Example Sheet 3

Every answer should include at least one relevant sketch

1. A rigid sphere of radius $a$ falls under gravity through a Newtonian fluid of viscosity $\mu$ towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap $h_0$ is very small, the speed of approach of the sphere is

$$h_0W/6\pi\mu a^2,$$

where $W$ is the weight of the sphere corrected for buoyancy.

2. A Newtonian fluid of viscosity $\mu$ is forced by a pressure difference $\Delta p$ through the narrow gap between two parallel circular cylinders of radius $a$ with axes $2a + b$ apart. Show that, provided $b \ll a$ and $\rho b^3\Delta p \ll \mu^2a$, the volume flux through the gap per unit length along the axis of the cylinders is approximately

$$\frac{2h^{5/2}\Delta p}{9\pi a^{1/2}\mu},$$

when the cylinders are fixed.

Show that when the two cylinders rotate with angular velocities $\Omega_1$ and $\Omega_2$ in opposite directions (i.e. one rotates $\Omega_1e_z$ while the other one $-\Omega_2e_z$, where $e_z$ is the unit vector along the axis of the cylinder), the change in the volume flux is given by

$$\frac{2}{3}ab(\Omega_1 + \Omega_2).$$

3. A viscous fluid coats the outer surface of a cylinder of radius $a$ which rotates with angular velocity $\Omega$ about its axis, which is horizontal. The angle $\theta$ is measured from the horizontal on the rising side. Show that the volume flux per unit length $Q(\theta,t)$ is related to the thickness $h(\theta,t) \ll a$ of the fluid layer by

$$Q = \Omega ah - \frac{g}{3\nu}h^3\cos\theta,$$

and deduce an evolution equation for $h(\theta,t)$.

Consider now the possibility of a steady state with $Q = \text{const}$, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and $2\pi$-periodic exists only if

$$\Omega a > (9Q^2g/4\nu)^{1/3}.$$  

[Hint: Consider a graph of $\cos\theta$ as a function of $h$.]

4. A drop of viscous fluid of thickness $h(r,t)$ spreads axisymmetrically on a horizontal surface. Explaining your reasoning carefully with the aid of a diagram, use mass conservation to show that

$$\frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(rq) = 0,$$

where $q(r,t) = \int_0^h u(r,z,t)\,dz$ in plane polar coordinates $(r,\theta,z)$ and $u$ is the radial velocity.

Solve the lubrication equations to determine the partial differential equation governing the evolution of $h$ once the drop has become a thin layer. Look for a similarity solution to the equation and apply a condition that the volume of the drop is constant to determine the radius of the drop $r_N(t)$.

5. A Newtonian fluid with dynamic viscosity $\mu$ flows in a shallow container with a free surface at $z = 0$. Using cartesian coordinates $(x,y,z)$, the fluid velocity is denoted $(u_x, u_y, u_z) \equiv (u_H, u_z)$. The base of
the container is rigid, and is located at \( z = -h(x, y) \). An external horizontal stress \( S(x, y) \) is applied at the free surface. Gravity may be neglected. Using lubrication theory, show that the two-dimensional horizontal volume flux \( q(x, y) \equiv \int_0^h u_H dz \) satisfies the equations

\[
\nabla \cdot q = 0, \quad \mu q = -\frac{1}{2} h^3 \nabla p + \frac{1}{2} h^2 S,
\]

where \( p(x, y) \) is the pressure. Find also an expression for the surface velocity \( u_0(x, y) \equiv u_H(x, y, 0) \) in terms of \( S, q \) and \( h \).

6. The walls of a straight two-dimensional channel are porous and separated by a distance \( d \). A Newtonian fluid of viscosity \( \mu \) is driven along the channel by a pressure gradient \( G = -\partial p / \partial x \). At the same time, suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity \( V > 0 \), with fluid being supplied at this rate to the other wall. Calculate the steady velocity and vorticity distributions in the fluid. Sketch them (i) when \( V d/\nu \ll 1 \) and (ii) when \( V d/\nu \gg 1 \).

7. A Newtonian fluid of viscosity \( \mu \) fills an annulus \( a < r < b \) between a long stationary cylinder \( r = b \) and a long cylinder \( r = a \) rotating at angular velocity \( \Omega \). Looking up the components of the Navier-Stokes equation in cylindrical coordinates, find the axisymmetric velocity field, ignoring end effects.

Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow \( -V a/r \) in the fluid annulus. Find an expression for the new steady flow around the cylinder when \( V a/\nu \neq 2 \). Comment on the flow structure when \( V a/\nu \gg 1 \).

Find the torque (per unit length along the cylinder axis) required to maintain the motion, and show that it is independent of \( b \) in the limit \( V a/\nu \to \infty \). [Check the dimensions and sign of your result.]

8. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \nabla^2 \omega.
\]

Interpret the terms in the equation.

At time \( t = 0 \) a concentration of vorticity is created along the \( z \)-axis, with the same circulation \( \Gamma \) around the axis at each \( z \). The fluid is viscous and incompressible, and for \( t > 0 \) has only an azimuthal velocity \( \omega \). Show that there is a similarity solution of the form \( \omega = f(\eta) \), where \( \eta = (x^2 + y^2)^{1/2} \) and \( \eta \) is a suitable similarity variable. Furthermore, show that all conditions are satisfied by

\[
f(\eta) = \frac{1}{2\pi}(1 - e^{-\eta^2}), \quad \eta = r/2\sqrt{\nu t}.
\]

Show also that the total vorticity in the flow remains constant at \( \Gamma \) for all \( t > 0 \). Sketch \( \nu \) as a function of \( r \).

9. Calculate the vorticity \( \omega \) associated with the velocity field

\[
u = (-\alpha x - y f(r, t), -\alpha y + x f(r, t), 2\alpha z),
\]

where \( \alpha \) is a positive constant, and \( f(r, t) \) depends on \( r = (x^2 + y^2)^{1/2} \) and time \( t \). Show that the velocity field represents a dynamically possible motion if \( f(r, t) \) satisfies

\[
2f + \frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},
\]

where

\[
\gamma(t) = \frac{\alpha}{2\nu} \left(1 + e^{-2\alpha(t-t_0)}\right)^{-1},
\]

and \( A \) and \( t_0 \) are constants.

Show that, in the case where the minus sign is taken, \( \gamma \) is approximately \( 1/[4\nu(t-t_0)] \) when \( t \) only just exceeds \( t_0 \). Which terms in the vorticity equation dominate when this approximation holds?