4.4 Flow past a sphere at low Reynolds number

Uniform flow $\mathbf{U}$ past a fixed rigid sphere, radius $a$. There are several methods, all of which have heavy algebra somewhere.

4.4.3 Method 3

The linearity of the Stokes equations means that $\mathbf{u}(\mathbf{x})$ must be linear in $\mathbf{U}$. Further, the problem has spherical symmetry about the centre of the sphere, which is taken as the origin. The velocity and pressure fields must therefore take the forms

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}f(r) + \mathbf{x}(\mathbf{U} \cdot \mathbf{x})g(r),$$

$$p(\mathbf{x}) = \mu(\mathbf{U} \cdot \mathbf{x})h(r),$$

where $r = |\mathbf{x}|$, and $f$, $g$ and $h$ are functions of scalar $r$ to be determined.

Now

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r + \delta_{ij} U_n x_n g + x_i U_j g + x_i x_j U_n x_n g'/r.$$  

Contracting $i$ with $j$, we have the incompressibility condition

$$0 = \nabla \cdot \mathbf{u} = U_n x_n (f'/r + 4g + rg').$$

Differentiating again

$$\mu \nabla^2 u_i = \mu U_i (f'' + 2f'/r + 2g) + \mu x_i U_n x_n (g'' + 6g'/r),$$

$$\nabla_i p(\mathbf{x}) = \mu U_i h + \mu x_i U_n x_n h'/r.$$  

Hence the governing equations give

$$f'/r + 4g + rg' = 0, \quad f'' + 2f'/r + 2g = h \quad \text{and} \quad g'' + 6g'/r = h'/r.$$  

Eliminating $h$ and then $f$ yields

$$r^2 g'' + 11rg'' + 24g' = 0.$$  

This differential equation is homogeneous in $r$ so there are solutions of the form $g = r^\alpha$. Substituting, one finds $\alpha = 0$, $-3$ and $-5$, with associated $f = -(\alpha + 4)r^{\alpha+2}/(\alpha + 2) + f_0$ and $h = -(\alpha + 5)/(\alpha + 2)r^{\alpha}$. Hence the general solution of the assumed form linear in $\mathbf{U}$ is

$$\mathbf{u}(\mathbf{x}) = \mathbf{U} \left( -2Ar^2 + B + Cr^{-1} - \frac{1}{3}Dr^{-3} \right) + \mathbf{x}(\mathbf{U} \cdot \mathbf{x}) \left( A + Cr^{-3} + Dr^{-5} \right),$$

$$p(\mathbf{x}) = \mu(\mathbf{U} \cdot \mathbf{x}) \left( -10A + 2Cr^{-3} \right).$$  

We shall need the stress exerted across a spherical surface with unit normal \( n = \mathbf{x}/r \)
\[
\begin{align*}
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mu \mathbf{U} \left( -3Ar + 2Dr^{-4} \right) + \mu \mathbf{x} \cdot \mathbf{x} \left( 9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6} \right)
\end{align*}
\]

Applying the boundary conditions on the rigid sphere and for the far field, we find the coefficients
\[
A = 0, \quad B = 1, \quad C = -\frac{3}{4}a \quad \text{and} \quad D = \frac{2}{3}a^3,
\]
so
\[
\begin{align*}
\mathbf{u} &= \mathbf{U} \left( 1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) + \mathbf{x} \mathbf{U} \cdot \mathbf{x} \left( -\frac{3a}{4r^3} + \frac{3a^3}{4r^5} \right), \\
p &= -\frac{3a\mu \mathbf{U} \cdot \mathbf{x}}{2r^3}
\end{align*}
\]
\[
\begin{align*}
\boldsymbol{\sigma} \cdot \mathbf{n} \bigg|_{r=a} &= \frac{3\mu}{2a} \mathbf{U}.
\end{align*}
\]

Hence the drag on the sphere is
\[
\int_{r=a} \boldsymbol{\sigma} \cdot \mathbf{n} \, dS = 4\pi a^2 \frac{3\mu}{2a} \mathbf{U} = 6\pi \mu a \mathbf{U}.
\]