1. From the vorticity equation, derive the equation satisfied by the streamfunction $\psi(r, \theta)$ for a steady two-dimensional flow in polar coordinates. Show further that this equation has solutions of the form $\psi = Qf(\theta)$, with $Q$ constant if $f$ satisfies an ordinary differential equation which you should determine.

2. Write down the equation satisfied by the vorticity $\omega(x, y, t)$ in a two-dimensional flow in Cartesian coordinates. Introduce a streamfunction $\psi$ and show that $\omega = -\nabla^2 \psi$. Show that the vorticity equation has a time-dependent similarity solution of the form $\psi = CxH(t)^{-1}\phi(\eta)$, $\omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta)$, for $\eta = yH(t)^{-1}$, if $H(t) = (2Ct)^{1/2}$ and if $\phi$ satisfies an ordinary differential equation which you should determine involving an effective Reynolds number, $R \equiv C/\nu$.

3. The concept of a boundary layer can be illustrated by ordinary differential equations. Consider the equation satisfied on the interval $[0, 1]$ by the function $f(x)$

$$\epsilon f'' + f = 1, \quad f(0) = 0.$$  (*)

Find the exact solution to (\*) and plot it for small values of $\epsilon$. Formally take $\epsilon = 0$ in (\*) and find its solution, $f_0(x)$. Is $f_0(x)$ compatible with the boundary condition? Compare the exact solution to $f_0(x)$ and explain what happens. What is the “size” of the boundary layer at $x = 0$?

4. Wind blowing over a deep reservoir exerts at the water surface a uniform tangential stress, $S$, which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based on (a) balancing the inertial and viscous forces in a thin boundary layer and (b) on the imposed boundary condition, to find order-of-magnitude estimates $\delta(x)$ for the boundary-layer thickness and $U(x)$ for the surface velocity as functions of distance $x$ from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function $f$ defined by $\psi(x, y) = U(x)\delta(x)f(\eta)$, where $\eta = y/\delta(x)$.

What are the boundary conditions on $f$?

5. A steady two-dimensional jet of fluid runs along a plane rigid wall. Use the boundary-layer equations to show that the quantity

$$P = \int_0^\infty u(y) \left( \int_y^\infty u(y')^2 \, dy' \right) \, dy$$

is independent of the distance $x$ along the wall. Find order-of-magnitude estimates for the boundary-layer thickness and velocity as functions of $x$.

Show that for the analogous axisymmetric wall jet spreading out radially the velocity varies like $r^{-3/2}$.
6. A vortex sheet of strength \( U \) is located at a distance \( h \) above a rigid wall \( y = 0 \) and is parallel to it, so that the fluid velocity \((u, 0, 0)\) is

\[
u = \begin{cases} 
U & \text{in } 0 < y < h, \\
0 & \text{in } y > h .
\end{cases}
\]

Suppose now that the sheet is perturbed slightly to the position \( y = h + R \left[ \eta_0 e^{i k(x - ct)} \right] \) where \( k > 0 \) is real but \( c \) may be complex. Show that

\[
c = U / (1 \pm i \sqrt{\tanh kh}).
\]

Deduce that: (i) the sheet is unstable to disturbances of all wavelengths; (ii) for short waves \((kh \gg 1)\) the growth rate \( k \text{Im}(c) \) is \( \frac{1}{2} U k \) and the wave propagation speed \( \text{Re}(c) \) is \( \frac{1}{2} U \), as if the wall were absent; (iii) for long waves \((kh \ll 1)\) the growth rate is \( U k \sqrt{kh} \) and the propagation speed is \( U \).

7. A two-dimensional jet in the \( x \)-direction has velocity profile

\[
u = \begin{cases} 
0 & \text{in } y > h, \\
U & \text{in } -h < y < h, \\
0 & \text{in } y < -h .
\end{cases}
\]

The vortex sheets at \( y = \pm h \) are perturbed to

\[
y = \begin{cases} 
+h + R \left[ \eta_1 e^{i k(x - ct)} \right], \\
-h + R \left[ \eta_2 e^{i k(x - ct)} \right].
\end{cases}
\]

Show that the jet is unstable to a ‘varicose’ instability for which \( \eta_1 = -\eta_2 \) (identical to that of question 6), and also to a ‘sinuous’ instability for which \( \eta_1 = \eta_2 \) and

\[
c = U / (1 \pm i \sqrt{\coth kh}).
\]

8. Two regions of the same inviscid fluid are separated by a thin membrane at \( y = 0 \). The fluid in \( y > 0 \) has the uniform velocity \((U, 0, 0)\) in cartesian coordinates, while the fluid at \( y < 0 \) is at rest. The membrane is now slightly perturbed to \( y = \eta(x,t) \). The dynamical effect of the membrane is to induce a pressure difference across it equal to \( \beta \partial^4 \eta / \partial x^4 \), where \( \beta \) is a constant such that the pressure is higher below the interface when \( \partial^4 \eta / \partial x^4 > 0 \). Assuming that the flow remains irrotational and all perturbations are small, derive the relation between \( \sigma \) and \( k \) for a disturbance of the form \( \eta(x,t) = R \left[ C e^{i k x + \sigma t} \right] \) where \( k > 0 \) is real but \( \sigma \) may be complex. Show that there is an instability only for \( k < k_{\text{max}} \) where \( k_{\text{max}} \) is to be determined. Find the maximum growth rate and the value of \( k \) for which this is obtained.

9. Show that the rate of dissipation of mechanical energy in an incompressible fluid is \( 2 \mu e_{ij} e_{ij} \) per unit volume, where \( e_{ij} \) is the rate-of-strain tensor and \( \mu \) is the dynamic viscosity.

A finite mass of incompressible fluid, of dynamic viscosity \( \mu \) and density \( \rho \) is held in the shape of a sphere \( r < a \) by surface tension. It is set into a mode of small oscillations in which the velocity field may be taken to have Cartesian components

\[
u = \beta x, \quad v = -\beta y, \quad w = 0,
\]

where \( \beta \propto \exp(-\epsilon t) \sin \omega t \). Assuming that \( \epsilon \ll \omega \), calculate the dissipation rate averaged over a cycle (ignoring the slowly varying factor \( \exp(-\epsilon t) \)) and hence show that \( \epsilon = 5\mu / \rho a^2 \). You may assume that the total energy of the oscillation is twice the kinetic energy averaged over a cycle. Why is it permissible to ignore the details of the boundary layer near \( r = a \)?