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1 Revision of main ideas and results from IB Fluids

1.1 Continuum hypothesis

We assume that at every point $\mathbf{x}$ of the fluid and at all times t we can define, by averaging over a small volume, "continuum" properties like density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$. Here $\mathbf{x}$ refers to a position in the laboratory frame (Eulerian description). We thus do not deal with the dynamics of individual molecules.

1.2 Time derivatives

A *fluid particle*, sometimes called a *material element* or a *Lagrangian point*, is one that moves with the fluid, so that its position $\mathbf{x}(t)$ satisfies

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t).$$

The rate of change of a quantity as seen by a fluid particle is written D/Dt, given by the chain rule as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla.$$

This is called the *material* (or *convected* or *substantial*) derivative. In particular, the *acceleration* of a fluid particle is

$$D\mathbf{u}/Dt = \partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}.$$

1.3 Mass conservation

Because matter is neither created nor destroyed, the mass density ρ satisfies

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \text{or equivalently} \quad D\rho/Dt + \rho\nabla\cdot\mathbf{u} = 0.$$

The quantity $\rho\mathbf{u}$ is called the *mass flux*. For an *incompressible* fluid, the density of each material element is constant, and so $D\rho/Dt = 0$. Hence

$$\nabla\cdot\mathbf{u} = 0.$$

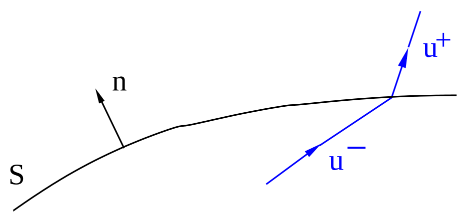
In this course, we shall also restrict attention to fluids that are incompressible and have uniform density, so that ρ is independent of both $\mathbf{x}$ and t.

For planar flows, the condition $\nabla\cdot\mathbf{u} = 0$ is automatically satisfied by $\mathbf{u} = (\psi_y, -\psi_x, 0)$, with *streamfunction* $\psi(x, y)$.

1.4 Kinematic boundary condition

Applying mass conservation to a region close to a boundary S, we have

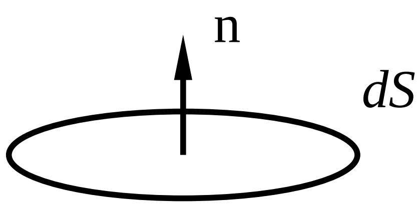
$$\rho\mathbf{u}^-\cdot\mathbf{n} = \rho\mathbf{u}^+\cdot\mathbf{n},$$



i.e. the normal component of velocity must be continuous across S. In particular at a fixed boundary, $\mathbf{u}\cdot\mathbf{n} = 0$. Equivalently, if the moving boundary of a fluid is given by the equation $F(\mathbf{x}, t) = 0$, then since the surface consists of material points, $DF/Dt = 0$. This form of the boundary condition is sometimes more convenient for free-surface problems.

1.5 Momentum conservation

On the *assumption* that the only surface force that acts across a material surface $\mathbf{n}dS$ is given by a pressure $p(\mathbf{x},t)$ as $-p\mathbf{n}dS$, then Newton's equation of motion is



$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}(\mathbf{x},t),$$

where $\mathbf{F}(\mathbf{x},t)$ is the force per unit volume (called 'force density' or 'body force', e.g. gravity $\rho\mathbf{g}$) that acts on the fluid. This is Euler's equation.

1.6 Dynamic boundary condition

On the same assumption, momentum conservation applied to a region close to the boundary S gives (in the absence of surface tension)

$$-p^{-}\mathbf{n} = -p^{+}\mathbf{n},$$

and thus the pressure must be continuous across S.

In this course, we abandon the *inviscid* assumption of §1.5 & §1.6, and will include tangential frictional forces (stresses) across material surfaces to derive new equations for momentum conservation with new boundary conditions. We will find that there is a viscous contribution to the normal force on a surface.

1.7 Example: Potential flow past a circular cylinder

The steady Euler equation with $\mathbf{F} = \mathbf{0}$ is satisfied by a potential flow $\mathbf{u} = \nabla\phi$ with $\nabla\cdot\mathbf{u} = \nabla^2\phi = 0$, and pressure from Bernoulli $p + \frac{1}{2}\rho u^2 = \text{const}$.

The solution with $\phi \sim Ux = Ur\cos\theta$ as $r \to \infty$ (i.e. uniform stream of velocity U) and $\mathbf{u}\cdot\mathbf{n} = \partial\phi/\partial r = 0$ on $r = R$ is

$$\phi = U(r + R^2/r)\cos\theta,$$

with associated streamfunction $\psi = U(r - R^2/r)\sin\theta$.

1.8 Other important ideas and results

- Introduction to viscosity and shear stresses
- Simple unidirectional flows (e.g. Couette)
- Introduction to vorticity
- Potential flows
- Bernoulli's equation
- Momentum integral [Integral form of Euler]
- Water waves
- Fluid dynamics in a rotating frame
- Shallow-water equations