Example Sheet 2

Every answer should include at least one relevant sketch

1. Show that in an unbounded Stokes flow at rest at infinity, two identical spheres, arbitrarily aligned, fall under gravity at constant separation, i.e. neither separating nor coming closer together.

2. An external force is applied at the centre of a cube in a direction normal to one flat surface. Show that in an unbounded Stokes flow, the cube moves in the direction of the applied force without rotating. Using linearity and rotational symmetry, deduce that, in all orientations, the terminal velocity of a cube of uniform density sedimenting in a fluid is vertical. Furthermore, using reflectional symmetry, show that it falls with no rotation. Using similar arguments, show that when the cube simply rotates about an axis through its centre, the resisting hydrodynamic torque is parallel to the angular velocity and the hydrodynamic force is zero.

Show that the same applies to a regular tetrahedron. How about an ellipsoid?

3. If the strain-rate tensor $\mathbf{e}(\mathbf{x})$ vanishes throughout a connected region, show that the flow is rigid-body motion. [Hint: you may first show that $\partial^2 u_1/\partial x_2 \partial x_3 \equiv 0$.]

Show that if the surface stress is specified on a bounding surface then the Stokes flow in the interior is unique to within the addition of a rigid body motion. What condition(s) must the prescribed surface stress satisfy for there to be a Stokes flow in the interior? [Hint: in the absence of body forces the Stokes equation can be written $\partial\sigma_{ij}/\partial x_j = 0$.]

4. If $\mathbf{A}(\mathbf{x})$ is a vector harmonic function, i.e. $\nabla^2\mathbf{A} = 0$, show that the flow

$$\mathbf{u} = 2\mathbf{A} - \nabla(\mathbf{A}\cdot\mathbf{x}), \quad p = -2\mu\nabla\cdot\mathbf{A}$$

is incompressible and satisfies the Stokes equation with no body force. Calculate the stress tensor.

For a sphere of radius a translating at velocity $\mathbf{V}$ through a fluid which is otherwise at rest, the harmonic function takes the form

$$\mathbf{A} = \alpha a \mathbf{V}\tfrac{1}{r} + \beta a^3 \left(\mathbf{V}\cdot\nabla\right)\nabla\tfrac{1}{r},$$

[Hint: Why? How many vector harmonic functions which are linear in $\mathbf{V}$ can you construct using the fundamental harmonic solution, $1/r$, and its derivatives?] Find the value of the constants α and β.

5. Consider an unbounded Stokes flow outside a rigid sphere of radius a rotating with angular velocity $\boldsymbol{\Omega}$. Show that the pressure gradient is zero. Then derive the velocity field as

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\Omega}\times\mathbf{x}\frac{a^3}{r^3}.$$

[Hint: What scalars p and *true* vector harmonic functions, $\mathbf{u}$, can you construct using the fundamental harmonic solution $1/r$ and its derivatives that are linear in the *pseudo* vector $\boldsymbol{\Omega}$?] Show that the torque exerted on the sphere by the flow is $-8\pi\mu a^3\boldsymbol{\Omega}$.

6. Consider a spherical bubble of radius a in a uniform flow $\mathbf{U}$. Recall that the expression obtained in lectures for the Stokes flow outside a sphere is of the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}f(r) + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})g(r).$$

Applying boundary conditions on $r = a$ of no normal component of velocity and no tangential component of surface traction (i.e. no tangential stress), find the flow $\mathbf{u}(\mathbf{x})$. Show that the drag force is $4\pi\mu a\mathbf{U}$.

7. Using the minimum dissipation theorem by stating carefully the flows you are comparing and exploiting the result from question 2, find upper and lower bounds for the hydrodynamic torque on a regular tetrahedron rotating about its centre in a viscous fluid.
[Hint: the radius of the inscribed sphere for a regular tetrahedron of edge length a is $a/\sqrt{24}$ while that of the circumscribed sphere is $\sqrt{6}a/4$.]

8. An incompressible, viscous fluid is contained in the two-dimensional region $-\alpha < \theta < \alpha$ between two rigid hinged plates rotating with equal and opposite angular velocity of magnitude ω. Therefore, in plane polar coordinates, the velocity components on the hinged plates satisfy

$$u_r = 0, \qquad u_\theta = \mp\omega r \quad \text{on} \quad \theta = \pm\alpha.$$

Neglecting all inertial forces, show that a solution to the Stokes problem is of the form

$$\psi = \tfrac{1}{2}\omega r^2 g(\theta)$$

(why?) and find the function $g(\theta)$. Deduce the pressure field $p(r,\theta)$. Is the logarithmic divergence of the pressure as $r \to 0$ an issue? What is the physical interpretation of the singularity in g?

9. An incompressible, viscous fluid occupies the region $0 < \theta < \alpha$, $0 < r < \infty$ in plane-polar coordinates (r,θ). It is bounded by a stationary, rigid plate at $\theta = \alpha$ and a rigid plate at $\theta = 0$ that translates with constant velocity U in its own plane in the negative r direction. Calculate the resulting Stokes flow of the fluid. Calculate the stresses on each of the plates and comment on the external forces required to sustain the flow.

10. A spherical annulus of incompressible viscous liquid of volume V occupies the region $R_1(t) < r < R_2(t)$ between two free surfaces on the outside of which pressures (i.e. normal stresses) $P_1(t)$ and $P_2(t)$ are applied. The resulting flow is spherically symmetric. Neglecting inertia, gravity and surface tension, show that

$$\frac{d}{dt}\left(R_1^3\right) = \frac{\pi(P_1 - P_2)}{\mu V} R_1^3 \left(R_1^3 + \frac{3V}{4\pi}\right).$$

[Hints: $u_r = f(t)/r^2$ (why?) and $\sigma_{rr} = -p + 2\mu\partial u_r/\partial r$ in this flow. Also, be careful to distinguish between pressure and normal stresses.]

Show that if $P_1 - P_2$ is maintained positive and constant, then R_1 becomes infinite in a finite time. What happens if $P_1 - P_2$ is maintained negative and constant?