1 Revision of main ideas and results from IB Fluids

1.1 Continuum hypothesis

We assume that at every point \( x \) of the fluid and at all times \( t \) we can define, by averaging over a small volume, “continuum” properties like density \( \rho(x,t) \), velocity \( u(x,t) \) and pressure \( p(x,t) \). Here \( x \) refers to a position in the laboratory frame (Eulerian description). We thus do not deal with the dynamics of individual molecules.

1.2 Time derivatives

A fluid particle, sometimes called a material element or a Lagrangian point, is one that moves with the fluid, so that its position \( x(t) \) satisfies

\[
\dot{x} = u(x,t).
\]

The rate of change of a quantity as seen by a fluid particle is written \( D/Dt \), given by the chain rule as

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla.
\]

This is called the material (or convected or substantial) derivative. In particular, the acceleration of a fluid particle is

\[
Du/Dt = \frac{\partial u}{\partial t} + u \cdot \nabla u.
\]

1.3 Mass conservation

Because matter is neither created nor destroyed, the mass density \( \rho \) satisfies

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad \text{or equivalently} \quad D\rho/Dt + \rho \nabla \cdot u = 0.
\]

The quantity \( \rho u \) is called the mass flux. For an incompressible fluid, the density of each material element is constant, and so \( D\rho/Dt = 0 \). Hence

\[
\nabla \cdot u = 0.
\]

In this course, we shall also restrict attention to fluids that are incompressible and have uniform density, so that \( \rho \) is independent of both \( x \) and \( t \).

For planar flows, the condition \( \nabla \cdot u = 0 \) is automatically satisfied by \( u = (\psi_y, -\psi_x, 0) \), with streamfunction \( \psi(x,y) \).

1.4 Kinematic boundary condition

Applying mass conservation to a region close to a boundary \( S \), we have

\[
\rho u^- \cdot n = \rho u^+ \cdot n,
\]

i.e. the normal component of velocity must be continuous across \( S \). In particular at a fixed boundary, \( u \cdot n = 0 \). Equivalently, if the moving boundary of a fluid is given by the equation \( F(x,t) = 0 \), then since the surface consists of material points, \( DF/Dt = 0 \). This form of the boundary condition is sometimes more convenient for free-surface problems.
1.5 Momentum conservation

On the assumption that the only surface force that acts across a material surface $ndS$ is given by a pressure $p(x, t)$ as $-p ndS$, then Newton’s equation of motion is

$$\rho \frac{Du}{Dt} = -\nabla p + F(x, t),$$

where $F(x, t)$ is the force per unit volume (called force density, e.g. gravity $\rho g$) that acts on the fluid. This is Euler’s equation.

1.6 Dynamic boundary condition

On the same assumption, apply momentum conservation to a region close to the boundary $S$ gives (in the absence of surface tension)

$$-p n = -p^* n,$$

and thus the pressure must be continuous across $S$.

In this course, we abandon the inviscid assumption of §1.5 & §1.6, and will include tangential frictional forces (stresses) across material surfaces to derive new equations for momentum conservation with new boundary conditions.

1.7 Example: Potential flow past a circular cylinder

The steady Euler equation with $F = 0$ is satisfied by a potential flow $u = \nabla \phi$ with $\nabla \cdot u = \nabla^2 \phi = 0$, and pressure from Bernoulli $p + \frac{1}{2} \rho u^2 = \text{const}$.

The solution with $\phi \sim U x = U r \cos \theta$ as $r \to \infty$ (i.e. uniform stream of velocity $U$) and $u \cdot n = \partial \phi / \partial r = 0$ on $r = R$ is

$$\phi = U(r + R^2 / r) \cos \theta,$$

with associated streamfunction $\psi = U(r - R^2 / r) \sin \theta$.

1.8 Other important ideas and results

- Introduction to viscosity and shear stresses
- Simple unidirectional flows (e.g. Couette)
- Introduction to vorticity
- Potential flows
- Bernoulli’s equation [Integral form of Euler]
- Water waves
- Fluid dynamics in a rotating frame
- Shallow-water equations