1 Revision of ideas and results from IB Fluids

1.1 Continuum hypothesis

We assume that at every point $\mathbf{x}$ of the fluid and at all times $t$ we can define, by averaging over a small volume, properties like density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$. Here $\mathbf{x}$ refers to a position in the laboratory frame (Eulerian description). We thus do not deal with the dynamics of individual molecules.

1.2 Time derivatives

A fluid particle, sometimes called a material element or a Lagrangian point, is one that moves with the fluid, so that its position $\mathbf{x}(t)$ satisfies

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t).$$

The rate of change of a quantity as seen by a fluid particle is written $D/Dt$, given by the chain rule as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

In particular, the acceleration of a fluid particle is

$$Du/Dt = \partial\mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u}.$$

1.3 Mass conservation

Because matter is neither created nor destroyed, the mass density $\rho$ satisfies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \text{or equivalently} \quad D\rho/Dt + \rho \nabla \cdot \mathbf{u} = 0.$$

The quantity $\rho \mathbf{u}$ is called the mass flux. For an incompressible fluid, the density of each material element is constant, and so $D\rho/Dt = 0$. Hence

$$\nabla \cdot \mathbf{u} = 0.$$

In this course, we shall also restrict attention to fluids that are incompressible and have uniform density, so that $\rho$ is independent of both $\mathbf{x}$ and $t$.

For planar flows, the condition $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied by $\mathbf{u} = (\psi_y, -\psi_x, 0)$, with streamfunction $\psi(x, y)$.

1.4 Kinematic boundary condition

Applying mass conservation to a region close to a boundary $S$, we have

$$\rho\mathbf{u}^- \cdot \mathbf{n} = \rho\mathbf{u}^+ \cdot \mathbf{n},$$

i.e. the normal component of velocity must be continuous across $S$. In particular at a fixed boundary, $\mathbf{u} \cdot \mathbf{n} = 0$. Equivalently, if the moving boundary of a fluid is given by the equation $F(\mathbf{x}, t) = 0$, then since the surface consists of material points, $DF/Dt = 0$. This form of the boundary condition is sometimes more convenient for free-surface problems.
1.5 Momentum conservation
On the assumption that the only surface force that acts across a material surface $\mathbf{n}dS$ is given by a pressure $p(x, t)$ as $-p\mathbf{n}dS$, then Newton’s equation of motion is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}(x, t),$$

where $\mathbf{F}(x, t)$ is the force per unit volume (called force density, e.g. gravity $\rho g$) that acts on the fluid. This is Euler’s equation.

1.6 Dynamic boundary condition
On the same assumption, apply momentum conservation to a region close to the boundary $S$ gives (in the absence of surface tension)

$$-p\mathbf{n} = -p^+\mathbf{n},$$

and thus the pressure must be continuous across $S$. In this course, we abandon the inviscid assumption of §1.5 & §1.6, and will include tangential frictional forces across material surfaces.

1.7 An example: steady flow past a circular cylinder
The steady Euler equation with $\mathbf{F} = 0$ is satisfied by a potential flow $\mathbf{u} = \nabla \phi$ with $\nabla \cdot \mathbf{u} = \nabla^2 \phi = 0$, and pressure from Bernoulli $p + \frac{1}{2} \rho u^2 = \text{const}$.

The solution with $\phi \sim Ux = Ur \cos \theta$ as $r \to \infty$ (i.e. uniform stream of velocity $U$) and $\mathbf{u} \cdot \mathbf{n} = \partial \phi / \partial r = 0$ on $r = R$ is

$$\phi = U(r + R^2 / r) \cos \theta,$$

with associated streamfunction $\psi = U(r - R^2 / r) \sin \theta$, and tangential velocity $2U \sin \theta$ on $r = R$. In the plot below (credit: Incredio) the iso-$\phi$ are shown in white with the streamfunction (iso-$\psi$) in black.