4.2 Flow past a sphere at low Reynolds number

Uniform flow $\mathbf{U}$ past a fixed rigid sphere, radius a. There are several methods, all of which have heavy algebra somewhere.

4.2.1 Method 1

The linearity of the Stokes equations means that $\mathbf{u}(\mathbf{x})$ must be linear in $\mathbf{U}$. Further, the problem has spherical symmetry about the centre of the sphere, which is taken as the origin. The velocity and pressure fields must therefore take the forms

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})h(r), \end{aligned}$$

where $r = |\mathbf{x}|$, and f, g and h are functions of scalar r to be determined.

Now

$$\frac{\partial u_i}{\partial x_j} = U_i x_j f'/r \ + \ \delta_{ij} U_n x_n g \ + \ x_i U_j g \ + \ x_i x_j U_n x_n g'/r.$$

Contracting i with j, we have the incompressibility condition

$$0 = \nabla\cdot\mathbf{u} = U_n x_n (f'/r + 4g + rg').$$

Differentiating again

$$\begin{aligned} \mu\nabla^2 u_i &= \mu U_i \left(f'' + 2f'/r + 2g\right) \ + \ \mu x_i U_n x_n \left(g'' + 6g'/r\right) \\ \nabla_i p(\mathbf{x}) &= \mu U_i h \ + \ \mu x_i U_n x_n h'/r \end{aligned}$$

Hence the governing equations give

$$f'/r + 4g + rg' = 0, \qquad f'' + 2f'/r + 2g = h \quad \text{and} \quad g'' + 6g'/r = h'/r.$$

Eliminating h and then f yields

$$r^2 g''' + 11rg'' + 24g' = 0.$$

This differential equation is homogeneous in r so that there are solutions of the form $g = r^\alpha$. Substituting, one finds $\alpha = 0$, -3 and -5, with associated $f = -(\alpha+4)r^{\alpha+2}/(\alpha+2)$ and $h = -(\alpha+5)(\alpha+2)r^\alpha$. Hence the general solution of the assumed form linear in $\mathbf{U}$ is

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{U}\left(-2Ar^2 + B + Cr^{-1} - \tfrac{1}{3}Dr^{-3}\right) \ + \ \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(A + Cr^{-3} + Dr^{-5}\right), \\ p(\mathbf{x}) &= \mu(\mathbf{U}\cdot\mathbf{x})\left(-10A + 2Cr^{-3}\right). \end{aligned}$$

We shall need the stress exerted across a spherical surface with unit normal $\mathbf{n} = \mathbf{x}/r$

$$\boldsymbol{\sigma}\cdot\mathbf{n} \ = \ \mathbf{U}\left(-3Ar + 2Dr^{-4}\right) \ + \ \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(9Ar^{-1} - 6Cr^{-4} - 6Dr^{-6}\right)$$

Applying the boundary conditions on the rigid sphere and for the far field, we find the coefficients

$$A = 0, \qquad B = 1, \qquad C = -\tfrac{3}{4}a \quad \text{and} \quad D = \tfrac{3}{4}a^3,$$

so

$$\mathbf{u} \ = \ \mathbf{U}\left(1 \ - \ \frac{3a}{4r} \ - \ \frac{a^3}{4r^3}\right) \ + \ \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\left(-\frac{3a}{4r^3} \ + \ \frac{3a^3}{4r^5}\right),$$

$$p \;=\; -\frac{3a\mu\mathbf{U}\cdot\mathbf{x}}{2r^3} \quad\text{and}\quad \boldsymbol{\sigma}\cdot\mathbf{n}|_{r=a} \;=\; \frac{3\mu}{2a}\mathbf{U}.$$

Hence the drag on the sphere is

$$\int_{r=a} \boldsymbol{\sigma}\cdot\mathbf{n}\,dS \;=\; 4\pi a^2\frac{3\mu}{2a}\mathbf{U} \;=\; 6\pi\mu a\mathbf{U}.$$

4.2.2 Method 2

Use a Stokes streamfunction for the axisymmetric flow

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \quad\text{and}\quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}.$$

The vorticity equation (curl of the momentum equation, to eliminate the pressure) is then at low Reynolds numbers

$$\mathcal{D}^2\mathcal{D}^2\Psi = 0 \quad\text{where}\quad \mathcal{D}^2 \;=\; \frac{\partial^2}{\partial r^2} \;+\; \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

The uniform flow at infinity has $\Psi = \frac{1}{2}Ur^2\sin^2\theta$, so one tries $\Psi = F(r)\sin^2\theta$, and finds $F = Ar^4 + Br^2 + Cr + D/r$.

4.2.3 Method 3

One can show that the general solution of the Stokes equation can be expressed in terms of a vector harmonic function $\boldsymbol{\phi}(\mathbf{x})$ (i.e. $\nabla^2\boldsymbol{\phi} = 0$)

$$\mathbf{u} \;=\; 2\boldsymbol{\phi} - \nabla(\mathbf{x}\cdot\boldsymbol{\phi}) \qquad p = -2\mu\nabla\cdot\boldsymbol{\phi}.$$

$$\sigma_{ij} = 2\mu\left(\delta_{ij}\frac{\partial\phi_n}{\partial x_n} - x_k\frac{\partial^2\phi_k}{\partial x_i\partial x_j}\right)$$

The fundamental harmonic functions (solid spherical harmonics) are denoted $\boldsymbol{\Phi}_{-(1+n)}$ and proportional to the nth gradient of $1/r$: $\Phi_{-1} = 1/r$ (scalar), $\boldsymbol{\Phi}_{-2} = \mathbf{x}/r^3$ (vector), $\boldsymbol{\Phi}_{-3} = \mathbf{1}/r^3 - 3\mathbf{x}\mathbf{x}/r^5$ (2nd order tensor) etc.

Linearity and spherical symmetry then give

$$\boldsymbol{\phi} \;=\; A\mathbf{U}\frac{1}{r} \;+\; B\mathbf{U}\cdot\nabla\nabla\frac{1}{r} \quad\text{or}\quad \boldsymbol{\phi} \;=\; C\mathbf{U}\Phi_{-1} \;+\; D\boldsymbol{\Phi}_{-3}\cdot\mathbf{U},$$

with coefficients to be determined by applying the boundary conditions.

4.2.4 Method 4

The pressure and vorticity are harmonic functions. Using linearity and spherical symmetry, they must take the form

$$p \;=\; \mu A\mathbf{U}\cdot\mathbf{x}/r^3 \quad\text{and}\quad \nabla\wedge\mathbf{u} \;=\; B\mathbf{U}\wedge\mathbf{x}/r^3.$$

The final step to $\mathbf{u}$ is tedious.

Note

Velocity is **true** vector. Vorticity/rotation are **pseudo** vectors. Sometimes linearity is not sufficient, you need to pay attention to the true vs. pseudo nature of the vectors involved. E.g. rotation of a sphere: $\mathbf{u}$ (true vector) linear in $\boldsymbol{\Omega}$ (pseudo vector)

$$\rightarrow \mathbf{u} = \boldsymbol{\Omega}\times\mathbf{x}\,f(r)$$