

## 1 Revision of main ideas and results from IB Fluids

### 1.1 Continuum hypothesis

We assume that at every point  $\mathbf{x}$  of the fluid and at all times  $t$  we can define, by averaging over a small volume, “continuum” properties like density  $\rho(\mathbf{x}, t)$ , velocity  $\mathbf{u}(\mathbf{x}, t)$  and pressure  $p(\mathbf{x}, t)$ . Here  $\mathbf{x}$  refers to a position in the laboratory frame (Eulerian description). We thus do not deal with the dynamics of individual molecules.

### 1.2 Time derivatives

A *fluid particle*, sometimes called a *material element* or a *Lagrangian point*, is one that moves with the fluid, so that its position  $\mathbf{x}(t)$  satisfies

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t).$$

The rate of change of a quantity as seen by a fluid particle is written  $D/Dt$ , given by the chain rule as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

This is called the *material* (or *convected* or *substantial*) derivative. In particular, the *acceleration* of a fluid particle is

$$D\mathbf{u}/Dt = \partial\mathbf{u}/\partial t + \mathbf{u} \cdot \nabla\mathbf{u}.$$

### 1.3 Mass conservation

Because matter is neither created nor destroyed, the mass density  $\rho$  satisfies

$$\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{u}) = 0, \quad \text{or equivalently} \quad D\rho/Dt + \rho\nabla \cdot \mathbf{u} = 0.$$

The quantity  $\rho\mathbf{u}$  is called the *mass flux*. For an *incompressible* fluid, the density of each material element is constant, and so  $D\rho/Dt = 0$ . Hence

$$\nabla \cdot \mathbf{u} = 0.$$

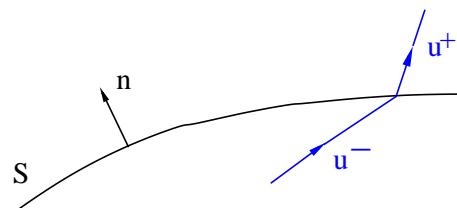
In this course, we shall also restrict attention to fluids that are incompressible and have uniform density, so that  $\rho$  is independent of both  $\mathbf{x}$  and  $t$ .

For planar flows, the condition  $\nabla \cdot \mathbf{u} = 0$  is automatically satisfied by  $\mathbf{u} = (\psi_y, -\psi_x, 0)$ , with *streamfunction*  $\psi(x, y)$ .

### 1.4 Kinematic boundary condition

Applying mass conservation to a region close to a boundary  $S$ , we have

$$\rho\mathbf{u}^- \cdot \mathbf{n} = \rho\mathbf{u}^+ \cdot \mathbf{n},$$



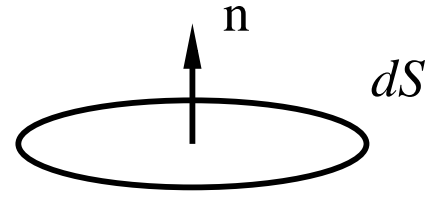
i.e. the normal component of velocity must be continuous across  $S$ . In particular at a fixed boundary,  $\mathbf{u} \cdot \mathbf{n} = 0$ . Equivalently, if the moving boundary of a fluid is given by the equation  $F(\mathbf{x}, t) = 0$ , then since the surface consists of material points,  $DF/Dt = 0$ . This form of the boundary condition is sometimes more convenient for free-surface problems.

## 1.5 Momentum conservation

On the *assumption* that the only surface force that acts across a material surface  $\mathbf{n}dS$  is given by a pressure  $p(\mathbf{x}, t)$  as  $-p\mathbf{n}dS$ , then Newton's equation of motion is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F}(\mathbf{x}, t),$$

where  $\mathbf{F}(\mathbf{x}, t)$  is the force per unit volume (called force density, e.g. gravity  $\rho\mathbf{g}$ ) that acts on the fluid. This is Euler's equation.



## 1.6 Dynamic boundary condition

On the same assumption, apply momentum conservation to a region close to the boundary  $S$  gives (in the absence of surface tension)

$$-p^- \mathbf{n} = -p^+ \mathbf{n},$$

and thus the pressure must be continuous across  $S$ .

In this course, we abandon the *inviscid* assumption of §1.5 & §1.6, and will include tangential frictional forces (stresses) across material surfaces to derive new equations for momentum conservation with new boundary conditions.

## 1.7 Example: Potential flow past a circular cylinder

The steady Euler equation with  $\mathbf{F} = \mathbf{0}$  is satisfied by a potential flow  $\mathbf{u} = \nabla\phi$  with  $\nabla \cdot \mathbf{u} = \nabla^2\phi = 0$ , and pressure from Bernoulli  $p + \frac{1}{2}\rho u^2 = \text{const.}$

The solution with  $\phi \sim Ux = Ur \cos\theta$  as  $r \rightarrow \infty$  (i.e. uniform stream of velocity  $U$ ) and  $\mathbf{u} \cdot \mathbf{n} = \partial\phi/\partial r = 0$  on  $r = R$  is

$$\phi = U(r + R^2/r) \cos\theta,$$

with associated streamfunction  $\psi = U(r - R^2/r) \sin\theta$ .

## 1.8 Other important ideas and results

- Introduction to viscosity and shear stresses
- Simple unidirectional flows (e.g. Couette)
- Introduction to vorticity
- Potential flows
- Bernoulli's equation [Integral form of Euler]
- Water waves
- Fluid dynamics in a rotating frame
- Shallow-water equations