1 A static space-time has line element

\[ ds^2 = -e^{2\phi/c^2}c^2dt^2 + h_{ij}dx^idx^j \quad (i, j = 1, 2, 3) \]

where \( \phi \) and \( h_{ij} \) are independent of \( t \). Show that

\[ \Gamma^0_{ab} = \frac{1}{c^2} \left( U_a \frac{\partial \phi}{\partial x^b} + U_b \frac{\partial \phi}{\partial x^a} \right) \]

and \( \Gamma^i_{00} = h_{ij} \frac{\partial \phi}{\partial x^j} e^{2\phi/c^2} \)

where \( U_a = (1, 0, 0, 0) \).

Let \( u^a \) be the 4-velocity of a co-moving observer (i.e. an observer at rest in these coordinates, so that \( u^t = 0 \) and \( u^0u_0 = -c^2 \)). Show that

\[ \nabla_b u_a = -\frac{1}{c^2} u_b \nabla_a \phi \]

and deduce that \( \nabla_a \phi = u^b \nabla_b u_a \). Show further that

\[ g^{ab} \nabla_a \nabla_b \phi = R_{ab} u^a u^b \]

and hence that

\[ h_{ij} \nabla_i \nabla_j \phi + \frac{1}{c^2} h_{ij} \nabla_i \phi \nabla_j \phi = R_{ab} u^a u^b \]

[The Ricci identity is \( u^a_{;bc} - u^a_{;cb} = R^d_{abc} \).] What does this reduce to in the Newtonian limit with \( T_{ab} = \rho u_a u_b \)?

2 A perfect fluid has 4-velocity \( u^a \) and particle number density \( n \), density \( \rho \) and pressure \( p \). The particle flux density \( N_a \) and energy-momentum tensor \( T_{ab} \) are given by

\[ N^a = nu^a, \quad T^{ab} = (\rho + p/c^2) u^a u^b + pg^{ab}, \]

and both are conserved: \( N^a_{;a} = T^{ab}_{;b} = 0 \).

(i) Suppose first that the fluid has zero pressure. Show that the fluid flow lines (integral curves of \( u^a \)) are geodesics and that \( \rho \) is proportional to \( n \) on each such geodesic.

(ii) Now consider a general perfect fluid and the weak-field metric

\[ ds^2 = -e^{2\varphi/c^2}c^2dt^2 + dx^2 + dy^2 + dz^2, \]

with \( \varphi/c^2 \sim v^2/c^2 \ll 1 \), where \( v \) is a typical speed, so that \( u^a \approx (1, u) \). Show that

\[ \frac{\partial n}{\partial t} + \nabla \cdot (nu) = 0, \]

where \( \nabla \) is the usual 3-dimensional flat space derivative. What is the corresponding equation for \( \rho \)? [Recall (sheet 2) that \( \Gamma^0_{ab} = \frac{1}{2} (\log g)_{ab} \).]

Show that

\[ (\rho + p/c^2) u_{a;b} u^b + p_{,a} + c^{-2} p_{,b} u^b u_a = 0 \]

and hence that, in the Newtonian limit, \( \rho u_{i;b} u^b = -p, (i = 1, 2, 3) \) i.e.

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla \phi - \frac{1}{\rho} \nabla p. \]
The FLRW metric is given by
\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \]
and
\[ G_{tt} = \frac{3(\dot{a}^2 + kc^2)}{a^2} \quad G_{rr} = -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{1 - kr^2}. \]

For a dust universe \((T_{tt} = \rho c^2)\), show that \(\rho a^3 = \rho_0\), where \(\rho_0\) is a constant.

(i) In the case \(k = 0\), show that \(a\dot{a}^2 = A^2\), where \(A\) is a constant and deduce that the universe expands for ever. Without further calculation, explain how this conclusion is affected in the case \(k < 0\).

(ii) In the case \(k > 0\), we define a new coordinate \(\eta\) by \(\frac{d\eta}{dt} = cRa\), where \(R^2 = k - 1\). Derive the equations
\[ a(\eta) = B(1 - \cos \eta) \quad c(t) = BR(\eta - \sin \eta), \]
where \(B\) is a constant. Hence show that the universe recollapses within a finite time.

Now set \(r = R \sin \chi\) in the line element and use the formula for the 3-space volume element \(dV = \sqrt{g_{\chi\chi} g_{\theta\theta} g_{\phi\phi}} d\chi d\theta d\phi\) to determine the volume of the universe at a given scale factor (the angular coordinates run from 0 to \(\pi\) for \(\chi\) and \(\theta\), and from 0 to \(2\pi\) for \(\phi\)). Hence find the maximum volume in terms of \(G, R, c\) and the total mass \(M\) of the universe.

Obtain the geodesic equations for the closed \((k = 1)\) FLRW dust universe, using \(\eta, \chi, \theta, \phi\) coordinates and show that there are null geodesics with \(\theta = \chi = \frac{1}{2}\pi\). How many times can a photon encircle the universe from the time of creation to the moment of annihilation?

Show that the Einstein-Maxwell equations (i.e. the Einstein equations with energy momentum tensor for an electromagnetic field \(T^{ab} = F^{ac}F_b{}^c - \frac{1}{4}F^{cd}F_{cd}g^{ab}\)) can be written
\[ R_{ab} = \kappa \left( F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd} \right). \]
You are given that, for a line element of the form
\[ ds^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]
the only non-zero components of the Ricci tensor are
\[ c^{-2} R_{tt}/f = -f R_{rr} = \frac{1}{2} f'' + f'/r, \quad R_{\theta\theta} = R_{\phi\phi}/\sin^2 \theta = 1 - rf'/f. \]

In the case
\[ F_{tr} = -F_{rt} = \frac{Q}{r^2}, \quad \text{with} \ F_{ab} = 0 \ \text{otherwise}, \]
show that a solution can be found that reduces to the Schwarzschild solution for \(Q = 0\).

Find an analogous solution in the case \(R_{ab} = \Lambda g_{ab}\).
A spacecraft is freely falling radially into a Schwarzschild black hole. It has 4-velocity $V^a$ and proper time $\tau$. It emits monochromatic radio of wavelength $\lambda_e$. Its signals propagate radially outwards and are received, with wavelength $\lambda_o$, by a distant observer who is at rest with respect to the Schwarzschild coordinates.

A retarded time coordinate $u$ is defined by $u = ct - r^*$ where $dr/dr^* = F(r)$ and $F(r) = 1 - r_s/r$. Show that

$$ds^2 = -F \, du^2 - 2du \, dr + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2).$$

Show that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{c \Delta \tau} = \frac{\Delta u_o}{c \Delta \tau} = \frac{\Delta u_e}{c \Delta \tau} = V^u/c$$

where, for example, $\Delta t_o$ is the proper time interval during which the observer receives one cycle of the signal and $\Delta \tau$ is the time for the spacecraft to emit one cycle.

Show next that $V^u = -K$, where $K$ is a constant and that

$$V^u = \frac{K + \sqrt{K^2 - F}}{F}, \quad V^r = -\sqrt{K^2 - F}.$$

Deduce that on the world line of the spacecraft near the horizon $du/dr \sim -2/F$, and that $u \sim -2r^*$ and $F \sim e^{-u/(2r_s)}$.

Conclude that, just as the transmitter is about to cross the event horizon, the observer sees the frequency red-shifted with an observer-time dependence $\propto \exp(-ct/(2r_s))$.

Show that, for an observer with proper time $\tau$ moving in the Schwarzschild space-time,

$$c^2 = F \, c^2 \dot{t}^2 - \dot{r}^2/F - r^2(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2),$$

where $\dot{t} = dt/d\tau$ etc., and $F = 1 - r_s/r$. Show that for an observer within the Schwarzschild horizon, $\dot{r}^2 \geq -F$ however the observer moves. Deduce that any observer crossing the Schwarzschild horizon will reach $r = 0$ within a proper time $\frac{1}{2} \pi r_s/c$.

Let $M$ be the torus ($S^1 \times S^1$) and define the metric $g_{ab}$ on $M$ by

$$ds^2 = \sin \theta (d\phi^2 - d\theta^2) + 2 \cos \theta d\theta d\phi,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. Show that, for a null geodesic,

$$\dot{\phi}^2 + 2\dot{\theta} \cot \theta - \dot{\theta}^2 = 0,$$

where dot is differentiation with respect to an affine parameter, and deduce that the two curves given by $\phi = -2 \ln \sin(\theta/2)$ and $\phi = -2 \ln \cos(\theta/2)$ are null geodesics. Use another first integral of Lagrange’s equations to show that in both cases $\theta = p\lambda$, where $\lambda$ an affine parameter and $p$ is a constant.

Show that one family of geodesics wraps round the torus an infinite number of times within a finite affine parameter, never reaching the null curve $\theta = 2\pi$, and that the other family of null geodesics crosses this curve.

Is this space geodesically complete? Is the Riemann tensor well-behaved (no calculation required)?
9 In the relativistic formulation of electromagnetism on Minkowski space-time, show that the field tensor $F_{ab} = 2A_b, a$ is left invariant under gauge transformations of the form $A_a \to A_a + f_a$. Show that we may choose an $f$ such that Maxwell’s equations $F^{ab}, _b = 0$ become $\Box A_a = 0$ where $\Box = \eta^{ab}\nabla_a \nabla_b$.

Given that $A_a = \epsilon_a e^{iks} x^k$ (a plane wave) in the above gauge, show that $k_\alpha \epsilon^\alpha = 0$, that $k^\alpha$ is null and that $\epsilon^\alpha$ is arbitrary. What effect has the remaining gauge freedom on $\epsilon^\alpha$? How many degrees of freedom are there for a wave propagating in a given direction? (i.e. for given $k^\alpha$, how many components $\epsilon^\alpha$ are left when the constraint is taken into account and the gauge freedom is used to reduce some components to zero?)

10 A weak gravitational field has the spacetime metric $g_{ab} = \eta_{ab} + \epsilon_{h_{ab}} + O(\epsilon^2)$, where $\eta_{ab}$ is the Minkowski metric and $\epsilon$ small. Show that

$$R_{abcd} = \frac{1}{2} \epsilon[\eta_{ad,bc} + \eta_{bc,ad} - \eta_{ac,bd} - \eta_{bd,ac}] + O(\epsilon^2).$$

Let $h = h^c_c$ and define $\overline{h}_{ab} = h_{ab} - \frac{1}{2} h \eta_{ab}$. Show that $h_{ab} = \overline{h}_{ab} - \frac{1}{2} \Box h$. Show also that

$$R_{ab} = \frac{1}{2} \epsilon[-\Box \overline{h}_{ab} + \overline{h}_{a,c} \overline{h}_{b,c} + \overline{h}_{b,c} \overline{h}_{a,c} + \frac{1}{2} \eta_{ac} \Box \overline{h}] + O(\epsilon^2).$$

where $\Box = \eta^{ab}\nabla_a \nabla_b$. What are the linear vacuum equations for $h_{ab}$?

An infinitesimal coordinate transformation (which may be called a gauge transformation) is given by $x^a \to x^a + \epsilon f^a(x)$. Show that $h_{ab} \to h_{ab} - f_{a,b} - f_{b,a} + O(\epsilon)$, but that the curvature tensors are unchanged (to leading order in $\epsilon$). Deduce that if $f^a$ is chosen to satisfy $\Box f^a = \overline{h}^{ab}, _b$, then in the new coordinates $\overline{h}_{ab,b} = 0$. Conclude that the linearized Einstein equation for weak fields in vacuum is the wave equation

$$\Box h_{ab} = 0.$$