1. A rigid sphere of radius $a$ falls through a fluid of viscosity $\mu$ under gravity towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap $h_0$ is very small, the speed of approach of the sphere is

$$h_0 W/6\pi \mu a^2,$$

where $W$ is the weight of the sphere corrected for buoyancy.

2. Oil is forced by a pressure difference $\Delta p$ through the narrow gap between two parallel circular cylinders of radius $a$ with axes $2a + b$ apart. Show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux is approximately

$$\frac{2b^{5/2} \Delta p}{9\pi a^{1/2} \mu}$$

when the cylinders are fixed.

Show also that when the two cylinders rotate with angular velocities $\Omega_1$ and $\Omega_2$ in opposite directions, the change in the volume flux is

$$\frac{2}{3} ab (\Omega_1 + \Omega_2).$$

3. A viscous fluid coats the outer surface of a cylinder of radius $a$ which rotates with angular velocity $\Omega$ about its axis which is horizontal. The angle $\theta$ is measured from the horizontal on the rising side. Show that the volume flux per unit length $Q(\theta, t)$ is related to the thickness $h(\theta, t)$ of the fluid layer by

$$Q = \Omega a h - \frac{g}{3 \nu} h^3 \cos \theta,$$

and deduce an evolution equation for $h(\theta, t)$.

Consider now the possibility of a steady state with $Q = \text{const}$, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and $2\pi$-periodic exists only if

$$\Omega a > (9Q^2 g / 4\nu)^{1/3}.$$ 

4. A two-dimensional drop $h(x, t)$ spreads on a horizontal table. Assuming that the drop has become a thin layer, find how the drops spreads. [It is not possible to integrate the volume in closed form.]

5. The walls of a channel are porous and separated by a distance $d$. Fluid is driven through the channel by a pressure gradient $G = -\partial p/\partial x$, and at the same time suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity $V$, fluid being supplied at this rate at the other wall. Find and sketch the steady velocity and vorticity distributions in the fluid (i) when $Vd/\nu \ll 1$ and (ii) when $Vd/\nu \gg 1$. 

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6. Viscous fluid fills an annulus $a < r < b$ between a long stationary cylinder $r = b$ and a long cylinder $r = a$ rotating at angular velocity $\Omega$. Find the axisymmetric velocity field, ignoring end effects.

Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow $-Va/r$. Find the new steady flow around the cylinder when $Va/\nu < 2$ and $Va/\nu > 2$. Comment on the flow structure when $Va/\nu \gg 1$.

Find the torque that must be applied to maintain the motion.

7. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \nabla^2 \omega.$$ 

Interpret the terms in the equation.

At time $t = 0$ a concentration of vorticity is created along the $z$-axis, with the same circulation $\Gamma$ around the axis at each $z$. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity $v$, say. Show that there is a similarity solution of the form $vr/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and $\eta$ is a suitable similarity variable. Further show that all conditions are satisfied by

$$f(\eta) = \frac{1}{2\pi}(1 - e^{-\eta}), \quad \eta = r^2/4\nu t.$$ 

Show also that the total vorticity in the flow remains constant at $\Gamma$ for all $t > 0$. Sketch $v$ as a function of $r$.

8. Calculate the vorticity $\omega$ associated with the velocity field

$$u = (\alpha x - yf(r, t), -\alpha y + xf(r, t), 2\alpha z),$$

where $\alpha$ is a positive constant, and $f(r, t)$ depends on $r = (x^2 + y^2)^{1/2}$ and time $t$. Hence show that the velocity field represents a dynamically possible motion if $f(r, t)$ satisfies

$$2f + r \frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu} \left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and $A$ and $t_0$ are constants.

Show that in the case where the minus sign is taken $\gamma$ is approximately $1/[4\nu(t-t_0)]$ when $t$ only just exceeds $t_0$. Which terms in the vorticity equation dominate when this approximation holds?