4.3 Stokes flow past a sphere

Uniform flow $\mathbf{U}$ past a fixed rigid sphere, radius a. There are several methods, all of which have heavy algebra somewhere or depend on familiarity with spherical polar coordinates.

4.3.2 Method 1 (semi-modern, linearity and vector method)

From the linearity of the Stokes equations, $\mathbf{u}(\mathbf{x})$ must be linear in $\mathbf{U}$. Taking the origin to be at the centre of the sphere, the spherically symmetric geometry means the only other vector ingredient for $\mathbf{u}$ is $\mathbf{x}$. Hence the velocity and pressure fields must take the forms

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r), \end{aligned}$$

where $r = |\mathbf{x}|$, and f, g and h are scalar functions of r to be determined.

Differentiate

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r + \delta_{ij} U_n x_n g + x_i U_j g + x_i x_j U_n x_n g'/r. \qquad (*)$$

Contract i with j and apply incompressibility:

$$\boldsymbol{\nabla}\cdot\mathbf{u} = U_n x_n \left(f'/r + 4g + rg'\right) = 0.$$

Differentiate again:

$$\begin{aligned} \mu\nabla^2 u_i &= \mu U_i \left(f'' + 2f'/r + 2g\right) + \mu x_i U_n x_n \left(g'' + 6g'/r\right) \\ \nabla_i p &= \mu U_i h + \mu x_i U_n x_n h'/r \end{aligned}$$

In $\mu\nabla^2\mathbf{u} = \boldsymbol{\nabla}p$ equate the bits with U_i and those with $x_i U_n x_n$. So the Stokes equations give

$$f'/r + 4g + rg' = 0, \qquad f'' + 2f'/r + 2g = h \quad \text{and} \quad g'' + 6g'/r = h'/r.$$

Eliminating h and then f yields

$$r^2 g''' + 11rg'' + 24g' = 0.$$

This ODE is equidimensional in r so look for solutions of the form $g = r^\alpha$. Substituting, we find $\alpha = 0$, -3 and -5, with associated $f = -(\alpha+4)r^{\alpha+2}/(\alpha+2)$ and $h = -(\alpha+5)(\alpha+2)r^\alpha$. Hence the general solution of the assumed form linear in $\mathbf{U}$ is

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{U}\left(-2Ar^2 + B + Cr^{-1} - \tfrac{1}{3}Dr^{-3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(A + Cr^{-3} + Dr^{-5}\right), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})\left(-10A + 2Cr^{-3}\right). \end{aligned}$$

[Find $\mathbf{e}$ from $(*)$ and thus $\boldsymbol{\sigma}$: the stress across a spherical surface with normal $\mathbf{n} = \mathbf{x}/r$ is

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \mu\mathbf{U}\left(-3Ar + 2Dr^{-4}\right) + \mu\mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6}\right). \quad]$$

Applying the boundary conditions $\mathbf{u} = \mathbf{0}$ on the rigid sphere and $\mathbf{u} \to \mathbf{U}$ in the far field, we find the coefficients

$$A = 0, \qquad B = 1, \qquad C = -\tfrac{3}{4}a \quad \text{and} \quad D = \tfrac{3}{4}a^3,$$

so

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} + \frac{3a^3}{4r^5}\right),$$

$$p = -\frac{3a\mu\mathbf{U}\cdot\mathbf{x}}{2r^3} \quad \text{and} \quad \boldsymbol{\sigma}\cdot\mathbf{n}|_{r=a} = \frac{3\mu}{2a}\mathbf{U}.$$

Hence the drag on the sphere is

$$\int_{r=a} \boldsymbol{\sigma}\cdot\mathbf{n}\, dS = 4\pi a^2 \frac{3\mu}{2a}\mathbf{U} = 6\pi\mu a\mathbf{U}.$$

4.3.3 Method 2 (classical, separable streamfunction in sphericals r, θ, ϕ)

Use a Stokes streamfunction $\Psi(r, \theta)$ for the axisymmetric flow with

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r}$$

(from $\mathbf{u} = \boldsymbol{\nabla} \wedge \mathbf{A}$ and $\mathbf{A} = \Psi \mathbf{e}_\phi / r \sin\theta$ so that $\mathbf{u} \cdot \boldsymbol{\nabla} \Psi = 0$). In Stokes flow $\nabla^2 \boldsymbol{\omega} = \mathbf{0} \Rightarrow \nabla^4 \mathbf{A} = (\boldsymbol{\nabla} \wedge)^4 \mathbf{A} = \mathbf{0}$ and (with patience or Wikipedia)

$$\mathcal{D}^2 \mathcal{D}^2 \Psi = 0 \quad \text{where} \quad \mathcal{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right).$$

The uniform flow at infinity (the nonzero condition) has $\Psi = \frac{1}{2} U r^2 \sin^2\theta$; try $\Psi = F(r) \sin^2\theta$ and find $F = Ar^4 + Br^2 + Cr + D/r$. Then $A = 0$, $B = \frac{1}{2}$, and apply the BCs on sphere.

4.3.4 Method 3 (semi-classical, hybrid)

The vorticity is a harmonic function (as is the pressure), decaying at infinity, linear in $\mathbf{U}$ and otherwise constructed only from $\mathbf{x}$ by spherical symmetry. Can be shown the decaying solutions of $\nabla^2(\cdot) = 0$ are $\frac{1}{r}$, $\boldsymbol{\nabla} \frac{1}{r} = -\mathbf{x}/r^3$, $\boldsymbol{\nabla}\boldsymbol{\nabla} \frac{1}{r}$, etc. Hence the vorticity, a pseudo vector, (and the pressure, a scalar) must have the form

$$\boldsymbol{\omega} = A \mathbf{U} \wedge \mathbf{x}/r^3 \qquad (\text{and } p = \mu B \mathbf{U} \cdot \mathbf{x}/r^3).$$

Return to spherical polars and solve $\mathcal{D}^2 \Psi = AU \sin^2\theta / r$ with $\Psi = F(r) \sin^2\theta$. From $\boldsymbol{\nabla} p = -\mu \boldsymbol{\nabla} \wedge \boldsymbol{\omega}$ get $B = \mu A$.

4.3.5 Method 4 (modern, Papkovich–Neuber)

If $\boldsymbol{\phi}(\mathbf{x})$ is a vector harmonic function ($\nabla^2 \boldsymbol{\phi} = \mathbf{0}$) then

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{x} \cdot \boldsymbol{\phi}) - 2\boldsymbol{\phi} \qquad p = 2\mu \boldsymbol{\nabla} \cdot \boldsymbol{\phi}.$$

is a solution of the Stokes equations (Sheet 2, Q5). It's not quite the general solution (see Part III), but does include this problem.

Linearity and spherical symmetry give (for the perturbation $\mathbf{u} - \mathbf{U}$, which decays at infinity)

$$\boldsymbol{\phi} = A \mathbf{U} \frac{1}{r} + B \mathbf{U} \cdot \boldsymbol{\nabla}\boldsymbol{\nabla} \frac{1}{r},$$

with coefficients to be determined by applying the boundary conditions.

Note

Velocity is a **true** vector. Vorticity/angular velocity are **pseudo** vectors. (Difference is behaviour under *reflections*.) Sometimes linearity is not sufficient, you need to pay attention to the true vs. pseudo nature of the vectors involved. E.g. rotation of a sphere: $\mathbf{u}$ (true vector) linear in $\boldsymbol{\Omega}$ (pseudo vector) gives (by Method 1)

$$\mathbf{u} = \boldsymbol{\Omega} \wedge \mathbf{x} \, f(r)$$

Note 2

Often deal with a sphere moving with velocity $\mathbf{U}$ in fluid otherwise at rest. Remove uniform flow from the above, and put $\mathbf{U} \to -\mathbf{U}$. Better, start from scratch with correct BCs. Can also use general algebra (A, B, C, D) with different BCs for a spherical bubble (Sheet 2, Q6), or a spherical drop.