where we started setting $\tilde{h}_{\alpha\beta} \propto H_{\alpha\beta}$). Show that $H_{\alpha\beta} k^\beta = \frac{1}{2} k^\alpha H_{\beta}^\beta$ and that $k^\alpha$ is null. Show also that through remaining gauge freedom there is arbitrariness in $H_{\alpha\beta} \to H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta$ for any $v_\alpha$. How many degrees of freedom are there for a gravitational wave propagating in a given direction?

Show that $R_{\alpha\beta\gamma\delta} k^\delta = 0$ to lowest order in $\epsilon$.

If $k^\alpha = k(1, 0, 0, 1)$, show that we may take the independent components to be $H_{11} = -H_{22}$, $H_{12} = H_{21}$. 

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