

1 A static space-time has line element

$$ds^2 = -e^{2\phi/c^2} c^2 dt^2 + h_{ij} dx^i dx^j \quad (i, j = 1, 2, 3)$$

where ϕ and h_{ij} are independent of t . Show that

$$\Gamma^0_{\alpha\beta} = \frac{1}{c^2} \left(U_\alpha \frac{\partial\phi}{\partial x^\beta} + U_\beta \frac{\partial\phi}{\partial x^\alpha} \right) \text{ and } \Gamma^i_{00} = h^{ij} \frac{\partial\phi}{\partial x^j} e^{2\phi/c^2}$$

where $U_\alpha = (1, 0, 0, 0)$.

Let u^α be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that $u^i = 0$ and $u^0 u_0 = -c^2$). Show that

$$\nabla_\beta u_\alpha = -\frac{1}{c^2} u_\beta \nabla_\alpha \phi$$

and deduce that $\nabla_\alpha \phi = u^\beta \nabla_\beta u_\alpha$. Show further that

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = R_{\alpha\beta} u^\alpha u^\beta$$

and hence that

$$h^{ij} \nabla_i \nabla_j \phi + \frac{1}{c^2} h^{ij} \nabla_i \phi \nabla_j \phi = R_{\alpha\beta} u^\alpha u^\beta$$

[The Ricci identity is $u_{\alpha;\beta\gamma} - u_{\alpha;\gamma\beta} = R^\delta_{\alpha\beta\gamma} u_\delta$.] What does this reduce to in the Newtonian limit with $T_{\alpha\beta} = \rho u_\alpha u_\beta$?

2 A perfect fluid has 4-velocity u^α and particle number density n , density ρ and pressure p . The particle flux density N^α and energy-momentum tensor $T^{\alpha\beta}$ are given by

$$N^\alpha = n u^\alpha, \quad T^{\alpha\beta} = (\rho + p/c^2) u^\alpha u^\beta + p g^{\alpha\beta},$$

and both are conserved: $N^\alpha_{;\alpha} = T^{\alpha\beta}_{;\beta} = 0$.

- (i) Suppose first that the fluid has zero pressure. Show that the fluid flow lines (integral curves of u^α) are geodesics and that ρ is proportional to n on each such geodesic.
- (ii) Now consider a general perfect fluid and the weak-field metric

$$ds^2 = -e^{2\varphi/c^2} c^2 dt^2 + dx^2 + dy^2 + dz^2,$$

with $\varphi/c^2 \sim v^2/c^2 \ll 1$, where v is a typical speed, so that $u^\alpha \approx (1, \mathbf{u})$. Show that, to lowest order,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0,$$

where ∇ is the usual 3-dimensional flat space derivative. What is the corresponding equation for ρ ? [Recall (sheet 2) that $\Gamma^\beta_{\beta\alpha} = \frac{1}{2}(\log(-g))_{,\alpha}$.]

Show that

$$(\rho + p/c^2) u_{\alpha;\beta} u^\beta + p_{,\alpha} + c^{-2} p_{,\beta} u^\beta u_\alpha = 0$$

and hence that, in the Newtonian limit, $\rho u_{i;\beta} u^\beta = -p_{,i}$ ($i = 1, 2, 3$) i.e. $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \phi - \frac{1}{\rho} \nabla p$.

3 The Friedman-Lemaître-Robertson-Walker (FLRW) metric is given by

$$ds^2 = -c^2 dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

and

$$G_{tt} = \frac{3(\dot{a}^2 + kc^2)}{a^2}, \quad c^2 G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{1 - kr^2}.$$

For a dust universe ($T_{tt} = \rho c^4$), show that $\rho a^3 = \rho_0$, where ρ_0 is a constant.

(i) In the case $k = 0$, show that $a\dot{a}^2 = A^2$, where A is a constant and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case $k < 0$.

(ii) In the case $k > 0$, we define a new coordinate η by $\frac{d\eta}{dt} = \frac{c}{Ra}$, where $R^2 = k^{-1}$. Derive the equations

$$a(\eta) = B(1 - \cos \eta) \quad ct(\eta) = BR(\eta - \sin \eta),$$

where B is a constant. Hence show that the universe recollapses within a finite time. Now set $r = R \sin \chi$ in the line element and use the formula for the 3-space volume element

$$dV = \sqrt{g_{\chi\chi} g_{\theta\theta} g_{\phi\phi}} d\chi d\theta d\phi$$

to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to π for χ and θ , and from 0 to 2π for ϕ). Hence find the maximum volume in terms of MG , where M is the total mass of the universe, and c .

4 Obtain the geodesic equations for the closed ($k = 1$) FLRW dust universe, using η, χ, θ, ϕ coordinates and show that there are null geodesics with $\theta = \chi = \frac{1}{2}\pi$. How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

5 Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field $T^{\alpha\beta} = F^{\alpha\gamma} F^{\beta}_{\gamma} - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta}$) can be written

$$R_{\alpha\beta} = \kappa (F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}).$$

You are given that, for a line element of the form

$$ds^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

the only non-zero components of the Ricci tensor are

$$c^{-2} R_{tt}/f = -f R_{rr} = \frac{1}{2} f'' + f'/r, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2 \theta = 1 - r f' - f.$$

In the case

$$F_{tr} = -F_{rt} = \frac{Q}{r^2}, \quad \text{with } F_{\alpha\beta} = 0 \text{ otherwise,}$$

show that a solution can be found that reduces to the Schwarzschild solution for $Q = 0$.

Find an analogous solution in the case $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$.

6 A spacecraft is freely falling radially into a Schwarzschild black hole. It has 4-velocity V^α and proper time τ . It emits monochromatic radio of wavelength λ_e . Its signals propagate radially outwards and are received, with wavelength λ_o , by a distant observer who is at rest with respect to the Schwarzschild coordinates.

A retarded time coordinate u is defined by $u = ct - r^*$ where $dr/dr^* = F(r)$ and $F(r) = 1 - 2M/r$. Show that

$$ds^2 = -F du^2 - 2du dr + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Show that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta \tau} = \frac{\Delta u_o}{c \Delta \tau} = \frac{\Delta u_e}{c \Delta \tau} \approx V^u/c$$

where, for example, Δt_o is the proper time interval during which the observer receives one cycle of the signal and $\Delta \tau$ is the time for the spacecraft to emit one cycle.

Show next that $V_u = -K$, where K is a constant, and that

$$V^u = \frac{K + \sqrt{K^2 - Fc^2}}{F}, \quad V^r = -\sqrt{K^2 - Fc^2}.$$

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(4M)}$.

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(4M))$.

7 Show that, for an observer with proper time τ moving in the Schwarzschild space-time,

$$c^2 = Fc^2 \dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2),$$

where $\dot{t} = dt/d\tau$ etc., and $F = 1 - 2M/r$. Show, that for an observer within the Schwarzschild horizon, $\dot{r}^2 \geq -c^2 F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\pi M/c$.

8 Let M be the torus ($S^1 \times S^1$) and define the metric $g_{\alpha\beta}$ on M by

$$ds^2 = \sin \theta (d\phi^2 - d\theta^2) + 2 \cos \theta d\theta d\phi,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,

$$\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} \cot \theta - \dot{\theta}^2 = 0,$$

where dot is differentiation with respect to an affine parameter, and deduce that the curves given by $\phi = -2 \ln \sin(\theta/2) + \phi_0$ and $\phi = -2 \ln \cos(\theta/2) + \phi_0$ are null geodesics. Use another first integral of Lagrange's equations to show that in both cases $\theta = p\lambda$, where λ an affine parameter and p is a constant.

Show that one family of null geodesics wraps round the torus an infinite number of times within a finite affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?

9 A weak gravitational field has the spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$, where $\eta_{\alpha\beta}$ is the Minkowski metric and ϵ small. Show that

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon[h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}] + O(\epsilon^2).$$

Let $h = h^\gamma{}_\gamma$ and define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}$. Show that $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\bar{h}\eta_{\alpha\beta}$. Show also that

$$R_{\alpha\beta} = \frac{1}{2}\epsilon[-\square\bar{h}_{\alpha\beta} + \bar{h}_{\alpha\gamma}{}^{,\beta\gamma} + \bar{h}_{\beta\gamma}{}^{,\alpha\gamma} + \frac{1}{2}\eta_{\alpha\beta}\square\bar{h}] + O(\epsilon^2).$$

where $\square = \eta^{\alpha\beta}\nabla_\alpha\nabla_\beta$. What are the linear vacuum equations for $\bar{h}_{\alpha\beta}$?

An infinitesimal coordinate transformation (which may be called a gauge transformation) is given by $x^\alpha \rightarrow x^\alpha + \epsilon f^\alpha(x)$. Show that

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - f_{\alpha,\beta} - f_{\beta,\alpha} + O(\epsilon),$$

but that the curvature tensors are unchanged (to leading order in ϵ). Deduce that if f^α is chosen to satisfy $\square f^\alpha = \bar{h}^{\alpha\beta}{}_{,\beta}$, then in the new coordinates $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$. Conclude that the linearised Einstein equation for weak fields in vacuum is the wave equation

$$\square\bar{h}_{\alpha\beta} = 0.$$

Consider a *gravitational wave* $h_{\alpha\beta} = H_{\alpha\beta}e^{ik_\beta x^\beta}$ in the above gauge, where $H_{ab,c} = 0$. (Note: we really mean $h_{\alpha\beta} \propto H_{\alpha\beta}$ here unlike in the lecture where we started setting $\bar{h}_{\alpha\beta} \propto H_{\alpha\beta}$). Show that $H_{\alpha\beta}k^\beta = \frac{1}{2}k_\alpha H_\beta{}^\beta$ and that k^α is null. Show also that through remaining gauge freedom there is arbitrariness in $H_{\alpha\beta} \rightarrow H_{\alpha\beta} + k_\alpha v_\beta + v_\alpha k_\beta$ for any v_α . How many degrees of freedom are there for a gravitational wave propagating in a given direction?

Show that $R_{\alpha\beta\gamma\delta}k^\delta = 0$ to lowest order in ϵ .

If $k^\alpha = k(1, 0, 0, 1)$, show that we may take the independent components to be $H_{11} = -H_{22}$, $H_{12} = H_{21}$.