Mathematical Tripos Part II **Michaelmas 2022**
D23 Fluid Dynamics II **Eric Lauga**

Example Sheet 4

Every answer should include at least one relevant sketch

1. Vorticity. From the vorticity equation, derive the equation satisfied by the streamfunction $\psi(r,\theta)$ for a steady two-dimensional flow in polar coordinates. Show further that this equation has solutions of the form $\psi = Qf(\theta)$, with Q constant if f satisfies an ordinary differential equation which you should determine.

2. Vorticity. Write down the equation satisfied by the vorticity $\omega(x,y,t)$ in a two-dimensional flow in Cartesian coordinates. Introduce a streamfunction ψ and show that $\omega = -\nabla^2\psi$. Show that the vorticity equation has a time-dependent similarity solution of the form

$$\psi = CxH(t)^{-1}\phi(\eta), \quad \omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta), \quad \text{for } \eta = yH(t)^{-1},$$

if $H(t) = (2Ct)^{1/2}$ and if ϕ satisfies an ordinary differential equation which you should determine involving an effective Reynolds number, $R \equiv C/\nu$.

3. Boundary layers. The concept of a boundary layer can be illustrated by ordinary differential equations. Consider the equation satisfied on the interval $[0,1]$ by the function $f(x)$

$$\epsilon f' + f = 1, \quad f(0) = 0. \qquad (*)$$

Find the exact solution to $(*)$ and plot it for small values of ϵ. Formally take $\epsilon = 0$ in $(*)$ and find its solution, $f_0(x)$. Is $f_0(x)$ compatible with the boundary condition? Compare the exact solution to $f_0(x)$ and explain what happens. What is the "size" of the boundary layer at $x = 0$?

4. Boundary layers. Wind blowing over a deep reservoir exerts at the water surface a uniform tangential stress, S, which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based on (a) balancing the inertial and viscous forces in a thin boundary layer and (b) on the imposed boundary condition, to find order-of-magnitude estimates $\delta(x)$ for the boundary-layer thickness and $U(x)$ for the surface velocity as functions of distance x from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function f defined by

$$\psi(x,y) = U(x)\delta(x)f(\eta), \quad \text{where} \quad \eta = y/\delta(x).$$

What are the boundary conditions on f?

5. Boundary layers. A steady two-dimensional jet of fluid runs along a plane rigid wall. Use the boundary-layer equations to show that the quantity

$$P = \int_0^\infty u(y)\left(\int_y^\infty u(y')^2\,dy'\right)dy$$

is independent of the distance x along the wall. Find order-of-magnitude estimates for the boundary-layer thickness and velocity as functions of x.
Show that for the analogous axisymmetric wall jet spreading out radially the velocity varies like $r^{-3/2}$.

6. Instabilities. A vortex sheet of strength U is located at a distance h above a rigid wall $y = 0$ and is parallel to it, so that the fluid velocity $(u, 0, 0)$ is

$$u = \begin{cases} U & \text{in } 0 < y < h, \\ 0 & \text{in } y > h \ . \end{cases}$$

Suppose now that the sheet is perturbed slightly to the position $y = h + \mathcal{R}\left[\eta_0 e^{ik(x-ct)}\right]$ where $k > 0$ is real but c may be complex. Show that

$$c = U/(1 \pm i\sqrt{\tanh kh}).$$

Deduce that: (i) the sheet is unstable to disturbances of all wavelengths; (ii) for short waves ($kh \gg 1$) the growth rate $k\mathrm{Im}(c)$ is $\frac{1}{2}Uk$ and the wave propagation speed $\mathrm{Re}(c)$ is $\frac{1}{2}U$, as if the wall were absent; (iii) for long waves ($kh \ll 1$) the growth rate is $Uk\sqrt{kh}$ and the propagation speed is U.

7. Instabilities. A two-dimensional jet in the x-direction has velocity profile

$$u = \begin{cases} 0 & \text{in } y > h, \\ U & \text{in } -h < y < h, \\ 0 & \text{in } y < -h \ . \end{cases}$$

The vortex sheets at $y = \pm h$ are perturbed to

$$y = \begin{cases} +h + \mathcal{R}\left[\eta_1 e^{ik(x-ct)}\right], \\ -h + \mathcal{R}\left[\eta_2 e^{ik(x-ct)}\right]. \end{cases}$$

Show that the jet is unstable to a 'varicose' instability for which $\eta_1 = -\eta_2$ (identical to that of question 6), and also to a 'sinuous' instability for which $\eta_1 = \eta_2$ and

$$c = U/(1 \pm i\sqrt{\coth kh}).$$

8. Instabilities. Two regions of the same inviscid fluid are separated by a thin membrane at $y = 0$. The fluid in $y > 0$ has the uniform velocity $(U, 0, 0)$ in cartesian coordinates, while the fluid at $y < 0$ is at rest. The membrane is now slightly perturbed to $y = \eta(x, t)$. The dynamical effect of the membrane is to induce a pressure difference across it equal to $\beta \partial^4 \eta / \partial x^4$, where β is a constant such that the pressure is higher below the interface when $\partial^4 \eta / \partial x^4 > 0$. Assuming that the flow remains irrotational and all perturbations are small, derive the relation between σ and k for a disturbance of the form $\eta(x, t) = \mathcal{R}\left[Ce^{ikx+\sigma t}\right]$ where $k > 0$ is real but σ may be complex. Show that there is an instability only for $k < k_{\max}$ where $k_{\max}$ is to be determined. Find the maximum growth rate and the value of k for which this is obtained.

9. Boundary layers. Show that the rate of dissipation of mechanical energy in an incompressible fluid is $2\mu e_{ij} e_{ij}$ per unit volume, where e_{ij} is the rate-of-strain tensor and μ is the dynamic viscosity.
A finite mass of incompressible fluid, of dynamic viscosity μ and density ρ is held in the shape of a sphere $r < a$ by surface tension. It is set into a mode of small oscillations in which the velocity field may be taken to have Cartesian components

$$u = \beta x, \quad v = -\beta y, \quad w = 0.$$

where $\beta \propto \exp(-\epsilon t) \sin \omega t$. Assuming that $\epsilon \ll \omega$, calculate the dissipation rate averaged over a cycle (ignoring the slowly varying factor $\exp(-\epsilon t)$) and hence show that $\epsilon = 5\mu/\rho a^2$. You may assume that the total energy of the oscillation is twice the kinetic energy averaged over a cycle. Why is it permissible to ignore the details of the boundary layer near $r = a$?