Fluid Dynamics II

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1 Introduction

1.1 Books, scope of the course

1.2 Revision of IB (handout)

These have already been handed out.

1.3 Stress and Rate of Strain

Rate of Strain tensor. Consider the velocity field $\mathbf{u}(\mathbf{x})$ close to a fixed point, (w.l.o.g.) the origin,

$$u_i(\mathbf{x}) - u_i(\mathbf{0}) = x_j \frac{\partial u_i}{\partial x_j}\bigg|_0 \quad + \text{higher order terms}$$

The velocity gradient tensor $\partial u_i / \partial x_j$ can be divided into a symmetric part, e_{ij}, the rate of strain tensor, and an antisymmetric part, Ω_{ij}, the vorticity tensor, Then

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\ &= \qquad e_{ij} \qquad + \qquad \Omega_{ij} \end{aligned}$$

with Ω_{ij} in 3-D being expressible in terms of three independent components

$$\Omega_{ij} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} = \epsilon_{ikj}\Omega_k \ .$$

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Note that $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = 2\boldsymbol{\Omega}$, so that $\boldsymbol{\Omega}$ is half the vorticity $\boldsymbol{\omega}$.
Thus

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{0}) + e_{ij}x_j + \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{x})_i + \dots \ ,$$

where $\frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{x})$ represents a solid body rotation.
Since e_{ij} is a symmetric, second order tensor, it has real, orthogonal eigenvectors. With respect to them as axes

$$\boldsymbol{e} = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

where the e_i are called the principal rates of strain. If the fluid is incompressible $e_{ii} = 0$.

Stress Tensor. The forces acting on a moving fluid can be divided into:

i) long-range forces, e.g. gravity and electromagnetic forces which change slowly with position. Thus the force on each part of a small volume is the same and the total force is proportional to δV. These are known as volume or body forces; and

ii) short-range forces, which penetrate only a few atomic distances into a volume. In a gas a molecule transports momentum through its mean free path before depositing this momentum in a collision. In a liquid the short-range forces are due to momentum transport and intermolecular Van der Waals forces as the molecules jostle and slide past each other. Whatever the mechanism, the force $\boldsymbol{F}$, is proportional to the surface area dA and a function of the direction of the orientation of dA (and of the fluid motion). By Newton's third law $\boldsymbol{F}(-\boldsymbol{n}) = -\boldsymbol{F}(\boldsymbol{n})$.
Consider the following tetrahedron:

On the oblique face, the force is $\boldsymbol{\tau}(\boldsymbol{n})\delta A$. The other faces have force $\boldsymbol{\tau}(\boldsymbol{n}^{(j)})\delta A^{(j)}$, for $j = 1, 2, 3$, where $n_i^{(j)} = -\delta_{ij}$.
As the volume $\delta V \to 0$, the forces must balance. All the δA's $\sim \delta V^{2/3}$, so the surface

forces are larger than the effects of any body force (which is $\propto \delta V$). So we must have

$$\boldsymbol{\tau}(\boldsymbol{n})\delta A + \sum_{j=1}^{3} \boldsymbol{\tau}(\boldsymbol{n}^{(j)})\delta A^{(j)} = 0,$$

but $\delta A^{(j)} = n_j \delta A$, so δA cancels and

$$\boldsymbol{\tau}(\boldsymbol{n}) = \boldsymbol{\tau}((1,0,0))n_1 + \boldsymbol{\tau}((0,1,0))n_2 + \boldsymbol{\tau}((0,0,1))n_3.$$

ie $\boldsymbol{\tau}$ is a linear function of the n_j's. We can write

$$\tau_j(\boldsymbol{n}) = \sigma_{ij} n_j,$$

where σ_{ij} is a *tensor* (by quotient theorem) and independent of $\boldsymbol{n}$. ie a property of the fluid.

Consider the total torque on a small volume δV

$$\begin{aligned} G_i &= \varepsilon_{ijk} \int \tau_j x_k \mathrm{d}S \\ &= \varepsilon_{ijk} \int \sigma_{jl} n_l x_k \mathrm{d}S \\ &= \varepsilon_{ijk} \int \frac{\partial}{\partial x_l}(\sigma_{jl} x_k)\mathrm{d}V \\ &= \varepsilon_{ijk} \int \sigma_{jk} \mathrm{d}V + \varepsilon_{ijk} \int x_k \frac{\partial \sigma_{jl}}{\partial x_k}\mathrm{d}V \\ &\sim \varepsilon_{ijk}\sigma_{jk}\delta V + \text{Torque due to body forces}(\sim \delta V^{4/3}). \end{aligned}$$

So for equilibrium we must have

$$\varepsilon_{ijk}\sigma_{jk} = 0, \qquad \sigma_{ij} = \sigma_{ji} \;\; \text{symmetric.}$$

We can separate σ_{ij}:

$$\sigma_{ij} = c\delta_{ij} + d_{ij},$$

where $3c = \sigma_{ii}$ and $d_{ii} = 0$. d_{ij} is called the *deviatoric stress*; $c\delta_{ij}$ is *isotropic*. If a fluid is at rest, then all directions are the same so we expect σ_{ij} to be isotropic. So, $d_{ij} = 0$ when $\boldsymbol{u} = 0$.

In this case the total force on the surface S is

$$\int_S \sigma_{ij} n_j \mathrm{d}S = \int c n_i \mathrm{d}S,$$

Comparing with the IB result $-\int_S p n_i \mathrm{d}S$, where p is the pressure, wesee that $c = -p$, $p = -\frac{1}{3}\sigma_{kk}$. Then

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}. \tag{1}$$

1.4 Equation of Motion

From IB we have the equation (Newton's Law) for a small volume element δV:

$$\rho \delta V \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \rho \delta V \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \boldsymbol{F}_{\mathrm{TOT}}, \tag{2}$$

where $\boldsymbol{F}_{\mathrm{TOT}}$ is the sum of the body forces and internal forces.
Consider a small fluid element δV. Then the force due to the body forces is $\boldsymbol{f}\delta V$, where $\boldsymbol{f}$ is the body force per unit volume. The (ith component of the) force due to tractions on the surface δS is

$$\begin{aligned} \int_{\delta S} \tau_i(\boldsymbol{n}) \mathrm{d}S &= \int_{\delta S} \sigma_{ij} n_j \mathrm{d}S \\ &= \int_{\delta V} \frac{\partial \sigma_{ij}}{\partial x_j} \mathrm{d}V \\ &\approx \left. \frac{\partial \sigma_{ij}}{\partial x_j} \right|_{\boldsymbol{x}} \delta V. \end{aligned}$$

So we have from (2), using $(\nabla \cdot \boldsymbol{\sigma})_i = \frac{\partial \sigma_{ij}}{\partial x_j}$,

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}.$$

Now differentiating equation (1) with respect to x_j gives

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial d_{ij}}{\partial x_j},$$

which in vector form becomes

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \nabla \cdot \boldsymbol{d}.$$

1.5 The relation between stress and rate-of-strain

In a real fluid, experimentally, it is found that d_{ij} depends on $\frac{\partial u_i}{\partial x_j}$ (the rate of deformaiton of the fluid element), and not on u_i (galilean invariance - adding velocity does not change deformation). For simple fluids (water, oil, etc) it is found that

1. d_{ij} is (approximately) a linear function of $\frac{\partial u_i}{\partial x_j}$;
2. d_{ij} does not depend on absolute displacements of fluid elements (no elastic effects);
3. no memory or long distance effects $\Big(d_{ij}(\boldsymbol{x}, t) = f_{ij}(\frac{\partial u_i}{\partial x_j}(\boldsymbol{x}, t))\Big)$;
4. isotropic - relation the same in all frames.

On these assumptions (these are called *Newtonian Fluids*)

$$d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l},$$

where A_{ijkl} is an isotropic tensor of order 4. A general isotropic tensor of order 4 is

$$A_{ijkl} = \nu\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk},$$

but $A_{ijkl} = A_{jikl}$ (as d_{ij} is symmetric), so $\mu = \mu'$. Therefore,

$$\begin{aligned} A_{ijkl}\frac{\partial u_k}{\partial x_l} &= (\nu\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{jk}\delta_{il})\frac{\partial u_k}{\partial x_l} \\ &= \nu\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_j}{\partial x_i} \\ &= \nu\delta_{ij}(\nabla \cdot \boldsymbol{u}) + 2\mu e_{ij}. \end{aligned}$$

And $d_{ii} = 0$ implies $3\nu\nabla \cdot \boldsymbol{u} + 2\mu e_{kk} = \nabla \cdot \boldsymbol{u}(3\nu + 2\mu) = 0$, so $\nu = -2\mu/3$. Therefore,

$$d_{ij} = 2\mu e_{ij} - \frac{2\mu}{3}e_{kk}\delta_{ij}.$$

Note that d_{ij} only depends on e_{ij} (which is reasonable as ω_{ij} does not lead to deformation locally).
Finally, assuming $\nabla \cdot \boldsymbol{u} = 0$, we obtain

$$d_{ij} = 2\mu e_{ij},$$

where μ is a scalar (and can be taken as constant for well mixed fluid, though many fluids have viscosity depending on temperature e.g. Golden Syrup). Then

$$\begin{aligned} \frac{\partial d_{ij}}{\partial x_j} &= \mu \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \\ &= \mu \nabla^2 u_i + \mu \nabla (\nabla \cdot \boldsymbol{u}). \end{aligned}$$

So

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{f} + \mu \nabla^2 \boldsymbol{u}.$$

This is the famous *Navier-Stokes equation*. The quantity μ is called the *dynamic viscosity*. We usually work with the *kinematic viscosity* $\nu = \mu/\rho$. ν has dimensions $[\mathrm{L}]^2[\mathrm{T}]^{-1}$. For water, $\nu \approx 1.1 \times 10^{-6} \mathrm{m}^2\mathrm{s}^{-1}$ (or one acre per century). For air $\nu \approx 1.5 \times 10^{-5} \mathrm{m}^2\mathrm{s}^{-1}$.

When can $\nabla \cdot \boldsymbol{u} = 0$ be justified? From inviscid, comressible flow we have compressive waves at the sound speed c, where

$$c^2 = \frac{\partial p}{\partial \rho},$$

so $\Delta \rho \sim c^2 \Delta p$. By Bernoulli, $\Delta p \sim \rho_o u^2$, so $\Delta \rho = \rho_o u^2/c^2$. Define the *Mach Number* $M = u/c$, such that $M^2 = \Delta \rho / \rho_o$. So a fluid can be considered incompressible at low speeds compared with c.

1.6 Boundary Conditions at an Interface

It is easily shown that at a interface between two fluids we have, using small cylinder argument,

$$\boldsymbol{u}_2 \cdot \boldsymbol{n} = \boldsymbol{u}_1 \cdot \boldsymbol{n}.$$

More generally, for a moving boundary, given by $F(\boldsymbol{x}, t) = 0$ we know that a particle on the surface stays on that surface. So, on each side we have ($\nabla F \parallel \boldsymbol{n}$)

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0 = \frac{\partial F}{\partial t} + \boldsymbol{u} \cdot \nabla F.$$

In the presence of viscosity, we may need more boundary conditions. At a boundary between two viscous fluids, we need to avoid ∞ stresses. This means that velocity (all components) is continuous at rigid boundary moving at velocity $\boldsymbol{V}$. So, $\boldsymbol{u} = \boldsymbol{V}$ at the boundary.
At a non-rigid boundary (eg air-water interface) traction on a small pillbox must be in balance

$$\boldsymbol{\tau}_2 = \boldsymbol{\sigma}_2 \cdot \boldsymbol{n},$$
$$\boldsymbol{\tau}_1 = -\boldsymbol{\sigma}_1 \cdot \boldsymbol{n},$$

and in the limit $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = 0$ so $\boldsymbol{\sigma}_1 \cdot \boldsymbol{n} = \boldsymbol{\sigma}_2 \cdot \boldsymbol{n}$, or $\sigma_{ij}^{(2)} n_j - \sigma_{ij}^{(1)} n_j \equiv [\sigma_{ij} n_j] = 0$.
In terms of the rate of strain for an incomressible fluid, we get

$$[-p\delta_{ij} + 2\mu e_{ij}] n_j = 0,$$

or

$$[-p n_i + 2\mu e_{ij} n_j] = 0.$$

eg at such a boundary in (x, y) plane the normal stress is

$$[-p + 2\mu e_{zz}] = [-p + 2\mu \frac{\partial u_z}{\partial z}] = 0,$$

and the tangential stresses are

$$[2\mu e_{xz}] = \left[\mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x})\right] = 0, \qquad [2\mu e_{yz}] = 0.$$

If boundaries are curved non-cartesian coordinates are appropriate, then we need an expression for e_{ij} in polars.
If the boundary is rigid, stress conditions are not needed.
If there is surface tension at an interface, then there is an extra normal surface tension force

$$[\sigma_{ij} n_j] = \lambda n_i \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \equiv T_i.$$

where $R_{1,2}$ are the principal radii of curvature.

1.7 Conservation Laws and dissipation of energy

The Navier-Stokes (N-S) equations can be written in conservation form. Already we have conservation of mass

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t} \mathrm{d}V = -\int_S \rho(\boldsymbol{u} \cdot \boldsymbol{n}) \mathrm{d}S = -\int_V \nabla \cdot (\rho \boldsymbol{n}) \mathrm{d}V$$

as no holes appear. So,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\underbrace{\rho \boldsymbol{u}}_{\text{mass flux}}) = 0$$

This is a typical *conservation form* $\dfrac{\partial f}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$, where $\boldsymbol{F}$ is the flux of f.
We can also write N-S in conservation form - conservation of momentum (assume $\rho =$ const and $\nabla \cdot \boldsymbol{u} = 0$)

$$\rho \left[\frac{\partial u_i}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_i \right] = F_i + \frac{\partial \sigma_{ij}}{\partial x_j},$$

or

$$\rho \left[\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}(u_i u_j) \right] = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}.$$

So,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\underbrace{\rho u_i u_j - \sigma_{ij}}_{\text{flux of momentum}}) = \underbrace{F_i}_{\text{body forces}}.$$

So if there are no body forces, then the equation is in conservation form.
In a fixed volume, V, with no body forces

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho u_i \, \mathrm{d}V + \int_S \rho u_i (\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}S = \int_S \sigma_{ij} n_j \, \mathrm{d}S.$$

So a change in momentum, apart from flux through the boundary, occurs only due to surface stresses, as expected.
N.B Energy is *not* conserved, even without body forces

$$E = \frac{KE}{\text{unit vol}} = \frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u}.$$

So,

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \rho \boldsymbol{u} \cdot \dot{\boldsymbol{u}} \\
&= u_i \left[-\rho (\boldsymbol{u} \cdot \nabla) u_i + \frac{\partial \sigma_{ij}}{\partial x_i} \right] \\
&= -\boldsymbol{u} \cdot \nabla E + \frac{\partial}{\partial x_i}(u_i \sigma_{ij}) - \sigma_{ij} \frac{\partial u_i}{\partial x_i}.
\end{aligned}$$

Thus

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(u_j E - u_i \sigma_{ij}) = -\sigma_{ij}\frac{\partial u_i}{\partial x_j}$$
$$= -\sigma_{ij}e_{ij} \quad \text{(as } \sigma_{ij} \text{ is symmetric)}$$
$$= pe_{ii} - \underbrace{2\mu e_{ij}e_{ij}}_{\text{viscous dissipation}},$$

and $pe_{ii} = 0$ as $\nabla \cdot \boldsymbol{u} = 0$. So for a fixed volume V

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}V + \int_S (\boldsymbol{u}\cdot\boldsymbol{n})E\mathrm{d}S = \int_S (\boldsymbol{u}\cdot\boldsymbol{\tau})\,\mathrm{d}S - 2\mu\int_V e_{ij}e_{ij}\,\mathrm{d}V + \int_V \boldsymbol{u}\cdot\boldsymbol{F}\,\mathrm{d}V,$$

where the first term on the RHS is the work dine by surface tractions, and the last term is the work done by body forces.

There is an alternative form for energy evolution, though it is not as clearly related to the various work terms. N-S equations can be written

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\boldsymbol{u}.$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}V + \int_S ((\boldsymbol{u}\cdot\boldsymbol{n})E + p(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}S = \mu\int_V u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j}\mathrm{d}V.$$

Now,

$$\int u_i \frac{\partial^2 u_i}{\partial x_j\partial x_j}\mathrm{d}V = -\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\mathrm{d}V + \int \boldsymbol{u}\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{u})\mathrm{d}S.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}V + \int_S (E(\boldsymbol{u}\cdot\boldsymbol{n}) + p(\boldsymbol{u}\cdot\boldsymbol{n}) - \mu(\boldsymbol{u}\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{u})))\,\mathrm{d}S = -\mu\int_V |\nabla\boldsymbol{u}|^2\,\mathrm{d}V.$$

1.8 Reynolds number and dynamical similarity

Behaviour of physical systems depends not on size and speed alone - these are measured in arbitrary units - but only on dimensionless quantities. We have already met the Mach Number $M = u/c$, where u is a typical velocity.

The importance of the viscosity is measured by the *Reynolds Number*. Suppose that a system has a typical size L and that a typical velocity is U. Then consider

$$\frac{\text{inertial forces}}{\text{viscous forces}} \sim \frac{\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u}}{\mu \frac{\partial e_{ij}}{\partial x_i}} \sim \frac{\rho U^2/L}{\mu U/L^2}.$$

The ratio is

$$\frac{UL\rho}{\mu} = \frac{UL}{\nu} = Re - \text{the Reynolds Number.}$$

For large Reynolds numbers the inertial forces dominate; for small Reynolds numbers the viscous forces dominate.

More carefully, we can non-dimensionalize the system. Write

$$\boldsymbol{x} = L\hat{\boldsymbol{x}}, \quad \boldsymbol{u} = U\hat{\boldsymbol{u}}, \quad p = \rho U^2 \hat{p}, \quad t = \frac{L}{U}\hat{t}.$$

(eg if $L = 1$m then length is measured in meters.) Then

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\rho U^2}{L}\frac{\partial \hat{\boldsymbol{u}}}{\partial \hat{t}}, \quad \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{\rho U^2}{L} \hat{\boldsymbol{u}} \cdot \hat{\nabla} \boldsymbol{u}, \quad \nabla p = \frac{\rho U^2}{L} \hat{\nabla} \hat{p}, \quad \mu \nabla^2 \boldsymbol{u} = \frac{\mu U}{L^2} \hat{\nabla}^2 \hat{\boldsymbol{u}}.$$

Then substituting in gives

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{\boldsymbol{u}} \cdot \hat{\nabla} \hat{\boldsymbol{u}} = -\frac{1}{\rho} \hat{\nabla} \hat{p} + \frac{1}{Re} \hat{\nabla}^2 \hat{\boldsymbol{u}},$$

where Re is the Reynolds number. Whatever the scale, flows with the same Re look similar.

N.B. L is usually a fixed scale depending on the size of the system (eg box size, body size) - not necessarily the actual size on which the velocity varies. Values of Re:

- Submarine:
  $L \sim 100$m,
  $\nu \sim 10^{-6}$m^{2}s^{-1} (water),
  $U \sim 10$km/h $\sim 10^4 m/10^4 s = 1$m/s.
  $\frac{UL}{\nu} \sim \frac{1 \times 10^2}{10^{-6}} \sim 10^8$
- Bubbles in Beer:
  $L \sim 10^{-4}$m
  $U \sim 10^{-3}$m/s
  $Re \sim 10^{-1}$
- Flow of rock, swimming micro-organisms, Re tiny.

2 Some Simple Flow Fields

2.1 Poiseuille and Couette Flow

There are some flows which are *rectilinear* and this leads to considerable simplifications.

Poiseuille flow is flow in a pipe (independent of time).

N-S equations

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u}, \quad \boldsymbol{u} = 0 \text{ at } r = a.$$

Look for solution of the form $\boldsymbol{u} = (0, 0, u(r))$ in polar coordinates (r, ϕ, z). Then $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = (0, 0, u\frac{\partial u}{\partial z}) = 0$. Also, r and ϕ components give $\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \phi} = 0$. So, $p = p(z)$ and z component gives

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}z} + \mu \left[\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}u}{\mathrm{d}r} \right) \right], \tag{3}$$

where the last component is independent of z.
Let $G = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}$, contant. So p depends linearly on z. Then (3) gives

$$\begin{aligned}
0 &= G + \left[\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}u}{\mathrm{d}r} \right) \right]. \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}u}{\mathrm{d}r} \right) &= -Gr \\
\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}r} &= -\frac{1}{2} Gr + \underbrace{\frac{A}{r}}_{\to 0, \text{ singular}} \\
\Rightarrow u &= -\frac{1}{4} Gr^2 + B = \frac{1}{4} G(a^2 - r^2),
\end{aligned}$$

a parabolic profile.

We can work out the mass flux

$$\begin{aligned}
\rho 2\pi \int_0^a ru \, \mathrm{d}r &= 2\pi\rho \int_0^a \frac{1}{4} G(a^2 r - r^3) \, \mathrm{d}r \\
&= \frac{\pi\rho G}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) \\
&= \frac{\pi\rho G a^4}{8}.
\end{aligned}$$

The the volume flux is $Q = \pi G a^4/8$. Also, we can work out the tangential tangential stress on the wall. Stress is

$$\begin{aligned}
2\mu e_{rz} &= \mu \frac{\partial u}{\partial r}\bigg|_{r=a} \\
&= -\mu G/, \frac{a}{2}/\text{unit area} \\
&= -\frac{\mathrm{d}p}{\mathrm{d}z}\frac{a}{2}\, 2\pi a/\text{unit length} \\
&= -\pi a^2 \frac{\mathrm{d}p}{\mathrm{d}z}/\text{unit length} \\
&= -\pi a^2 \frac{\Delta p}{L},
\end{aligned}$$

which equals net pressure force on the ends (must do so to have equilibrium). Clearly, Re does not come into the solution!

Since either $\boldsymbol{u}$ or $\frac{\partial \boldsymbol{u}}{\partial n}$ vanishes on boundary, we can work out the disspiation using either formula of the last section. The total dissipation in the fluid is

$$\begin{aligned}
\int_0^L \mathrm{d}z \cdot 2\pi \int_0^a r\mathrm{d}r \cdot \mu |\nabla u|^2 &= \mu\, 2\pi L \int_0^a r\mathrm{d}r \left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 \\
&= 2\pi\mu L \int_0^a r\mathrm{d}r\, \frac{1}{4} G^2 r^2, \qquad \text{since } \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{2} G r, \\
&= 2\pi\mu L \frac{G^2 a^4}{16} \\
&= Q\Delta p,
\end{aligned}$$

which equals the rate of working of the surface forces (at ends only).

N.B. This is a solution of particularly simple form. It is a *laminar flow*. But when $Re \gtrsim 10^4$, instabilities appear leading to turbulence in the pipe - most fascinating part of fluid mechanics is nonlinearity.

Couette Flow between two translating plates, distance a apart. At $y = \pm a/2$ we have $\boldsymbol{u} = (\pm U/2, 0, 0)$

Look for solution of the form $\boldsymbol{u} = (u(y), 0, 0)$. No imposed pressure gradient. Same arguments as above lead to p is constant everywhere (exercise). So, (now in cartesian coordinates),

$$0 = \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} \quad \Rightarrow u = \frac{Uy}{a}.$$

There is no net traction on the fluid as tangetial stresses on the two plates cancel.The dissipation per unit area in x, z, namely $\mu \int_{-a/2}^{a/2} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \mathrm{d}y, is$ balanced by rate of working of tractions (these now add up rather than cancel – exercise). This flow too can be unstable if $Re = Ua/\nu$ is large enough.

Flow down a sloping plate needs gravity as it drives the flow. The plate makes an angle θ with horizonta. Use cartesian axes (x, y) downstream and perpendicular to the plate.Free surface at $y = a$. Pressure above fuid surface is p_0 (const).

The y component of the equation of motion is

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta,$$

so

$$p = -\rho g \cos\theta (y - h) + p_0 + f(x),$$

where $f(x) = 0$ as no applied pressure. The x component is

$$0 = \rho g \sin\theta + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2},$$

where the first term on the right hand side is a body force. The boundary conditions are $u = 0$ at $y = 0$ and $\frac{\partial u}{\partial y} = 0$ at $y = h$, so

$$u = \frac{1}{2\mu} \rho g \sin\theta \cdot y(2h - y),$$

(no tangential stress), so $u(h) = \frac{1}{2\mu} \rho g \sin\theta \cdot h^2$ and

$$\begin{aligned} Q = \int_0^h u \, \mathrm{d}y &= \frac{1}{1\mu} \rho g \sin\theta \int_0^h (2hy - y^2) \, \mathrm{d}y \\ &= \frac{1}{3\mu} \rho g \sin\theta h^3. \end{aligned}$$

2.2 Time dependent problems

<u>**Stokes Flow**</u> with a harmonically oscillating plate at $y = 0$.

As before, we can look for a solution where $\boldsymbol{u} = (u(y,t), 0, 0)$. Then

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4}$$

with $u = U \cos\omega t$ at $y = 0$ and $u \to 0$ as $y \to -\infty$. Using the substitution $u = \Re[\hat{u} \mathrm{e}^{i\omega t}]$, the first boundary condition becomes $\hat{u} = U$ at $y = 0$, and the time derivative is

$$\frac{\partial u}{\partial t} = \Re[i\omega \hat{y}(y) \mathrm{e}^{i\omega t}].$$

So equation (4) becomes

$$i\omega\hat{u} = \nu \frac{\partial^2 \hat{u}}{\partial y^2}.$$

Therefore, using the identity $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$,

$$\hat{u} = U\mathrm{e}^{\sqrt{\frac{i\omega}{\nu}}y} = U\mathrm{e}^{\sqrt{\frac{\omega}{2\nu}}(1+i)y},$$

Or, taking the real part,

$$u = U\mathrm{e}^{\sqrt{\frac{\omega}{2\nu}}y}\cos\left(\omega t + \sqrt{\frac{\omega}{2\nu}}y\right).$$

This defines a new length scale $\sqrt{2\nu/\omega}$.

The total dissipation per unit horizontal area is given by

$$\rho\nu\int_{-\infty}^{0}\left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}.$$

This is periodic in time, so take a time average. If $c = \Re(\hat{c}\mathrm{e}^{i\omega t})$ and $d = \Re(\hat{d}\mathrm{e}^{i\omega t})$, then

$$\begin{aligned}
\frac{\omega}{2\pi}\int_0^{2\pi/\omega} cd\,\mathrm{d}t &= \int_0^{2\pi/\omega}\frac{\omega}{2\pi}(\hat{c}_R\cos\omega t - \hat{c}_I\sin\omega t)(\hat{d}_R\cos\omega t - \hat{d}_I\sin\omega t)\mathrm{d}t \\
&= \int_0^{2\pi/\omega}\frac{\omega}{2\pi}[\hat{c}_R\hat{d}_R\cos^2\omega t + \hat{c}_I\hat{d}_I\sin^2\omega t + \underbrace{\ldots}_{\text{vanish}}] \\
&= \frac{1}{2}(\hat{c}_R\hat{d}_R + \hat{c}_I\hat{d}_I) \\
&= \frac{1}{2}\Re[cd^*].
\end{aligned}$$

So,

$$\begin{aligned}
\nu \int_{-\infty}^{0} \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}y &= \frac{\nu}{2} \int_{-\infty}^{0} \left|\frac{\partial \hat{u}}{\partial y}\right|^2 \mathrm{d}y \\
&= \frac{\nu}{2} \int_{-\infty}^{0} \sqrt{\frac{i\omega}{\nu}} U \mathrm{e}^{\sqrt{\frac{\omega}{2\nu}}(1+i)y} \cdot \sqrt{\frac{-i\omega}{\nu}} U \mathrm{e}^{\sqrt{\frac{\omega}{2\nu}}(1-i)y} \, \mathrm{d}y \\
&= \frac{\nu}{2}\frac{\omega}{\nu} U^2 \int_{-\infty}^{0} \mathrm{e}^{2\sqrt{\frac{\omega}{2\nu}}y} \, \mathrm{d}y \\
&= \frac{\omega U^2}{2} \frac{1}{2} \sqrt{\frac{2\nu}{\omega}}.
\end{aligned}$$

It should be checked that this is the same as the time and horizontal space average of the work done by the tractions on $y = 0$, namely the time average of $\rho\nu u \frac{\partial u}{\partial y}|_{y=0}$.

The Rayleigh Problem with an impulsively started flat plate.

This is another example of rectilinear flow, where $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = 0$, $\boldsymbol{u} = (u(y,t), 0, 0)$, $\boldsymbol{u}(y = 0) = (UH(t), 0, 0)$, ($H =$ Heaviside function). As before, we have to solve

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

with $u(0,t) = UH(t)$, $u \to 0$ as $y \to \infty$, and $u \equiv 0$ at $t = 0^-$. We could solve this by Laplace Transform. But in fact we can use a *similarity solution*. There is no natural scale of motion given in the problem. The only way to make a length scale is out of t, ν, with dimensions $[t] = T$ and $[\nu] = L^2T^{-1}$. So, $\sqrt{\nu t}$ has dimensions of length (this is the diffusion length).

So, try $u = Uf(t)g\left(\frac{y}{2\sqrt{\nu t}}\right)$, $t > 0$, with $u = U$ at $y = 0$ so $f(t)$ is constant. Then u becomes

$$u = Ug(\eta), \quad \text{where } \eta = \frac{y}{2\sqrt{\nu t}}.$$

The y derivatives are

$$\frac{\partial u}{\partial y} = \frac{U}{2\sqrt{\nu t}} g'(\eta), \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{4\nu t} g''(\eta),$$

and t derivative is

$$\frac{\partial u}{\partial t} = U g'(\eta) \frac{\partial \eta}{\partial t} = -\frac{yU}{4t\sqrt{\nu t}} g'(\eta).$$

Substituting these into the equation above gives

$$-\frac{yU}{4t\sqrt{\nu t}} g'(\eta) = \frac{\nu U}{4\nu t} g''(\eta),$$

or

$$-\frac{1}{2}\frac{\eta}{t} g'(\eta) = \frac{\nu}{4\nu t} g''(\eta)$$
$$\Rightarrow g''(\eta) = -2\eta g'(\eta),$$

thus justifying the choice. Solving this gives

$$g'(\eta) = -c\mathrm{e}^{-\eta^2},$$

as $g, g' \to 0$ as $\eta \to \infty$. Integrating this gives

$$g = c \int_\eta^\infty \mathrm{e}^{-\xi^2} \mathrm{d}\xi.$$

Applying the boundary condition $g = 1$ at $\eta = 0$ gives

$$1 = c \int_0^\infty \mathrm{e}^{-\xi^2} \mathrm{d}\xi,$$

so $c = 2/\sqrt{\pi}$. Finally,

$$u(y,t) = \frac{2U}{\sqrt{\pi}} \int_{y/2\sqrt{\nu t}}^\infty \mathrm{e}^{-\xi^2} \mathrm{d}\xi \equiv U \mathrm{erfc}\left(\frac{y}{2\nu t}\right).$$

We can work out the stress at $y = 0$

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu U}{2\sqrt{\nu t}} g'(0) = -\frac{\nu U}{2\sqrt{\nu t}} \rho \frac{2}{\sqrt{\pi}} = -\rho \sqrt{\frac{\nu}{\pi t}} U$$

3 Flows at very low Reynolds number

3.1 Introduction

If the flow has zero inertia, then the force balance is entirely between pressure and viscous forces (and any body forces). Is is assumed that the time and velocity scales are such that

$$|\boldsymbol{u} \cdot \nabla \boldsymbol{u}| \sim \frac{U^2}{L} \ll \frac{\nu U}{L^2}, \quad \text{or} \quad Re \ll 1,$$

and

$$\left|\frac{\partial u}{\partial t}\right| \sim \frac{U}{T} \ll \frac{\nu U}{L^2}, \quad \text{or} \quad \tilde{Re} \equiv \frac{L^2}{\nu T} \ll 1$$

(usually take $T \sim \frac{L}{U}$ so $\tilde{Re}$ same as Re). So we are left with

$$0 = -\nabla p + \rho \boldsymbol{F} + \mu \nabla^2 \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0.$$

This simplification has several properites:

1. Instantaneity - instant response to changes in $\boldsymbol{F}$ or in boundaries;

2. Linearity - any two solutions can be added together to give a third solution. Thus solutions satisfying boundary conditions can be built up by superposition;

3. Reversibility - if boundaries move at velocity $\boldsymbol{V}(t)$ implies the flow $\boldsymbol{u}(\boldsymbol{x}, t)$, then changing the boundary velocity to $-\boldsymbol{V}$ leads to flow $-\boldsymbol{u}$.

It follows from point 3 that flow past an object that is symmetric under reflection in a plane perpendicular to distant flow field (e.g. a sphere about $y - z$ plane through centre with velocity in x-direction at ∞) is anti-symmetric under reflection.

Another result: sphere falling near a wall.

If body force changes sign, then so will $\boldsymbol{u}$. So there is a contradiction unless $\boldsymbol{u}$ is parallel to the wall.

Proof of uniqueness of Stokes flow

In V

$$0 = -\nabla p_1 + \rho \boldsymbol{F} + \mu \nabla^2 \boldsymbol{u}_1, \qquad 0 = -\nabla p_2 + \rho \boldsymbol{F} + \mu \nabla^2 \boldsymbol{u}_2,$$

and on ∂V

$$\boldsymbol{u}_{1,2} = \boldsymbol{U}.$$

Let $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ and $\pi = p_2 - p_1$. Then

$$0 = -\nabla \pi + \mu \nabla^2 \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v} = 0 \text{ on } \partial V.$$

So,

$$\begin{aligned} 0 &= \int_V (-\boldsymbol{v} \cdot \nabla \pi + \mu \boldsymbol{v} \cdot \nabla^2 \boldsymbol{v}) \mathrm{d}V, \\ \Rightarrow \quad 0 &= -\int_V \left(\nabla \cdot (\boldsymbol{v} \pi) - \mu \frac{\partial}{\partial x_i} \left(v_j \frac{\partial v_j}{\partial x_i} \right) \right) \mathrm{d}V - \mu \int_V \left(\frac{\partial v_j}{\partial x_i} \right)^2 \mathrm{d}V, \\ \Rightarrow \quad 0 &= -\int_{\partial V} \boldsymbol{v} \cdot \boldsymbol{n} \pi \, \mathrm{d}S + \mu \int_{\partial V} n_i v_j \frac{\partial v_j}{\partial x_i} \mathrm{d}S - \mu \int_V |\nabla \boldsymbol{v}|^2 \, \mathrm{d}V. \end{aligned}$$

The first two terms on the right hand side are zero. Hence $\nabla \boldsymbol{v} = 0$, so $\boldsymbol{v} =$const$= 0$.

Minimum Dissipation Theorem

Let $\boldsymbol{u}$ be a solution for Stokes flow with given boundary conditions, with rate of strain tensor e_{ij}^u. Let $\boldsymbol{v}$ satisfy $\nabla \cdot \boldsymbol{v} = 0$ and the same boundary conditions as $\boldsymbol{u}$, with corresponding tensor e_{ij}^v. Let $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ (with corresponding tensor e_{ij}^w), where $\nabla \cdot \boldsymbol{w} = 0$,

$\boldsymbol{w} =$ on boundaries. Then

$$\int_V e^v_{ij}e^v_{ij}\mathrm{d}V = \int_V e^u_{ij}e^u_{ij}\mathrm{d}V + \int_V e^w_{ij}e^w_{ij}\mathrm{d}V + \underbrace{2\int_V e^u_{ij}\frac{\partial w_i}{\partial x_j}\mathrm{d}V}$$

$$= 2\int_V \frac{\partial}{\partial x_j}\left(w_i e^u_{ij}\right)\mathrm{d}V - 2\int_V \boldsymbol{w}\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}V$$

$$= \underbrace{2\int_{\partial V} w_i n_j e^u_{ij}\mathrm{d}S}_{=0,\text{ as } w_i=0 \text{ on } \partial V} \underbrace{-2\frac{1}{\mu}\int_V \boldsymbol{w}\cdot\nabla p\,\mathrm{d}V}_{=0 \text{ by usual calculation}}$$

So

$$\int_V e^v_{ij}e^v_{ij}\mathrm{d}V = \int_V e^u_{ij}e^u_{ij}\mathrm{d}V + \int_V e^w_{ij}e^w_{ij}\mathrm{d}V \geq \int_V e^u_{ij}e^u_{ij}\mathrm{d}V.$$

So Stokes flow has minimum dissipation for given boundary conditions.

3.2 Two-dimensional Flows

N-S equations at low inertia give

$$0 = -\nabla p + \mu\nabla^2\boldsymbol{u},$$

so if μ is constant then $\nabla^2 p = 0$ and

$$0 = \nabla^2\boldsymbol{\omega}.$$

If the flow is 2-D we can write $\boldsymbol{u}(x,y) = \nabla\times(\psi\hat{\boldsymbol{z}})$

$$\boldsymbol{u} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\phi}{\partial x}, 0\right)$$
$$\boldsymbol{\omega} = (0, 0, -\nabla^2\psi).$$

So, $\nabla^2\psi = 0$ (*biharmonic* equation). We can solve this in several interesting cases

1. Hinged plates separating,
2. Flow in a corner - Moffatt eddies,
3. Paintbrush or knife.

Hinged Plates Separating

Two rigid plates, one at $\phi = 0$, one at $\phi = \alpha(t)$, $\dot{\alpha} = \Omega(t)$.Use polars: $u_r = (1/r)\partial\psi/\partial\phi = (1/r)\psi_\phi$, $u_\phi = -\partial\psi/\partial r = -\psi_r$.

At $\phi = 0$, $u_r = u_\phi = 0$. So $\psi_r = 0 \Rightarrow \psi =$ const. Take $\psi = 0$. Also, $\psi_\phi = 0$.
At $\phi = \alpha$, $u_r = 0$ and $u_\phi = \Omega r$. So $\psi_\phi = 0$, $\psi_r = -\Omega r \quad \Rightarrow \quad \psi = -(1/2)\Omega r^2$.
This suggests we can find a solution of the form $\psi = \Omega r^2 f(\phi)$. So, at $\phi = 0$, $f = 0 = f_\phi$.
And at $\phi = \alpha$, $f = -1/2$ and $f_\phi = 0$.
Before solving this consider a general class of *separable* solutions of $\nabla^4\psi = 0$,

$$\psi = r^\lambda f(\phi),$$

(cf $\nabla^2\Phi = 0$, $\Phi = r^{\pm a}\{\sin a\phi, \cos a\phi\}$.) Then

$$\begin{aligned}
\nabla^2\psi &= \left(\lambda^2 f + f_{\phi\phi}\right) r^{\lambda-2} \\
\nabla^4\psi &= \left[(\lambda-2)^2(\lambda^2 f + f_{\phi\phi}) + \lambda^2 f_{\phi\phi} + f_{\phi\phi\phi\phi}\right] r^{\lambda-4}.
\end{aligned}$$

So

$$\begin{aligned}
&(\lambda-2)^2\lambda^2 f + \left[(\lambda-2)^2 + \lambda^2\right] f_{\phi\phi} + f_{\phi\phi\phi\phi} = 0 \\
\Rightarrow \quad &\left((\lambda-2)^2 + \frac{\partial^2}{\partial\phi^2}\right)\left(\lambda^2 + \frac{\partial^2}{\partial\phi^2}\right) f = 0,
\end{aligned}$$

In general the solution is

$$f = A\cos\lambda\phi + B\sin\lambda\phi + C\cos(\lambda-2)\phi + D\sin(\lambda-2)\phi, \quad \lambda \neq 0, 1, 2.$$

If $\lambda = 1$, we have

$$\left(1 + \frac{\partial^2}{\partial\phi^2}\right)^2 f = 0,$$

and

$$f = A\cos\phi + B\sin\phi + C\phi\cos\phi + D\phi\sin\phi.$$

For $\lambda = 0, 2$,

$$\frac{\partial^2}{\partial\phi^2}\left[4 + \frac{\partial^2}{\partial\phi^2}\right] f = 0.$$

and

$$f = A\cos 2\phi + B\sin 2\phi + C + D\phi.$$

For a hinged plate, choose $\lambda = 2$, $\psi = \Omega r^2 f(\phi)$.

$$\begin{aligned} &\text{at } \phi = 0, \quad f = 0 = f_\phi, \\ &\text{at } \phi = \alpha, \quad f = -\frac{1}{2}, f_\phi = 0. \end{aligned}$$

The boundary condition at $\phi = 0$ gives

$$A + C = 0, \quad 2B + D = 0.$$

So general solution becomes

$$f = A(\cos 2\phi - 1) + B(\sin 2\phi - 2\phi).$$

The boundary condition at $\phi = \alpha$ gives

$$\begin{aligned} &A(\cos 2\alpha - 1) + B(\sin 2\alpha - 2\alpha) = -\frac{1}{2} \\ &-2A\sin 2\alpha + 2B(\cos 2\alpha - 1) = 0, \end{aligned}$$

so

$$A = \frac{1 - \cos 2\alpha}{4(1 - \cos 2\alpha - \alpha\sin 2\alpha)}, \quad B = \frac{-\sin 2\alpha}{4(1 - \cos 2\alpha - \alpha\sin 2\alpha)}.$$

When α is sufficiently small, the denominator is positive. When denominator is zero, $\alpha = 257°$, there exists a solution with $\lambda = 2$ satisfying $f = f_\phi = 0$ at $\phi = 0, \alpha$ called a *free corner flow*

$$
\begin{aligned}
& A + C = 0, \quad 2B + D = 0 \\
\Rightarrow \quad & f = A(\cos 2\phi - 1) + B(\sin^2 2\phi - 2\phi) \\
\Rightarrow \quad & A(\cos 2\alpha - 1) + B(\sin^2 \alpha - 2\alpha) = 0, \\
& -2A \sin 2\alpha + 2B(\cos 2\alpha - 1) = 0,
\end{aligned}
$$

which is consistent. If α is greater than this value, the free corner flow induced by the outer solution is bigger than the flow forced by the plate motion.

Moffatt Eddies

A more interesting flow is the induced free corner flow itself, in a narrow corner - the so-called *Moffatt eddies*.

Consider two fixed plates at $\phi = \pm\alpha$. Then $\boldsymbol{u} = 0$ at $\phi = \pm\alpha$. Try a solution of the form $f = r^\lambda f(\phi)$. Then $f = f_\phi = 0$ at $\phi = \pm\alpha$. Look for a symmetric f, then $u_r \propto \psi_r$ is antisymmetric. So

$$f = A \cos \lambda\phi + C \cos(\lambda - 2)\phi.$$

Boundary conditions give

$$
\begin{aligned}
& A \cos \lambda\alpha + C \cos(\lambda - 2)\alpha = 0, \\
& A\lambda \sin \lambda\alpha + (\lambda - 2)C \sin(\lambda - 2)\alpha = 0
\end{aligned}
$$

So

$$(\lambda - 2) \cos \lambda\alpha \sin(\lambda - 2)\alpha = \lambda \sin \lambda\alpha \cos(\lambda - 2)\alpha.$$

This equation determines λ as a function of α.

Are there real solutions for λ? Let $\lambda = 1 + \beta$. Then

$$\begin{aligned}
&(\beta - 1)\cos(\beta+1)\alpha\sin(\beta-1)\alpha = (1+\beta)\cos(\beta-1)\alpha\sin(\beta+1)\alpha \\
\Rightarrow\quad &(\beta-1)\left[\sin 2\beta\alpha - \sin 2\alpha\right] = (\beta+1)\left[\sin 2\beta\alpha + \sin 2\alpha\right] \\
\Rightarrow\quad &-2\beta\sin 2\alpha = 2\sin 2\beta\alpha \\
\Rightarrow\quad &\frac{\sin 2\alpha}{2\alpha} = -\frac{\sin 2\beta\alpha}{2\beta\alpha}.
\end{aligned}$$

To find real solutions for β we need two equal and opposite intercepts on the curve $y = \sin x/x$. The minimum possible value for real β is when $-y = \min(\sin x/x)$, $y = .2172$, which gives $\alpha \approx 73°$.

When λ is complex, $\lambda = p + iq$, consider

$$u_\phi(r, 0) = \text{const} \cdot r^{\lambda-1}(\cos(\lambda-2)\alpha - \cos\lambda\alpha) \propto r^{p-1}\cos(q\ln r + \epsilon),$$

where ϵ depends on α. So we have alternating eddies with size decreasing exponentially as $r \to 0$:

$$q\ln r_{N+1} = q\ln r_N - \pi,$$

so

$$\frac{r_{N+1}}{r_N} = \mathrm{e}^{-\pi/q}.$$

The relative strengths are

$$\left(\frac{r_{N+1}}{r_N}\right)^{p-1} = \mathrm{e}^{-\pi(p-1)/q}.$$

eg if $2\alpha = \pi/2$, then $\lambda = 3.74 \pm 1.13i$, so $\mathrm{e}^{-\pi(p-1)/q} = \mathrm{e}^{-\pi 2.74/1.13} \approx 1/2000$.
If $u \sim f^{\lambda-1}$, $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \sim r^{2\lambda-3}$, $\nabla^2 u \sim r^{\lambda-3}$. So as $\lambda > 0$, inertia can be neglected for sufficiently small r.

Paintbrush or Knife

On $\phi = 0$, $\psi = 0, (1/r)\psi_\phi = -U$. On $\phi = \alpha$, $\psi = \psi_\phi = 0$. Suggest $\psi = Urf(r)$. Then

$$f(\phi) = \frac{-\alpha(\alpha-\phi)\sin\phi + \phi\sin\alpha\sin(\alpha-\phi)}{\alpha^2 - \sin^2\alpha},$$

(exercise).

3.3 Forces and torques on rigid bodies

The linearity of the Stokes equations menas that we can make progress in understanding forces and torques on arbitrary bodies. Take an arbitrary body

The flow outside is given by

$$\nabla p = \mu\nabla^2\boldsymbol{u},$$

with $\boldsymbol{u} = \boldsymbol{U}(t) + \boldsymbol{\Omega}(t)\times\boldsymbol{x}$ on S, and $\boldsymbol{u} \to 0$ as $|\boldsymbol{x}| \to \infty$.
Because of linearity, we know that the drag $\boldsymbol{F}$ and couple $\boldsymbol{G}$ on the body depend linearly

on $\boldsymbol{U}$ and $\boldsymbol{\Omega}$. So we must have the relation.

$$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\Omega} \end{pmatrix},$$

where $\boldsymbol{A}, \boldsymbol{B}$, etc depend only on μ and the shape etc of V. We can learn a lot about the nature of these tensors by using the *reciprocal theorem.*

Consider the flows $\boldsymbol{u}$ (and p) with $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{x}$ on S, and $\boldsymbol{u}^*$ (and p^*) with $\boldsymbol{u}^* = \boldsymbol{U}^* + \boldsymbol{\Omega}^* \times \boldsymbol{x}$ on S. Then

$$\begin{aligned} &\int_V u_i^* \frac{\partial \sigma_{ij}}{\partial x_j} \mathrm{d}V = 0 \\ &= \int_S u_i^* \sigma_{ij} n_j \mathrm{d}S - \int_V \sigma_{ij} \frac{\partial u_i^*}{\partial x_j} \mathrm{d}V, \end{aligned}$$

where

$$\begin{aligned} \int_V \sigma_{ij} \frac{\partial u_i^*}{\partial x_j} \mathrm{d}V &= \int_V \sigma_{ij} e_{ij}^* \mathrm{d}V = \int_V (-p^* \delta_{ij} + 2\mu e_{ij}) e_{ij}^* \mathrm{d}V \\ &= \int_V 2\mu e_{ij} e_{ij}^* \mathrm{d}V = \int_V \sigma_{ij}^* \frac{\partial u_i}{\partial x_j} \mathrm{d}V \\ &= \int_S u_i \sigma_{ij}^* n_j \mathrm{d}S. \end{aligned}$$

But, using $\int_S \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = \boldsymbol{F}$ and $\int_S \boldsymbol{x} \times \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = \boldsymbol{G}$,

$$\begin{aligned} \int_S \boldsymbol{u}^* \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S &= \int_S (\boldsymbol{U}^* + (\boldsymbol{\Omega}^* \times \boldsymbol{x})) \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S \\ &= \int_S \boldsymbol{U}^* \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} + \boldsymbol{\Omega}^* \cdot (\boldsymbol{x} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}) \, \mathrm{d}S \\ &= \boldsymbol{U}^* \cdot \boldsymbol{F} + \boldsymbol{\Omega}^* \cdot \boldsymbol{G} \\ &= \boldsymbol{U} \cdot \boldsymbol{F}^* + \boldsymbol{\Omega} \cdot \boldsymbol{G}^* \end{aligned}$$

by the reciprocal theorem. Thus, for arbitrary $\boldsymbol{U}$ and $\boldsymbol{\Omega}$,

$$\begin{aligned} U_i A_{ij} U_j^* + &U_i B_{ij} \Omega_j^* + \Omega_i C_{ij} U_j^* + \Omega_i D_{ij} \Omega_j^* \\ &= U_i^* A_{ij} U_j + U_i^* B_{ij} \Omega_j + \Omega_i^* C_{ij} U_j + \Omega_i^* D_{ij} \Omega_j. \end{aligned}$$

So $A_{ij} = A_{ji}$, $B_{ij} = C_{ji}$, and $D_{ij} = D_{ji}$ irrespective of any symmetries of the body. Clearly symmetries of body will be reflected in additional symmetries of A, B, etc.

eg A cube clearly has identical values of A, B, C, D about any axis, so $\boldsymbol{A}$, $\boldsymbol{D}$ must be isotropic. By choosing appropriate rotations and reflections we can show that

$$\boldsymbol{A} = A\delta_{ij}, \quad \boldsymbol{D} = D\delta_{ij}, \quad \text{and } \boldsymbol{B} = \boldsymbol{C} = 0.$$

So $\boldsymbol{F} = A\boldsymbol{U}$ and $\boldsymbol{G} = D\boldsymbol{\Omega}$. A cube falling under gravity ($\boldsymbol{F} = \rho\boldsymbol{g}$, $\boldsymbol{G} = 0$) falls vertically without rotation.

3.4 Flows due to moving bodies

Rigid Sphere Uniformly Translating

Go into a reference frame in which a sphere is at rest.

Assume that inertia can be neglected and look for an axisymmetric flow (no ϕ component). So, and $\nabla \cdot \boldsymbol{u} = 0$, adopt a *stokes stream function* in spherical polar coordinates (R, θ, ϕ)

$$u_R = \frac{1}{R^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \quad u_\theta = -\frac{1}{R\sin\theta}\frac{\partial \Psi}{\partial R}.$$

Then we have $\nabla^2 \boldsymbol{\omega} = \nabla \times \nabla \times \boldsymbol{\omega} = 0$, where $\omega = \nabla \times \boldsymbol{u}$,

$$\boldsymbol{u} = \nabla \times \left(0, 0, \frac{\Psi}{R\sin\theta}\right) \quad \text{and} \quad \boldsymbol{\omega} = \left(0, 0, -\frac{1}{R\sin\theta}D^2\Psi\right),$$

where $D^2\Psi = \frac{\partial^2 \Psi}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)$. NB $D^2 \neq \nabla^2$!! Thus

$$\nabla \times \nabla \times \boldsymbol{\omega} = \nabla \times \nabla \times \left(0, 0, -\frac{D^2\Psi}{R\sin\theta}\right) = \left(0, 0, \frac{D^4\Psi}{R\sin\theta}\right) = 0.$$

Thus the equation to be solved for Ψ is

$$D^4\Psi = 0,$$

with $\Psi = 0$ at $\theta = 0$; at $R = a$, $u_R = u_\theta = 0$ so $\Psi = 0$ and $\frac{\partial \Psi}{\partial R} = 0$; and at $R \to \infty$, $u_R \to U\cos\theta$ and $u_\theta \to -U\sin\theta$. Thus

$$\frac{1}{R^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} \sim U\cos\theta,$$

so

$$\Psi \sim \frac{1}{2} U R^2 \sin^2\theta, \quad \text{as } R \to \infty.$$

Seek a separable solution of the form

$$\Psi = \frac{U}{2} f(R) \sin^2\theta.$$

Then (exercise)

$$D^2\Psi = \frac{U}{2}\left[\left(f_{RR} - \frac{2f}{R^2}\right)\sin^2\theta\right]$$

$$D^4\Psi = \frac{U}{2}\left[\left(\partial^2_{RR} - \frac{2}{R^2}\right)^2 f \sin^2\theta\right] = 0$$

So

$$\left(\partial^2_{RR} - \frac{2}{R^2}\right)^2 f = 0.$$

Assume f is a polynomial in R, and first find for what α

$$\left(\partial^2_{RR} - \frac{2}{R^2}\right) R^\alpha = 0.$$

We find $\alpha = 2$ or -1, and now we solve

$$\left(\partial^2_{RR} - \frac{2}{R^2}\right) f = AR^2 + \frac{B}{R},$$

to get

$$f = \frac{A}{10}R^4 - \frac{B}{2}R + CR^2 + \frac{D}{R}.$$

The boundary condition at ∞ gives $A = 0$, $C = 1$. At $R = a$

$$f = -\frac{B}{2}a + a^2 + \frac{D}{2a} = 0$$

$$f_R = -\frac{B}{2} + 2a - \frac{D}{a^2} = 0.$$

We finally get

$$f = \left(R^2 - \frac{3}{2}aR + \frac{1}{2}\frac{a^3}{R} \right),$$

and

$$u_R = U\left(1 - \frac{3a}{2R} + \frac{a^3}{2R^3}\right)\cos\theta,$$
$$u_\theta = U\left(-1 + \frac{3a}{4R} + \frac{a^3}{4R^3}\right)\sin\theta.$$

We can also calculate the pressure as $-\nabla p + \mu\nabla^2 \boldsymbol{u} = 0$,

$$p = p_o - \frac{3U}{R^2}a\cos\theta$$

To ensure consistency, we have to check that as R becomes large, $|\boldsymbol{u}\cdot\nabla\boldsymbol{u}| \ll |\nu\nabla^2\boldsymbol{u}|$ where $Re = Ua/\nu \ll 1$. Consider frame where $\boldsymbol{u} = -\boldsymbol{U}$ at $R = a$, $\boldsymbol{u} = 0$ as $R \to \infty$. Then $u_r' \sim Ua/R$. So $\nabla^2 u' \sim Ua/R^3$ and $|\boldsymbol{u}\cdot\nabla\boldsymbol{u}| \sim U^2a^2/R^3$, so ok for large R.
Next we calculate the drag on the sphere.

We need the z component of traction

$$\boldsymbol{\tau} = (\sigma_{RR}, \sigma_{R\theta}, 0)$$

on $\boldsymbol{n} = (1, 0, 0)$. So

$$\tau_z = \sigma_{RR}\cos\theta - \sigma_{R\theta}\sin\theta.$$

The total drag is

$$2\pi\int_{R=a}(\sigma_{RR}\cos\theta - \sigma_{R\theta}\sin\theta)\cdot a^2\sin\theta \, \mathrm{d}\theta.$$

Referring to Batchelor or the handout

$$\sigma_{RR} = -p + 2\mu\frac{\partial u_R}{\partial R}, \quad \sigma_{R\theta} = \mu\left[R\frac{\partial}{\partial R}\left(\frac{u\theta}{R}\right) + \frac{1}{R}\frac{\partial u_R}{\partial\theta}\right],$$

but at $R = a$ $\frac{\partial u_R}{\partial \theta} = 0$, and also $\frac{\partial u_R}{\partial R} = 0$ (why?). So

$$\sigma_{RR} = -p_o + \mu \frac{3U}{2a} \cos\theta, \quad \sigma_{R\theta} = -\mu \frac{3U}{2a} \sin\theta.$$

Therefore

$$\begin{aligned} \text{drag} &= 2\pi \int \left(\left(-p_o + \mu \frac{3U}{2a} \cos\theta \right) \sin\theta \cos\theta + \mu \frac{3U}{2a} \sin^3\theta \right) \mathrm{d}\theta a^2 \\ &= 6\pi U a \mu \end{aligned}$$

(Stokes 1857).
eg for a solid sphere of density ρ' falling in a fluid of density ρ,

$$\begin{aligned} \text{drag} &= \text{gravity} - \text{buoyancy} \\ 6\pi a \boldsymbol{U} \mu &= \frac{4}{3}\pi \rho' a^3 \boldsymbol{g} - \frac{4}{3}\pi \rho a^3 \boldsymbol{g} \\ &= \frac{4}{3}\pi (\rho' - \rho) a^3 \boldsymbol{g}. \end{aligned}$$

So, $\boldsymbol{U} = \frac{2}{9}\frac{a^2}{\nu}\left(\frac{\rho'}{\rho} - 1\right)\boldsymbol{g}$.
We can define a *drag coefficient* c_D by

$$F = c_D \frac{1}{2} \rho U^2 \pi a^2.$$

Then $c_D = 12/Re$, where $Re = aU/\nu$. In experiments $Re \times c_D$ increases - for large Re c_D becomes independent of Re.

Flow past a bubble

Suppose a bubble is essentially spherical due to surface tension forces. Then we still have $u_R = 0$ at $R = a$. We can't use normal strss condition as surface tension is now the dominant force. Ignoring viscosity inside the bubble, we have zero tangential stress at $R = a$

$$\sigma_{R\theta}|_{R=1} = \mu a \frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right)\Big|_{R=a} \left(+ \frac{\mu}{a} \underbrace{\frac{\partial u_R}{\partial \theta}\Big|_{R=a}}_{=0} \right).$$

So instead of $u_\theta = 0$ at $R = a$, we have $\frac{\partial u_\theta}{\partial R} = u_\theta / a$ at $R = a$. This leads to a similar but different flow field.

It may be verified that

$$\tau_z = 2\pi \int \sigma_{RR} \cos\theta \cdot a^2 \sin\theta \, \mathrm{d}\theta = 4\pi a \mu U,$$

(less than for a rigid sphere). So a bubble of density $\rho' \ll \rho$ rising under gravity has velocity

$$\boldsymbol{u} = -\frac{1}{3}\frac{a^2}{\nu}\boldsymbol{g}.$$

More generally, if a sphere has a density ρ', contains fluid of viscosity μ', then we can find the Stokes flow inside the sphere, provided we satisfy matching conditions on tangential velocity and stress at $r = a$. We get

$$\boldsymbol{u} = \frac{2}{9}a^2\nu\boldsymbol{g}\left(\frac{\rho'}{\rho} - 1\right)\left(\frac{\mu + \mu'}{(2/3)\mu + \mu'}\right).$$

Flow past a cube

We have seen tha the cube having symmetry, the drag is parallel to $\boldsymbol{U}$.

Consider the flow that is the Stokes flow solution outside the circumscribing sphere with radius $a = L\sqrt{3}$ and zero inside the gap. This satisfies the boundary conditions on cube plus $\nabla \cdot \boldsymbol{u} = 0$ etc.
Then the dissipation due to the actual solution FU $<$the dissipation due to any other flow (min dissipation theorem). So $FU < U6\pi\mu L\sqrt{3}U$, so $F < 6\pi\mu L\sqrt{3}U$.
Similarly, consider the inscribed sphere,

and consider real flow to the cube boundaries and zero inside gap. By the same method $F > 6\pi\mu LU$.
So for the cube, $6\pi\mu LU < F < 6\pi\mu L\sqrt{3}U$.

4 Lubrication Theory

4.1 Viscous flow in a narrow gap

Examples:

- oil in a bearing, $h = b - a \ll a$

- drops on surface of large aspect ratio

- flow between close sheets - Hele-shaw cell

Basic principle: Re based on gap width/height $Uh^2/L\nu \ll 1$ and variations in other direction are slow $\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \sim 1/L \ll 1/h \sim \frac{\partial}{\partial y}$ because $\nabla \cdot \boldsymbol{u} = 0$, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ so $v \sim (h/L)(u, w)$ (ie flow almost horizontal.

Ignore z-dependence (easy to generalize)
The x component of the NS equations is

$$\rho\left(\frac{\partial u}{\partial t}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\to 0,\ \mathcal{O}(\frac{h}{L})^2}.$$

Ignore intertial term if $U^2/l \ll \nu U/h^2$ or $Uh^2/L\nu \ll 1$.
The y component is

$$0 \simeq -\frac{\partial p}{\partial y}+\mu\frac{\partial^2 v}{\partial y^2} \quad \Rightarrow \quad \frac{\partial p}{\partial y}=0$$

at leading order, so $p = p(x)$.
So

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x},$$

with $u = U$ at $y = 0$, and $u = 0$ at $y = h(x)$. So,

$$u=\frac{1}{2\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)y(h(x)-y)+U\frac{h(x)-y}{h(x)}.$$

We can find the pressure from mass conservation

$$\begin{aligned}\int_0^h u\ \mathrm{d}y &= Q \quad \text{(independent of } x\text{)}\\ &= \frac{h^3}{12\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)+\frac{1}{2}Uh.\end{aligned}$$

So,

$$\frac{\mathrm{d}p}{\mathrm{d}x}=-\frac{12\mu Q}{h^3}+\frac{6\mu U}{h^2}.$$

If for example there is no net pressure drop along the gap then

$$\int_0^L \frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}x = 0 \quad \Rightarrow \quad 2Q\int_0^L h^{-3}\mathrm{d}x = U\int_0^L h^{-2}\mathrm{d}x.$$

Example: the thrust bearing

$d_1 = d_2 + \alpha L$ and $h = d_1 - \alpha x$, say.

Then

$$\int_0^L \frac{1}{(d_1-\alpha x)^3}\,\mathrm{d}x = \frac{1}{2\alpha}(d_2^{-2} - d_1^{-2}),$$

and

$$\int_0^L \frac{1}{(d_1-\alpha x)^2}\,\mathrm{d}x = \frac{1}{\alpha}(d_2^{-1} - d_1^{-1}),$$

so

$$\frac{Q}{U} = \frac{1/d_2 - 1/d_1}{1/d_2^2 - 1/d_1^2} = \frac{d_1 d_2}{d_1+d_2}.$$

We can calculate the maximum pressure. When $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$, $2Q/h^3 = U/h^2$, so $h = 2Q/U = 2d_1d_2/(d_1+d_2)$. Then

$$\begin{aligned} p(x) &= p_o + \mu\int_0^x \left(-\frac{12Q}{h^3} + \frac{6U}{h^2}\right)\,\mathrm{d}x \\ &= p_o + \frac{\mu}{\alpha}\left(\frac{6Q}{h^2} - \frac{6U}{h}\right) - \frac{6Q}{d_1^2} + \frac{6U}{d_1}. \end{aligned}$$

(exercise)

$$\begin{aligned} p - p_o &= \frac{6\mu U}{\alpha(d_1+d_2)}\left[-\frac{d_1d_2}{h^2} + \frac{d_1+d_2}{h} - 1\right] \\ &= \frac{6\mu U}{\alpha}\frac{(d_1-h)(h-d_2)}{h^2(d_1+d_2)} \\ &\sim \frac{\mu U}{\alpha h} \\ &\sim \frac{\mu U}{h^2}L. \end{aligned}$$

So the total force in the normal direction is

$$
\begin{aligned}
\int_0^L (p-p_o)\,\mathrm{d}x &= \frac{6\mu U}{\alpha(d_1+d_2)}\int_0^L \left(-1+h^{-1}(d_1+d_2)-\frac{d_1d_2}{h^2}\right)\,\mathrm{d}x \\
&= \frac{6\mu U}{\alpha(d_1+d_2)}\left[-x-\frac{d_1+d_2}{\alpha}\ln h-\frac{d_1d_2}{\alpha h}\right]_0^L \\
&= \frac{6\mu U}{\alpha(d_1+d_2)}\left(-L+\frac{d_1+d_2}{\alpha}\ln\frac{d_1}{d_2}+\frac{d_1d_2}{\alpha}\left[\frac{1}{d_1}-\frac{1}{d_2}\right]\right) \\
&\sim \frac{6\mu U}{\alpha^2(d_1+d_2)}(d_1d_2)\left(\frac{1}{d_2}-\frac{1}{d_1}\right) \\
&\sim \frac{6\mu U}{\alpha^2}\frac{(d_1-d_2)}{(d_1+d_2)}.
\end{aligned}
$$

The tangential force it

$$
\mu\int_0^L \frac{\partial u}{\partial y}\,\mathrm{d}x = \mu\left(-\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)\int_0^L h\,\mathrm{d}x - \mu U\int_0^L \frac{1}{h}\,\mathrm{d}x.
$$

These terms are both of order $\mu U/\alpha$.

4.2 Time dependent problems

Disc moving towards plane wall

Cirular disc, small gap is $h(t)$

Consider circular surface at $r = x$.

Mass flux out of sides is $2\pi x Q$, where

$$Q(x) = \int_0^h u\,\mathrm{d}y = -\dot{h}\pi x^2$$

is the rate of change of volume. So

$$Q = -\frac{\dot{h}x}{2}.$$

Flow $u(r, y)$ (radial), by same argument as above. We ignore $v(r, y)$ cf u

$$0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 u}{\partial y^2}$$

with $u = 0$ on $y = 0, h$. So

$$u = \left(-\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}r}\right) y(h - y), \quad (p = p(r)).$$

So

$$\int_0^h u \, \mathrm{d}y = \left(-\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\frac{h^3}{12},$$

and

$$\left(-\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\frac{h^3}{12} = -\frac{\dot{h}r}{2}.$$

Then

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{6\mu r\dot{h}}{h^3}$$

with $p = p_o$ at $r = a$ gives

$$p = \frac{3\mu\dot{h}}{h^3}(r^2 - a^2) + p_o.$$

The total upward force is

$$\begin{aligned} 2\pi\int_0^a (p - p_o)r \, \mathrm{d}r &= \frac{2\pi \cdot 3\mu\dot{h}}{h^3}\left(\frac{1}{4}a^4 - \frac{1}{2}a^4\right) \\ &= -\frac{3\pi\mu}{2}\frac{a^4\dot{h}}{h^3}. \end{aligned}$$

Thus if the disc weighs Mg, we have

$$-\frac{3\mu\pi a^4}{2}\frac{\dot{h}}{h^3} = Mg.$$

Rearrange the constants

$$\begin{aligned} & \frac{\dot{h}}{h^3} = -k \\ \Rightarrow \quad & \frac{1}{2}\left(-\frac{1}{h_o^2} + \frac{1}{h^2}\right) = kt \\ \Rightarrow \quad & h \sim t^{-1/2} \text{ as } t \to \infty. \end{aligned}$$

Peristalsis

$$h = f(x - ct), k = f'/f \ll 1/h.$$

No imposed pressure gradient.

$$0 = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial^2 u}{\partial y^2}, \qquad u = \left(-\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right) y(h - y)$$

Mass flux Q satisfies

$$\dot{h} + \frac{\partial Q}{\partial x} = 0, \tag{5}$$

Then

$$Q(x) = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{h^3}{12}. \tag{6}$$

Integrating (6) over one period P of h,

$$\int_0^P \frac{Q(x)}{h^3}\,\mathrm{d}x = 0.$$

Let $h = h(x - ct) = f(x - ct)$. Now $\dot{h} = -cf' = -cf_x = -Q_x$ using equation (5). Then

$$Q = Q_o + cf, \quad \bar{Q} = Q_o + c\bar{f},$$

and thus

$$\int_0^P \frac{Q_o}{h^3}\mathrm{d}x + \int_0^P \frac{c}{h^2}\mathrm{d}x = 0.$$

eg $f = h_o + h_1 \sin kx$, $\bar{f} = h_o$.

$$Q_o \int \frac{\mathrm{d}x}{(h_o + h_1 \sin kx)^3} + c \int \frac{\mathrm{d}x}{(h_o + h_1 \sin kx)^2} = 0$$

is difficult to integrate. But suppose $h_1 \ll h_0$, then

$$\frac{1}{2\pi}\int \frac{1}{(h_o + h_1 \sin kx)^3}\mathrm{d}x = \frac{1}{2\pi h_o^3}\int \left(1 - \frac{3h_1}{h_o}\sin kx + \frac{6h_1^2}{h_o^2}\sin^2 kx + \dots\right)\mathrm{d}x$$
$$= \frac{1}{h_o^3}\left(1 + \frac{3h_i^2}{h_o^2} + \dots\right),$$

and

$$\frac{1}{2\pi}\int \frac{1}{(h_o + h_1 \sin kx)^2}\mathrm{d}x = \frac{1}{h_o^2}\left(1 + \frac{3}{2}\frac{h_1^2}{h_o^2} + \dots\right).$$

So,

$$\frac{(\bar{Q} - ch_o)}{h_o^3}\left(1 + \frac{3h_1^2}{h_o^2} + \dots\right) + \frac{c}{h_o^2}\left(1 + \frac{3}{2}\frac{h_1^2}{h_o^2} + \dots\right) = 0$$
$$\Rightarrow \quad \bar{Q} = \frac{3}{2}\frac{ch_1^2}{h_o} + \dots$$

Note that $\bar{Q}$ changes sign with c, but not with h_1 as might have been predicted on symmetry grounds.

<u>**Spreading drop**</u>

Now include the effect of gravity. 2-D case

$$0 = -\frac{\partial p}{\partial y} - \rho g, \quad p = p_o \text{ at } y = h,$$

gives

$$p = p_o + \rho g(h(x) - y)$$

(*hydrostatic*). So

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x}.$$

The horizontal component

$$0 = -g\frac{\partial h}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2},$$

with $u = 0$ at $y = 0$ and $\frac{\partial u}{\partial y} = 0$ at $y = h$ (no stress). So

$$u = -\frac{g}{2\nu}\frac{\partial h}{\partial x} y(2h - y).$$

$$Q = \int_0^h u \, \mathrm{d}y = -\frac{g}{3\nu} h^3 \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0.$$

So,

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}(h^3 h_x)_x,$$

non-linear diffusion equation cf $\dot{h} = h_{xx}$.
This problem has solutions in a finite domain $|x| < L(t)$, $h(\pm L) = 0$, $V = \int_{-L}^{L} h \mathrm{d}x$ =const.
Suppose $L \sim t^{-\alpha}$, then $h \sim t^{-\alpha}$, $\frac{\partial}{\partial x} \sim 1/L \sim t^{-\alpha}$. So

$$(h^3 h_x)_x \sim t^{-6\alpha}, \qquad \dot{h} \sim t^{-1-\alpha}$$

so $\alpha = 1/5$. Try a solution of the form

$$h(x,t) = t^{-1/5} f(\eta), \quad \eta = Axt^{-1/5}.$$

Edge of drop $x = L$, $\eta = 1$. $L = (1/A)t^{-1/5}$.

$$\begin{aligned} \dot{h} &= -\frac{1}{5t^{6/5}}\left(f + \eta\frac{\partial f}{\partial \eta}\right), \\ h_x &= Af_\eta, \\ (h^3 h_x)_x &= A^2 t^{-6/5}(f^3 f_\eta)_\eta. \end{aligned}$$

So

$$\begin{aligned} -\frac{1}{5t^{6/5}}(f + \eta f_\eta) &= \frac{g}{3\nu}A^2 t^{-6/5}(f^3 f_\eta)_\eta \\ (f^3 f_\eta)_\eta &= -k(\eta f)_\eta, \quad k = \frac{3\nu}{5gA^2}. \end{aligned}$$

So we have $f^3 f_\eta = -k\eta f$. Constant of integration zero: need to check OK with full solution as $f_\eta \to \infty$ as $\eta \to 1$. Solution with $f = 0$, $\eta = 1$

$$f = \left(\frac{3k}{2}\right)^{1/3}(1-\eta^2)^{1/3}$$

where $V = \frac{1}{A}\int_{-1}^{1} f \mathrm{d}\eta$ determines A in terms of V, ν, g.

$$V = \frac{1}{A}\left(\frac{3}{2}\cdot\frac{3\nu}{5gA^2}\right)^{1/3}\int_{-1}^{1}(1-\eta^2)^{1/3}\mathrm{d}\eta,$$

5 Vorticity dynamics

5.1 Introduction

NS equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} + \boldsymbol{F} + \nu \nabla^2 \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0.$$

Take the curl

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \nabla \times \boldsymbol{F} + \nu \nabla^2 \boldsymbol{\omega},$$

or

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \nabla \times \boldsymbol{F} + \nu \nabla^2 \boldsymbol{\omega}.$$

This is the vorticity equation. In the absence of body forces, vorticity is enganced by stretching in the body of the fluid. It is created at the boundaries as flow has to make boundary velocity to free-stress velocity.

Stretching: $\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ (no ν). We have Kelvin's circulation theorem ($\nu = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_C \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{l} = 0$$

where C is a material curve

$$\begin{aligned}
&= \oint \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathbf{d}\boldsymbol{l} + \oint \boldsymbol{u} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{d}\boldsymbol{l}) \\
&= \int \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathbf{d}\boldsymbol{l} + \oint \boldsymbol{u} \cdot (\mathbf{d}\boldsymbol{l} \cdot \nabla \boldsymbol{u}) \\
&= \int -\nabla p \mathbf{d}\boldsymbol{l} \\
&= \int \mathbf{d}\boldsymbol{l} \cdot \nabla (\frac{1}{2}\boldsymbol{u}) \\
&= 0
\end{aligned}$$

if there are no body forces.

In particular if $\oint_C \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{l} \equiv 0$, then $\int_S \boldsymbol{\omega} \cdot \mathbf{d}\boldsymbol{S} = 0$. Vorticity can not be created from nothing. But action of viscosity at boundaries allows vorticity to appear.

Rayleigh problem

Recall $u(y,t) = Uf(y/\sqrt{2\nu t})$. So $\frac{\partial u}{\partial y} \sim t^{-1/2}$, but vorticity distribution $\sim t^{1/2}$. Vorticity is created at $t = 0$ and diffused into the interior.

Vorticity can be kept in the neighbourhood of the boundary by flow towards the boundary.

eq Flow towards a rigid boundary.

Far away from boundary, stagnation point flow

$$u = Ax, \quad v = -Ay, \quad \Psi \sim -Axy,$$

but this does not satisfy the boundary condition at $y = 0$. Try a solution with $\Psi \propto x$. In fact, let $\Psi = -xg(y)$, with $g = \frac{\mathrm{d}g}{\mathrm{d}y} = 0$ at $y = 0$ and $g \sim Ay$ as $y \to \infty$, then

$$u = xg',$$
$$v = -g.$$

Then

$$\omega = g''x, \qquad \boldsymbol{u}\cdot\nabla\omega = xg'g'' - xg'''$$
$$\nabla^2\omega = g''''x.$$

So

$$\boldsymbol{u}\cdot\nabla\omega = \nu\nabla^2\omega,$$

gives

$$x(g'g'' - gg''') = \nu g''''x$$

or

$$g'^2 - gg'' = \nu g''' + A^2$$

(as $g' \to +A$, $g'' \to 0$). Can simplify this. Let $y = (\nu/A)^{1/2}\eta$,

$$g = \sqrt{A\nu}G(\eta),$$
$$\Rightarrow \quad G_\eta^2 - GG_{\eta\eta} = G_{\eta\eta\eta} + 1,$$
$$G = G_\eta = 0 \text{ on } \eta = 0, \quad G \to \eta \text{ as } \eta \to \infty.$$

Can't solve this exactly, but numerical solution can be found

So there is a layer next to the wall of thickness $\sim (A/\nu)^{-1/2}$ (so like $Re^{-1/2}$), outside of which the flow is like stagnation point flow, displaced by a distance $.65(\nu/A)^{1/2}$.

Flow on a plane wall with suction

This is a simpler example of vorticity confinement.

Assume flow passes into wall (through a small hole for example) at speed v. So boundary condition at $y = 0$ is $u = 0, v = -W$, and at $y = a$ $u = 0, v = -W$. Suppose we seek a steady flow field with $u = u(y)$, $v = -W$ and constant pressure gradient $-\frac{\mathrm{d}p}{\mathrm{d}x} = G$ (independent of y). Clearly, the y-component on momentum equation is satisfied. The x-component is

$$\boldsymbol{u} \cdot \nabla u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -W\frac{\partial u}{\partial y}.$$

So

$$\begin{aligned}
&-W\frac{\partial u}{\partial y} = G + \nu\frac{\partial^2 u}{\partial y^2}, \quad u = 0, y = 0, a \\
\Rightarrow \quad &\frac{\partial u}{\partial y} = -\frac{G}{W} + c\mathrm{e}^{-Wy/\nu} \\
\Rightarrow \quad &u = -\frac{Gy}{W} + \frac{\nu c}{W}\left(1 - \mathrm{e}^{-Wy/\nu}\right), \quad u(0) = 0.
\end{aligned}$$

And so

$$\begin{aligned}
0 &= -\frac{Ga}{W} + \frac{\nu c}{W}\left(1 - \mathrm{e}^{-Wy/\nu}\right) \\
u &= -\frac{G}{W}(y - a) + \frac{\nu c}{W}\left(\mathrm{e}^{-Wa/\nu} - \mathrm{e}^{-Wy/\nu}\right)
\end{aligned}$$

and

$$\begin{aligned}
u &= \frac{G}{W}(a - y) + \frac{Ga}{W}\frac{\left(\mathrm{e}^{-Wa/\nu} - \mathrm{e}^{-Wy/\nu}\right)}{\left(1 - \mathrm{e}^{-Wa/\nu}\right)} \\
\frac{\partial u}{\partial y} &= \omega(y) = -\frac{G}{W} + \frac{Ga}{\nu}\frac{\mathrm{e}^{-Wy/\nu}}{\left(1 - \mathrm{e}^{-Wa/\nu}\right)}.
\end{aligned}$$

Let $Wa/\nu = Re$,

$$u = \frac{G}{Wa}(1 - y/a) + \frac{Ga}{W}\frac{\left(\mathrm{e}^{-Re} - \mathrm{e}^{-Re(y/a)}\right)}{\left(1 - \mathrm{e}^{-Re}\right)}$$

For small Re

$$
\begin{aligned}
u &= \frac{G}{W}(a-y) + \frac{Ga}{W}\frac{(1 - Re + \frac{1}{2}Re^2 - 1 + Re(y/a) - \frac{1}{2}Re^2(y/a)^2)}{1 - 1 + Re - \frac{1}{2}Re^2 + \ldots} \\
&= \frac{G}{W}(a-y) + \frac{Ga}{W}(1 + \frac{1}{2}Re + \ldots)(-1 + y/a + \frac{1}{2}Re(1 - y^2/a^2)) \\
&= \frac{G}{W}(a-y) + \frac{Ga}{W}\left[-1 + y/a + \frac{1}{2}Re(1 - y^2/a^2) + \frac{1}{2}Re(-1 + y/a) + \ldots\right] \\
&= \frac{Ga^2}{2\nu}(1 - y^2/a^2 - 1 + y/a + \ldots).
\end{aligned}
$$

This is the usual Poiseuille flow solution, correct as $Re \to 0$.

For large Re

If $Re \gg 1$, the second term is very small unless $y/a \sim 1/Re$, $\mathrm{e}^{-Re} \ll 1$. So if $y = (a/Re)z$ near $y = 0$, $u \sim (Ga/W)(1 - \mathrm{e}^{-z})$. When $y/a = \mathcal{O}(1)$, the second term is negligible and $u \sim (G/W)(a-y)$.
In the large Re case we get a boundary layer of thickness a/Re. Outside this, very small dissipation.

Note: boundary layer only at $y = 0$. At $y = a$ both diffusion and negative suction act to move vorticity away from the wall.

5.2 Joint effect of stretching and diffusion on a straight line vortex

Consider an axisymmetric flow of the form

$$
\boldsymbol{u} = \left(-\frac{\alpha r}{2}, u_\phi(r), \alpha z\right).
$$

Certainly $\nabla \cdot \boldsymbol{u} = 0$. Then $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = (0, 0, \omega(r))$, check!

The z-component of the vorticity equation is

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla \omega = \boldsymbol{\omega} \cdot \nabla(u_z) + \nu \nabla^2 \omega.$$

Other components, for example ω_x, $\frac{\partial \omega_x}{\partial t} + \boldsymbol{u} \cdot \nabla \omega_x - \nu \nabla^2 \omega_x = \boldsymbol{\omega} \cdot \nabla(u_x) = 0$, as u_x is not a function of z. THus we can take $\omega_x = 0$ consitently. Similarly for ω_y.
For ω, we get

$$\frac{\partial \omega}{\partial t} - \frac{\alpha r}{2} \frac{\partial \omega}{\partial r} = \alpha \omega + \nu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega}{\partial r} \right) \right]$$

General solution very difficult, but can find *vortex tube* solution in form $\omega = g(t) \mathrm{e}^{-r^2 f(t)}$,

$$\frac{\partial}{\partial t} \int_0^\infty \omega r \, \mathrm{d}r = \int_0^\infty \alpha \left(\frac{r^2}{2} \frac{\partial \omega}{\partial r} + r\omega \right) \, \mathrm{d}r + \nu \int_0^\infty \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \omega}{\partial r} \right) \, \mathrm{d}r.$$

The first term on the right hand side is $[r^2 \omega/2]_0^\infty = 0$. The second term is $\left[(1/r) \frac{\partial \omega}{\partial r} \right]_0^\infty = 0$, if $\frac{\partial \omega}{\partial r} \propto r^2$ at 0. So total vortex strength is conserved

$$\int_0^\infty g \mathrm{e}^{-r^2 f} r \, \mathrm{d}r = \frac{g}{f} \int_0^\infty \mathrm{e}^{-x^2} x \, \mathrm{d}x,$$

independent of time. So $g \propto f$ (take $g = f$).
So $\omega = f(t) \mathrm{e}^{-r^2 f(t)} \Rightarrow$

$$\begin{aligned} \dot{\omega} &= (\dot{f} - r^2 f \dot{f}) \mathrm{e}^{-r^2 f} \\ &= \alpha \left[f + \frac{1}{2} r f (-2rf) \right] \mathrm{e}^{-r^2 f} + \nu (-4f^2 + 4r^2 f^3) \mathrm{e}^{-r^2 f}. \end{aligned}$$

So there are two types of terms, which must separately balance.

$$\left. \begin{aligned} \dot{f} &= \alpha f - 4\nu f^2 \\ -r^2 f \dot{f} &= -r^2 f^2 \alpha + 4\nu r^2 f^3 \end{aligned} \right\} \text{these are the same.}$$

So need to solve

$$\dot{f} = \alpha f - 4\nu f^2.$$

Let $p = 1/f$, then $\dot{p} = -\dot{f}/f = -\alpha p + 4\nu$, and $p = 4\nu/\alpha + p_o \mathrm{e}^{-\alpha t}$. So

$$f = \frac{1}{4\nu/\alpha + p_o \mathrm{e}^{-\alpha t}} = \frac{\mathrm{e}^{\alpha t}}{(4\nu/\alpha)\mathrm{e}^{\alpha t} + p_o}.$$

So, as $t \to \infty$, $f \to \alpha/4\nu$.

The final state is

$$\omega = \frac{\alpha}{4\nu} \mathrm{e}^{-r^2\alpha/4\nu},$$

(constant arbitrary). So the size of the vortex tube $\propto (\nu/\alpha)^{1/2} \sim Re^{-1/2}$.

5.3 Hele-Shaw cell

Flow in a narrow gap if width h and uniform thickness, subject to imposed pressure gradients.

$$\frac{\partial}{\partial z} \sim \frac{1}{h} \gg \frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \quad \boldsymbol{u} = (u, v, w), \quad w \ll u, v.$$

Then,

$$\begin{aligned} 0 &= -\nabla p + \mu \nabla^2 \boldsymbol{u} \\ 0 &= -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial z^2}, \end{aligned}$$

and

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \\ 0 &= -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2}. \end{aligned}$$

As before, at leading order $\frac{\partial p}{\partial z} \approx 0$, as $w \ll u, v$ and $\frac{\partial p}{\partial z} \gg \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$. So,

$$\begin{aligned} & p \approx p(x, y) \\ & 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}, \quad u = 0, \ z = 0, h, \end{aligned}$$

gives

$$u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} z(1 - z).$$

Similarly,

$$v = -\frac{1}{2\mu} \frac{\partial p}{\partial y} z(1 - z).$$

Thus,

$$\bar{u} = \frac{1}{h}\int_0^h u \, \mathrm{d}z = -\frac{1}{12\mu}h^2\frac{\partial p}{\partial x}$$
$$\bar{v} = -\frac{1}{12\mu}h^2\frac{\partial p}{\partial y}.$$

So $\nabla \times (\bar{u}, \bar{v}, 0) = 0$. $\bar{\boldsymbol{u}}$ is *irrotational.* Can be used to simulate irrotational flow past 2-d bodies. But note that because p is single valued such flows will have no circulation.

6 Flow at Large Reynolds Number

6.1 The Prandtl and Euler limits

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p\ [+\boldsymbol{F}] + \mu\nabla^2\boldsymbol{u}.$$

What happens as $\mu \to 0$? We have seen that in the absence of boundaries, viscous forces can be very small. But viscosity can be introduced at boundaries, even in the limit $\mu \to 0$ ($Re \to \infty$). Euler equations are of lower order, can't describe flows satisfying all boundary conditions. If length scales become small as $Re \to \infty$ then viscosity is always important, even as $Re \to \infty$: This is the *Prandtl limit.* (*Euler limit* is $\mu = 0$).

At large Re, the viscosity appears in narrow layers. These are generally, though not always, to be found at boundaries, so called *boundary layers.* These have a definite small scale (the inner scale) that depends on Re. Outside the boundary layer the flow has length scale independent of Re (often called the outer solution).

We can solve problems involving large Re using *singular perturbation theory.*

6.2 Regular and Singular Perturbations

Consider the problem for $y = y(x)$

$$y'' + \epsilon y' = 1, \quad y(0) = y(1) = 0.$$

When $\epsilon = 0$, $y = (1/2)x(x-1)$. This satisfies all boundary conditions as y'' term is kept. For $\epsilon \neq 0$, can look for solution of the form

$$y = y_o + \epsilon y_1 + \ldots$$

At $\mathcal{O}(\epsilon^0)$, $y_o'' = 1$. At $\mathcal{O}(\epsilon)$, $y_1'' + y_o' = 0$. In general $y_{i+1}'' + y_i' = 0$. So

$$y_1 = -\int_0^x y_o(x')\,\mathrm{d}x' + \int_0^1 y_o \mathrm{d}x'$$

etc, etc. So solution is a regular expansion in powers on ϵ.

Now consider

$$-\epsilon y'' + y = x, \quad y(0) = y(1) = 0.$$

This has the exact solution

$$y = x - \frac{\sinh x/\sqrt{\epsilon}}{\sinh 1/\sqrt{\epsilon}}.$$

Note the non-integer power of ϵ. Put $\epsilon = 0$, then $y = x$ satisfies only one boundary condition. Near $x = 1$ we get a so called *inner solution.* Choose $\xi = \epsilon^{-1/2}(x-1)$. Then equation becomes

$$-y_{\xi\xi} + y = 1 + \epsilon^{-1/2}\xi,$$

Then $y = A\sinh\xi + B\cosh\xi + 1 + \epsilon^{-1/2}\xi$. Ignoring the $\epsilon^{-1/2}$ term,

$$y = A\sinh\xi + (1 - \cosh\xi).$$

Now we find A from *matching.* "The outer limit of the inner solution $\sim$ the inner limit of the outer solution" (Van Dyke's matching condition). The limit as $x \to 1$ of outer solution ~ 1. The limit as $\xi \to -\infty$ of inner solution $\sim 1 - ((A+1)/2)\mathrm{e}^{\xi}$. So $A = -1$. Then for $\xi = \mathcal{O}(1)$, $x - 1 = \mathcal{O}(\epsilon)$

$$y \sim -\sinh\left(\frac{x-1}{\sqrt{\epsilon}}\right) + 1 - \cosh\left(\frac{x-1}{\sqrt{\epsilon}}\right) + \dots$$

The actual solution is

$$\begin{aligned}
y &= x - \frac{\sinh(x/\sqrt{\epsilon})}{\sinh(1/\sqrt{\epsilon})} \\
&\sim 1 - \frac{\sinh\left((x-1)/\sqrt{\epsilon} + 1/\sqrt{\epsilon}\right)}{\sinh(1/\sqrt{\epsilon})} \\
&= 1 - \left[\frac{\sinh((x-1)/\sqrt{\epsilon})\cosh(1/\sqrt{\epsilon})}{\sinh(1/\sqrt{\epsilon})} + \frac{\cosh((x-1)/\sqrt{\epsilon})\sinh(1/\sqrt{\epsilon})}{\sinh(1/\sqrt{\epsilon})}\right] \\
&\sim 1 - \sinh\left(\frac{x-1}{\sqrt{\epsilon}}\right) - \cosh\left(\frac{x-1}{\sqrt{\epsilon}}\right).
\end{aligned}$$

So we have a boundary layer of thickness $\sim \sqrt{\epsilon}$ as $\epsilon \to 0$. Quite often the boundary layer $\sim Re^{-1/2}$ at large Re.

6.3 The boundary layer equation for steady flow

Prototype Problem: The Blasius boundary layer

We want to solve steady state NS equations for $\boldsymbol{u} = (u, v, w)$ with $u = v = 0$ at $y = 0$, $x > 0$ and $(u, v) \to (U, 0)$ as $y \to \infty$. We have the x-components

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u,$$

and the y-component

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v.$$

We imagine that there is a layer of thickness $\delta(x)$ in which u differs from uniform flow. Suppose $\delta(x) \ll x$ (verify later).

Then

$$\frac{\partial^2 u}{\partial y^2} \sim \frac{u}{\delta^2} \gg \frac{\partial^2 u}{\partial x^2},$$

etc. $u/x \sim v/\delta$ so $v \ll u$. So

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \sim \frac{u^2}{x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \sim \frac{u^2\delta}{x^2} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{8}$$

So if $-\frac{\partial p}{\partial x} \sim \frac{p}{x}$ balances other terms in (7) then $-\frac{\partial p}{\partial y} \sim \frac{p}{\delta}$ is unbalanced. So p does not

vary across boundary layer. At the edge fo the layer, no pressure gradient at leading order so can take $\frac{\partial p}{\partial x} = 0$ throughout.
The equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}.$$

So $|u\frac{\partial u}{\partial x}| = |v\frac{\partial u}{\partial y}| \sim U^2/x \sim \nu U/\delta^2$. Then $\delta = \sqrt{\nu x/U}$ (or $\delta/x \sim Re(x)^{-1/2}$, $Re(x) = U/\nu x$).
Define a stream function $\psi = U\delta(x)f(\eta)$, $\eta = y/\delta(x)$, $u = Uf_\eta$.

$$\begin{aligned} u_x &= -\frac{U\delta'}{\delta}\eta f_{\eta\eta} \\ u_y &= \frac{U}{\delta}f_{\eta\eta} \\ u_{yy} &= \frac{U}{\delta^2}f_{\eta\eta\eta} \end{aligned}$$

$$\begin{aligned} v &= -\frac{\partial\psi}{\partial x} \\ &= -U\delta' f + U\delta'\eta f_\eta \end{aligned}$$

So

$$\left(-\frac{U^2\delta'}{\delta}\eta f_\eta f_{\eta\eta}\right) + \left(-\frac{U^2\delta'}{\delta}f_{\eta\eta}f + \frac{U^2\delta'}{\delta}\eta f_\eta f_{\eta\eta}\right) = \frac{\nu U}{\delta^2}f_{\eta\eta\eta}.$$

Or

$$-\frac{U^2\delta'}{\delta}ff_{\eta\eta} = \frac{\nu U}{\delta^2}f_{\eta\eta\eta}.$$

Thus we need $\delta'/\delta \sim \nu U/\delta^2$. Let $\delta \sim x^m$, then $m/x \sim \nu U/x^{2m}$, $m = 1/2$. Let $\delta = (\nu x/U)^{1/2}$ as previously suggested. Then $\delta'/\delta = 1/2x$, $1/\delta^2 = U/\nu x$. So

$$\frac{1}{2}(-\eta f_\eta f_{\eta\eta} - ff_{\eta\eta} + \eta f_\eta f'_{\eta\eta}) = f_{\eta\eta\eta},$$

or

$$-\frac{1}{2}ff_{\eta\eta} = f_{\eta\eta\eta}$$

with $f = f_\eta = 0$ at $\eta = 0$ and $f_\eta \to 1$ as $\eta \to \infty$. We can't solve this!
The thickness of the boundary layer is provided by the quantity

$$\delta = \int_0^\infty \left(1 - \frac{u(y)}{U}\right) \mathrm{d}y \approx 1.72\left(\frac{\nu x}{U}\right)^{1/2}.$$

More general boundary layers

Suppose the flow outside the boundary layer is $(U(x), V(x)), 0)$. Then to a good approximation outside the boundary layer we can ignore y-derivatives and get

$$U\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

The x-component of the momentum equations in the boundary layer is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$

and the y-component

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2}.$$

As for the Blasius layer, $v \ll u$, $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, so $p = p(x)$ in the boundary layer. Thus x-component gives

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

For simple form of U, eg $U = cx^m$, coresponding to eg flow past a wedge

we can find a similarity solution of the form

$$\psi = U(x)\delta(x)f(\eta/\delta(x)).$$

So,

$$u = \frac{\partial \psi}{\partial y} = U(x)f'(\eta), \quad \eta = y/\delta.$$

In boundary layer equation

$$uu_x \sim \frac{U^2}{x}$$
$$vu_y \sim \frac{U^2}{x}.$$

So $U\delta^2/\nu x = 1$, $\delta = \sqrt{\nu x/U(x)}$.
Let $\psi = U(x)\delta(x)f(\eta)$, $\eta = y/\delta(x)$, $U = cx^m$, say. Then

$$u = \frac{\partial\psi}{\partial y} = U(x)f',$$

and

$$\begin{aligned} v = -\frac{\partial\psi}{\partial x} &= -cx^m\left(\frac{m\delta}{x} + \delta'\right)f - cx^m\delta\cdot\left(-\frac{y\delta'}{\delta^2}\right)f' \\ &= -cx^m\left[\left(\frac{m\delta}{x} + \delta'\right)f + \eta f'\delta'\right]. \end{aligned}$$

So

$$\begin{aligned} uu_x + vu_y &= Uf'\left(U'f' - \frac{Uy\delta'}{\delta^2}f''\right) + \frac{Uf''}{\delta}\left[-U\left(\frac{m\delta}{x} + \delta'\right)f + U\eta\delta' f'\right] \\ &= \frac{mU^2}{x}f'^2 - \frac{\delta' U^2\eta}{\delta}f'f'' - \frac{U^2 f''}{\delta}\left(\frac{m\delta}{x} + \delta'\right)f + \frac{U^2\delta'}{\delta}\eta f' f''. \end{aligned}$$

$\delta \propto x^k$, $\sqrt{x/U} \propto x^{(1-m)/2}$, so $k = (1-m)/2$.

$$\begin{aligned} u_x &= U' f_\eta^2 - \frac{U\delta'}{\delta}\eta f_{\eta\eta} \\ u_y &= \frac{U}{\delta}f_{\eta\eta} \\ u_{yy} &= \frac{U}{\delta^2}f_{\eta\eta\eta} \end{aligned}$$

$$\begin{aligned} v &= -\frac{\partial\psi}{\partial x} \\ &= -(U'\delta + U\delta')f + U\delta'\eta f_\eta \end{aligned}$$

So

$$\left(UU'f_\eta^2 - \frac{U^2\delta'}{\delta}\eta f_\eta f_{\eta\eta}\right) + \left(-\frac{U(U'\delta + U\delta')}{\delta}f_{\eta\eta}f + \frac{U^2\delta'}{\delta}\eta f_\eta f_{\eta\eta}\right) = UU' + \frac{\nu U}{\delta^2}f_{\eta\eta\eta}.$$

Or

$$UU'(f_\eta^2 - ff_{\eta\eta}) - \frac{U^2\delta'}{\delta}ff_{\eta\eta} = UU' + \frac{\nu U}{\delta^2}f_{\eta\eta\eta}.$$

Or

$$mc^2x^{2m-1}(f_\eta^2 - ff_{\eta\eta}) - kc^2x^{2m-1}ff_{\eta\eta} = c^2mx^{2m-1} + c^2x^{2m-1}f_{\eta\eta\eta}.$$

Or, cancelling

$$f_{\eta\eta\eta} + m - mf_\eta^2 + \frac{1}{2}(m+1)ff'' = 0.$$

The *Falkner - Skan* equation (1930) boundary conditions $f = f_\eta = 0$ at $\eta = 0$, $f_\eta \to 1$ as $\eta \to \infty$. $m = 1$, $\alpha = \pi/2$ (already done), $k = 0$ boundary layer of constant thickness.

Problems with negative m

If $U \propto cx^m$, $m < 0$, then the external shear decelerates (pressure increases with x) and if $m = -0.0904$ there is no stress at the wall at all $f_{\eta\eta} = 0$. No sensible solutions (with flow in one direction) can be found for $m < -0.0904$. For $-0.0904 < m < 0$ there are some other solutions found, which reverse direction and are not observed. $m < -1$ no sensible solutions at all.

Jeffery-Hamel Flow (in diverging channel)

Another example of failure of loundary layer theory to account for actual solution. (cf example sheet question - different scaling.) Assume radial flow

$$u_r = \frac{1}{r}F(\phi), \quad F(0) = F_o.$$

$R = \alpha F_o/\nu$ and $\phi = \alpha\eta$. So $-1 < \eta < 1$, $F = F_o f(\eta)$. Substitutiong in as before, get *exact* equation

$$f_{\eta\eta\eta} + 2\alpha R f f_\eta + 4\alpha^2 f_\eta = 0,$$

$f(0) = 1$, $f(1) = f(-1) = 0$. Suppose symmetric flow profile so $f_\eta(0) = 0$,

$$\begin{aligned} &f'' + \alpha R f^2 + 4\alpha^2 f + d = 0, \\ &f'(\cdot) = 0, \\ &f'^2 + \frac{2\alpha R}{3} f^3 + 4\alpha^2 f^2 + 2df - c = 0, \end{aligned}$$

where $c = f'(1)^2 \geq 0$. Since $f'(0) = 0$,

$$0 = \frac{2\alpha R}{3} + 4\alpha^2 + 2d - c,$$

and so eliminating d, get

$$\begin{aligned} f'^2 &= -\frac{2\alpha R}{3} f^3 - 4\alpha^2 f^2 + c + f\left(\frac{2\alpha R}{3} + 4\alpha - c\right), \\ &= (1-f)\left[\frac{2}{3}\alpha R(f^2 + f) + 2\alpha f + c\right] \\ &= R(f). \end{aligned}$$

Suppose we have a normal type of "boundary layer" flow, with $f \leq 1$ everywhere. Then

$$\int_0^1 \frac{1}{\sqrt{R}}\,\mathrm{d}f = \int_o^1 \mathrm{d}\eta = 1 = \int_0^1 \frac{\mathrm{d}f}{\sqrt{1-f}\sqrt{\frac{2}{3}\alpha R(f^2+f) + 2\alpha^2 f + c}}.$$

Since $c \geq 0$ and $0 \leq f \leq 1$,

$$1 < \int_0^1 \frac{\mathrm{d}f}{\sqrt{f(1-f^2)}\sqrt{2\alpha R/3}},$$

or $\alpha R \leq 10.31$.
For larger αR flow must reverse directions.

The momentum Jet - A solvable example. Flow through a nozzle

Assume that force applied at $x = 0$ to produce momentum flux F. So at $x \approx 0$,

$$F = \rho \int_{-\infty}^{\infty} u^2 \, \mathrm{d}y.$$

This is in fact a constant:

$$\begin{aligned}
\frac{1}{2\rho}\frac{\mathrm{d}F}{\mathrm{d}x} &= \int_{-\infty}^{\infty} u u_x \, \mathrm{d}y \\
&= \nu \underbrace{\int_{-\infty}^{\infty} u_{yy} \, \mathrm{d}y}_{=0} - \int_{-\infty}^{\infty} v u_y \, \mathrm{d}y \\
&= \underbrace{[vu]_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} u v_y \, \mathrm{d}y \\
&= -\int_{-\infty}^{\infty} u u_x \, \mathrm{d}y \\
&= 0.
\end{aligned}$$

So indeed F is constant, as is physically sensible. So as before $\psi = U(x)\delta(x)f(\eta)$, $\eta = x/\delta$, $u = U(x)f^*_\eta$. So $u^2 \sim U^2$, $(1/\rho)F \sim U^2\delta$ =const. Also, $\delta^2 = \nu x/U$, $U \sim x^k$, $k = (1/2)(1-m)$, and $2m + k = 0$, $2m + (1/2)(1-m) = 0$, $m = -1/3$, $k = -2/3$. So $\mathcal{F} = F/\rho \sim U^2\sqrt{\nu x/U} \Rightarrow U = (\mathcal{F}/\nu x)^{1/3}$. Similarly $\delta = (\nu^2 x^2/\mathcal{F})^{1/3}$. F-S equation with no flow at ∞, $m = -1/3$,

$$f_{\eta\eta\eta} + \frac{1}{3}f'^2 + \frac{1}{3}ff'' = 0,$$

f odd in η (as u is even), $f(0) = 0$, $\int_{-\infty}^{\infty} f_\eta^2 \,\mathrm{d}\eta = 1$ (exercise). And so

$$\begin{aligned} f_{\eta\eta} + \frac{1}{3}(f f_\eta) &= 0 \\ \Rightarrow \quad f_\eta + \frac{1}{6} f^2 &= \frac{1}{6} k^2, \end{aligned}$$

say.

$$\frac{\mathrm{d}f}{k^2 - f^2} = \frac{\eta}{6},$$

so $1/k \tanh^{-1} f/k = \eta/6$ (const $= 0$), $f = k \tanh(ky/6)$. And

$$\int_{-\infty}^{\infty} f'^2 \,\mathrm{d}\eta = \frac{k^5}{36} \int_{-\infty}^{\infty} \operatorname{sech}^4 \frac{k\eta}{6} \,\mathrm{d}\eta = 1 = \frac{2k^3}{9}.$$

$f = (9/2)^{1/3} \tanh\left[(9/2)^{1/3}(1/6)\eta\right]$, $f_\eta = k^2/6 \sinh^2 k\eta$.

Mass flux

$$M = \rho \int_{-\infty}^{\infty} u \,\mathrm{d}y = \rho U \delta \int_{-\infty}^{\infty} f_\eta^p \,\mathrm{d}y = 2k\rho U\delta \propto x^{1/3},$$

increases with x due to entrainment of flow outside layer.

Effective Reynolds number of the momentum jet

$$u \sim \nu x^{-1/3} \quad \delta \sim x^{2/3}.$$

So the effective $Re \propto u\delta \sim x^{1/3}$. In fact the maximum velocity times jet width can be derived to be $\propto (\mathcal{F}x/\nu^2)^{1/3}$ (exercise). So effective Re increases with x and boundary layer approximation gets better and better. Since Re increases, we may expect the possibility of instability (as shear flow becomes unstable at sufficiently large Re). In fact, jets of this kind are always tubulent for enough shear downstream.

Note that the geometry is crucial. For a cylindrical momentum jet

$$\mathcal{F} = \int_0^\infty u^2 r \, \mathrm{d}r.$$

If the jet has thickness $\delta(x)$, then $\mathcal{F} \sim U^2(x)\delta^2(x)$. Can show (exercise) that in the bounary layer approximate $u = (u(x,t), v(x,t), 0)$,

$$uu_x + vu_r \approx \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

We can show that $\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}x} = 2\int_0^\infty uu_x r \, \mathrm{d}r = 0$. So $U\delta = \mathcal{F}^{1/2}$ and as always $U^2/x \sim \nu U/\delta^2$. So $U = \mathcal{F}/\nu x$, $\delta = \nu x/\mathcal{F}^{1/2}$. Jet with increases linearly. For boundary layer approximation to be valid $\nu \ll \mathcal{F}^{1/2}$. Then we can find solution with $u = (1/r)\frac{\partial \Psi}{\partial r}$, $v = -(1/r)\frac{\partial \Psi}{\partial x}$, $\Psi = U\delta^2 f(\eta)$, $u = \mathcal{F}^{1/2}/\delta \cdot (1/\eta) f_\eta$, and $\int_0^\infty (1/\eta) f_\eta^2 \, \mathrm{d}\eta = 1$. Hard to solve though.

6.4 Boundary layers at a free surface

Suppose we have a free surface that for some reason (eg large gravity, large surface tension) may be considered flat.

Then at the surface

$$v = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

So $\frac{\partial v}{\partial x} = 0$, so consistent to take $w_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$. A flat surface with viscosity is consistant with irrotational flow. So a boundary layer can only develop when the free surface is curved.

Since the boundary layer acts to maintain correct values of the stress (including $\frac{\partial u}{\partial n}$ tangential) rather than allowing the u_t itself to match, we have a much weaker layer in which the velocity canges by δ (boundary layer thickness) rather than $\mathcal{O}(1)$. How much dissipation in such a boundary layer?

$$\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 \, \mathrm{d}y \sim U^2\delta$$

as $\frac{\partial u}{\partial y}$ is of order 1.
Compare with dissipation for a boundary layer at a rigid boundary where u_{bl} is $\mathcal{O}(1)$, so $\frac{\partial u}{\partial y} \sim U/\delta$,

$$\int_0^y \left(\frac{\partial u}{\partial y}\right)^2 \, \mathrm{d}y \sim \frac{U^2}{\delta},$$

which is much larger than mainstream dissipation.
Consider the use of a spherical bubble with radius a. Assume that $Re = Ua/\nu$, but that the surface tension forces to keep the bubble spherical. This works ok in water for bubble sizes up to about 0.05 cm. Assume that the Re is large for these bubbles (true for the larger ones).
Experimentally, we see that no boundary layer separation occurs and that we have

So in steady motion, if drag is D, then UD =energy dissipated in free stream (larger than bubble). So calculate dissipation due to irrotational flow from potential theory for irrotational flow past a sphere of radius a.
We have

$$\phi = -\frac{1}{2}\frac{Ua^3}{R^2}\cos\theta.$$

So total dissipation is

$$\int e_{ij}e_{ij} \, \mathrm{d}V = \int \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \, \mathrm{d}V$$

$$2\pi \int_a^\infty \int_0^\pi R^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}R \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)$$

NB

$$\int \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} = \int \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right) - \underbrace{\int u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{=0 \text{ as } \nabla^2\phi=0}$$

$$= \int \nabla^2 \left(\frac{1}{2}q^2\right) \mathrm{d}V, \quad \text{where } q = |\boldsymbol{u}|$$

$$= \int_{R=a} \boldsymbol{n}\cdot\nabla\left(\frac{1}{2}q^2\right) \mathrm{d}S$$

$$= \int_{R=a} -\frac{\partial}{\partial R}\left(\frac{1}{2}q^2\right) \mathrm{d}S$$

$$\phi = -\frac{1}{2}\frac{Ua^3}{R^2}, \quad u_R = \frac{Ua^3}{R^2}\cos\theta, \quad \frac{1}{2}\frac{Ua^3}{R^2}\sin\theta.$$

So,

$$q^2|_{R=a} = \frac{a^3U^2}{R^3}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right), \quad \left.\frac{\partial}{\partial R}(q^2)\right|_{R=a} = -\frac{3U^2}{a}(\cos^2\theta + \frac{1}{4}\sin^2\theta)$$

So,

$$6\pi a^2 \int \frac{U}{a}\sin\theta(\cos^2\theta + \frac{1}{4}\sin^2\theta)\,\mathrm{d}\theta = 12\pi\mu a U^2.$$

So $D = 12\pi\mu a U$. Note that this is three times the dissipation for stokes flow. We knew it had to be larger as Stokes flow has minimum dissipation, by earlier result.

6.5 Boundary layer separation

We have seen how when the free stream velocity decelerates, the boundary layer equations do not work well. Large *Re* flow behind a bluff body usually involves a wake with reversed flow behind the body (often unsteady) - see handouts earlier in the term.

The tangential component of the free stream decreases sufficiently rapidly that boundary layer can no longer be sustained. In fact for general incoming boundary layer, very little deceleration is needed for separation - special Stokes flow solution. Assume a particular form of input and very gradual deceleration. Separation usually occurs near point of maximum cross stream distance, or at a sharp edge.

Need a straight line here as otherwise flow would tend to zero at separation point and that would lead to earlier separation. It seems probable that the point of separation is point of zero wall friction, but not clear.

Actual flow field appears to be affected by history of the boundary layer - when separation occurs global flow field changes, as does the pressure distribution. Using conventional boundary layer theory, we get a singularity near the stagnation point.

Some progress was made in the last 30 years - need to look at mulit-boundary layer theory (so called *triple-deck* theory). Outer scale - mainstream flow upper deck adverse pressure gradient modified. Main deck - $Re^{-1/2}$ - usual boundary layer scale. Lower deck - $Re^{-5/8}$(!) - local Reynolds number small. Can solve near stagntaion point.

Wakes behind bodies at large Re

Eventually the direct effect of body has disappeared so flow outside wake $\sim$ free shear flow.

Observed that the velocity in the wake does not differ much from the free stream velocity U - this could not happen for a rigid body as $u \to 0$ at boundary. So ignore down stream diffusion compared with advection (y, z scales are small compared with x scale). Also, $uu_x \sim UU_x$ if $|u - U| \ll U$. Then we can solve for the down stream flow (no imposed pressure gradient) by investigating.

$$U\frac{\partial u}{\partial x} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

At large x, solution can be found to which observation indicate all solution tend, in form

$$U - u \to \frac{QU}{4\pi\nu x}\mathrm{e}^{-Ur^2/4\nu x},$$

where Q is a contant. NB $2\pi\int_0^\infty (U-u)r\ \mathrm{d}r = Q$, independent of x. So velocity deficit is defined by Q and so presumably by the boundary conditions upstream. Q can be related to the drag D on the body using the momentum theorem (Bathcelor p350).

There is a flux of momentum into the control surface which is related to the total force on the body. So in fact $D = \rho U Q$.

7 Shear Flow Instabilities

7.1 Instability of a vortex sheet

Now for the time being, abandon viscosity and consider a tangential discontinuity in the flow (allowed without diffusion).

$$\begin{aligned} &u = \mp U/2,\ x \gtrless 0 \\ &v = 0 \\ &p = p_o \text{ (uniform)}. \end{aligned}$$

Most such shear layers are not planar, but solve this problem first - then consider this a local solution. There is no vorticity for $y \neq 0$, but $\int_e \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = U\delta x = w\delta x\delta y$. So $\int_{0-}^{0+} w \ \mathrm{d}y = U$ and so the vorticity is all contained in the interface.
In reality the interface diffuses due to vorticity, and we have seen this happen at a diffusive rate so that thickness $h \propto \sqrt{\nu t}$. Ignore this for the moment and treat interface as a line - check validity later. Now suppose the interface displaced to $y = \eta(x, z, t)$, or that interface satisfies $F(\boldsymbol{x}, t) = y - \eta(x, z, t) = 0$. On each side of the interface, there is no vorticity and so

$$\begin{aligned} y > 0 \quad \boldsymbol{u} &= \left(-\frac{1}{2}U + \frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_1}{\partial y}, \frac{\partial \phi_1}{\partial z}\right) \\ y < 0 \quad \boldsymbol{u} &= \left(\frac{1}{2}U + \frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_2}{\partial y}, \frac{\partial \phi_2}{\partial z}\right). \end{aligned}$$

And $\nabla \cdot \boldsymbol{u} = 0 \Rightarrow \nabla^2 \phi_1 = \nabla^2 \phi_2 = 0$,

$$\left.\begin{aligned} \phi_1 \to 0,\ y \to +\infty \\ \phi_2 \to 0,\ y \to -\infty \end{aligned}\right\}.$$

Need to apply boundary conditions at $y = \eta$.

First condition: kinematic - particle at $F = 0^+$ remains at $F = 0^+$, particle at $F = 0^-$ remains at $F = 0^-$.

$$\begin{aligned}
\Rightarrow \quad & \frac{\mathrm{D}_1 F}{\mathrm{D}t} = (\partial_t + \boldsymbol{u}_1 \cdot \nabla)F = 0, \quad F = 0^+ \\
& \frac{\mathrm{D}_2 F}{\mathrm{D}t} = (\partial_t + \boldsymbol{u}_2 \cdot \nabla)F = 0, \quad F = 0^- \\
& \frac{\partial F}{\partial t} - \frac{1}{2}U\frac{\partial F}{\partial x} + (\nabla\phi_1 \cdot \nabla)F = 0 \\
& -\eta_t + \frac{1}{2}U\eta_x + \frac{\partial \phi_1}{\partial y} - \nabla\phi_1 \cdot \nabla\eta = 0, \quad \text{at } y = \eta^+
\end{aligned}$$

Now suppose η is small, then perturbation velocity is small, of order η. So if we neglect products of small quantities we get

$$\eta_t - \frac{1}{2}U\eta_x = \frac{\partial \phi_1}{\partial y}, \quad \text{at } y = \eta^+.$$

But

$$\left.\frac{\partial \phi_1}{\partial y}\right|_{y=\eta} = \left.\frac{\partial \phi_1}{\partial y}\right|_{y=0} + \eta \left.\frac{\partial^2 \phi_1}{\partial y^2}\right|_{y=0} + \dots$$

neglect all of the terms but the first. So we finally have a boundary condition at $y = \eta^+$

$$\frac{\partial \eta}{\partial t} - \frac{1}{2}U\frac{\partial \eta}{\partial x} = \frac{\partial \phi_1}{\partial y} + \mathcal{O}(\eta^2).$$

Similarly at $y = 0^-$

$$\frac{\partial \eta}{\partial t} + \frac{1}{2}U\frac{\partial \eta}{\partial x} = \frac{\partial \phi_2}{\partial y} + \mathcal{O}(\eta^2)$$

Next boundary condition: Bernoulli. Recall

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} = \text{const},$$

$(\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla\boldsymbol{u} = -\nabla p/\rho)$. So at $y = \eta^+$

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2} \underbrace{\left(-\frac{U}{2} + \frac{\partial \phi_1}{\partial x}\right)^2}_{\underbrace{\frac{U^2}{4}}_{\text{const}} - 2U\frac{\partial \phi}{\partial x} + \underbrace{\left(\frac{\partial \phi}{\partial x}\right)^2}_{\text{small}}} + \frac{1}{2}\frac{\partial \phi}{\partial y} + \frac{p}{\rho} = \text{const}.$$

So neglecting small terms

$$\frac{\partial \phi_1}{\partial t} - \frac{U}{2}\frac{\partial \phi_1}{\partial x} = p_i \quad \text{at } y = \eta^+,$$

and so at $y = 0^+$

$$\frac{\partial \phi_1}{\partial t} - \frac{U}{2}\frac{\partial \phi_1}{\partial x} = \frac{p_1}{\rho} + \text{terms of order } \eta^2.$$

Similarly at $y = 0^-$

$$\frac{\partial \phi_2}{\partial t} + \frac{U}{2}\frac{\partial \phi_2}{\partial x} = \frac{p_2}{\rho}.$$

And $p_1 = p_2$ (no surface tension etc). Can generalize this later.
So the problem to solve is

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0$$
$$(\partial_t - \frac{U}{2}\partial_x)\eta = \left.\frac{\partial \phi_1}{\partial y}\right|_{y=0}, \quad (\partial_t - \frac{U}{2}\partial_x)\eta = \left.\frac{\partial \phi_2}{\partial y}\right|_{y=0}.$$

So

$$\left.\left(\partial_t - \frac{U}{2}\partial_x\right)\phi_1\right|_{y=0} = \left.\left(\partial_t - \frac{U}{2}\partial_x\right)\phi_2\right|_{y=0}$$

and

$$\phi_1 \to 0, \quad y \to \infty$$
$$\phi_2 \to 0, \quad y \to -\infty.$$

This is a linear problem. Seek a seperable solution of the form

$$\eta = \Re\hat{\eta}\mathrm{e}^{\sigma t + ikx + imz}.$$

Then we can look for $\phi_1, \phi_2 \propto \mathrm{e}^{\sigma t+ikx+imz}$ too. eg $\phi_1 = \Re\hat{\phi}_1(y)\mathrm{e}^{\sigma t+ikx+imz}$.

$$\nabla\phi_1 = 0 \quad \Rightarrow \quad \left(\frac{\mathrm{d}^2\hat{\phi}_1}{\mathrm{d}y^2} - (k^2+m^2)\hat{\phi}_1\right)\mathrm{e}^{\sigma t+ikx+imz} = 0.$$

So $\hat{\phi}_1 = A_1\mathrm{e}^{-\gamma y}$ as $\phi_1 \to 0$, $y \to +\infty$. Similarly

$$\frac{\mathrm{d}^2\hat{\phi}_2}{\mathrm{d}y^2} - \gamma^2\hat{\phi}_2 = 0, \qquad \hat{\phi}_2 = A_2\mathrm{e}^{\gamma y}.$$

Substitute in to get

$$\left(\sigma - \frac{iUk}{2}\right)\hat{\eta} = -\gamma A_1$$
$$\left(\sigma + \frac{iUk}{2}\right)\hat{\eta} = \gamma A_2$$
$$\left(\sigma - \frac{iUk}{2}\right)A_1 = \left(\sigma + \frac{iUk}{2}\right)A_2.$$

$$\left(\sigma - \frac{iUk}{2}\right)^2\hat{\eta} = -\gamma\left(\sigma - \frac{iUk}{2}\right)A_1$$
$$\left(\sigma - \frac{iUk}{2}\right)^2\hat{\eta} = \gamma\left(\sigma + \frac{iUk}{2}\right)A_2,$$

or

$$\left(\sigma + \frac{iUk}{2}\right)^2 + \left(\sigma + \frac{iUk}{2}\right)^2 = 0,$$

so $\sigma^2 = U^2k^2/4$, $\sigma = \pm Uk/2$. Then

$$\eta = \left(\hat{\eta}_1 e^{Ukt/2} + \hat{\eta}_2 e^{-Ukt/2}\right) e^{ikx+imz},$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are determined by the initial consitions

$$\eta(0) = (\hat{\eta}_1 + \hat{\eta}_2)e^{(\quad)}, \quad \dot{\eta}(0) = \frac{Uk}{2}(\hat{\eta}_1 = \hat{\eta}_2)e^{(\quad)}.$$

7.2 Ultra-violet catastrophe - how to remove

Clearly all disturbances grow - largest growth rate for smallest wavelength! - *utraviolet catastrophe*. This is a highly singular situation, and we expect that other physcial effects will resolve the situation. Clearly viscosity has to play a role - see later. Another mechanism that can stabilize short wavelengths is *surface tension*. If the two fluids are different, then there is a pressure discontinuity across the surface proportional to the local curvature. Consider a simple case where $y = \eta(x,t)$ (no z-dependence). Then if η is small the curvature $\sim \eta_{xx}$.

Effects of surface tension between two fluids

For simplicity, assume same density, $\rho_1 = \rho_2$. The pressure difference across curved boundary propotional to curvature. (assume now $m = 0$, so 2-d disturbance $\eta = \eta(x,t)$.)

$\rho_1 - \rho_2 = \gamma k$. For $y = \eta(x,t)$, with η small, $k \sim \eta_{xx}$. So

$$\rho\left(\frac{\partial \phi_i}{\partial t} - \frac{U}{2}\frac{\partial \phi}{\partial x}\right) = p_1 = \rho\left(\frac{\partial \phi_2}{\partial t} + \frac{U}{2}\frac{\partial \phi_2}{\partial x}\right) + \gamma \eta_{xx}.$$

So

$$\left(\sigma - \frac{iUk}{2}\right)\hat{\eta} = -|k|A_1$$
$$\left(\sigma + \frac{iUk}{2}\right)\hat{\eta} = |k|A_2$$

$$\left(\sigma - \frac{iUk}{2}\right)A_1 - \left(\sigma + \frac{iUk}{2}\right)A_2 = -\frac{\gamma k^2}{\rho}.$$

So

$$\left(\sigma - \frac{iUk}{2}\right)^2 + \left(\sigma + \frac{iUk}{2}\right)^2 = -T|k|^3,$$

where $T = \gamma/\rho$, or

$$2\sigma^2 - \frac{U^2k^2}{2} = -T|k|^3.$$

For sufficiently large $|k|$ we find that σ imaginary - so these modes are stable (capillary waves). We can also consider effects of viscosity, and perform a boundary layer analysis for a very thin sheet, in presence of viscosity. This gives a modified dispersion relation for the growing mode.

$$\sigma^+ = \frac{1}{2}\left(\sqrt{k^4\nu^2 + U^2k^2} - k^2\nu\right).$$

All modes are now unstable, but this too has a maximum at finite k, real for all k.
When there is a mode of maximum growth rate, a general initial condition has all Fourier modes, and this mode will grow most rapidly. So the growing disturbance will have a definite scale corresponding to this wavenumber.

Effect of viscosity when layer has a finite thickness d ($\propto \sqrt{\nu t}$)

Then disturbances with $kd \ll 1$ will see discontinuous shear layer. For inviscid theory to be relevant, we must have rate of growth of disturbance with $kd \ll 1$ faster than growth rate of inteface

$$\frac{\dot{d}}{d} = \frac{1}{2t} \sim \frac{\nu}{2d^2}.$$

So all is ok if $Uk \gg \nu/d^2$ or $Ukd^2/\nu \gg 1$. So if $kd \ll 1$, we need $Re = Ud^2/\nu \gg 1/(kd)$. Gets better and better as d increases.

7.3 Shear flow and buoyancy

2 Fluids of different densities ρ_1, ρ_2.

Ignore surface tension. We now have a *hydrostatic pressure gradient.* So Bernoulli gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \frac{p}{\rho} = \text{const.}$$

So at $y = \eta^+$

$$\frac{\partial \phi_1}{\partial t} - \frac{1}{2}U\frac{\partial \phi_1}{\partial x} + g\eta + \frac{p_1}{\rho_1} = \text{const},$$

and at $y = \eta^-$

$$\frac{\partial \phi_2}{\partial t} + \frac{1}{2}U\frac{\partial \phi_2}{\partial x} + g\eta + \frac{p_2}{\rho_2} = \text{const.}$$

In the absence of gravity, we just have

$$
\begin{aligned}
&\rho_1\left(\frac{\partial\phi_1}{\partial t}-\frac{U}{2}\frac{\partial\phi_2}{\partial x}\right)=\rho_2\left(\frac{\partial\phi_2}{\partial t}+\frac{U}{2}\frac{\partial\phi_2}{\partial x}\right)\\
&\rho_1\left(\sigma-\frac{iUk}{2}\right)A_1=\rho_2\left(\sigma+\frac{iUk}{2}\right)A_2\\
&\rho_1\left(\sigma-\frac{iUk}{2}\right)^2+\rho_2\left(\sigma+\frac{iUk}{2}\right)^2=0\\
&(\rho_1+\rho_2)\left(\sigma^2-\frac{U^2k^2}{4}\right)+(\rho_2-\rho_1)iUk\sigma=0
\end{aligned}
$$

And p is continuous (no surface tension). Retain $g\eta$ term as linear when movint to $y=0^{\pm}$. So (assume $m=0$ again)

$$
\begin{aligned}
&\left(\sigma-\frac{iUk}{2}\right)\hat{\eta}=-|k|A_1\\
&\left(\sigma+\frac{iUk}{2}\right)\hat{\eta}=|k|A_2\\
&\rho_1\left[\left(\sigma-\frac{iUk}{2}\right)|k|A_1+|k|g\hat{\eta}\right]-\rho_2\left[\left(\sigma+\frac{iUk}{2}\right)|k|A_2+|k|g\hat{\eta}\right]=0\\
&\rho_1\left[-\left(\sigma-\frac{iUk}{2}\right)^2\hat{\eta}+|k|g\hat{\eta}\right]-\rho_2\left[\left(\sigma+\frac{iUk}{2}\right)^2\hat{\eta}+|k|h\hat{\eta}\right]=0\\
&-(\rho_1+\rho_2)\left(\sigma^2-\frac{U^2k^2}{4}\right)+(\rho_1-\rho_2)iUk\sigma+(\rho_1-\rho_2)|k|g=0\\
&\sigma^2-\frac{U^2k^2}{4}=-\frac{\rho_2-\rho_1}{\rho_1+\rho_2}ikU\sigma-\frac{\rho_2-\rho_2}{\rho_1+\rho_2}|k|g
\end{aligned}
$$

Write $\sigma=\sigma_R+i\sigma_I$, $\Delta=(\rho_2-\rho_1)/(\rho_2+\rho_1)$.

$$
\begin{aligned}
&\sigma_R^2-\sigma_I^2-\frac{U^2k^2}{4}=\Delta kU\sigma_I-\Delta|k|g\\
&2\sigma_R\sigma_I=-\Delta kU\sigma_R\\
&\sigma_R^2=-\frac{\Delta^2k^2U^2}{4}+\frac{U^2k^2}{4}-\Delta|k|g
\end{aligned}
$$

So if $\Delta>0$, long waves (small k) are stabilized.