Example Sheet 1: Sound Waves

1. **Plane waves and radiation.** A thin piston executes very small oscillations about \( x = 0 \) in a long straight fluid-filled tube with cross-sectional area \( A \) and rigid walls aligned with the \( x \)-axis. Given the piston velocity \( \dot{X}(t) \), find the velocity potential \( \phi(x,t) \) for the (linearised) sound waves generated in \( x > 0 \) and \( x < 0 \) (linearising \( X \approx 0 \)). Show that if \( x_0 > 0 \) the total power \( AI_x \) radiated across \( x = x_0 \) is

\[
(q(t - x_0/c_0))^2 c_0/\rho_0 A,
\]

where \( q(t) = \rho_0 A \dot{X}(t) \) is the rate at which mass is displaced on one side of the piston. What is the corresponding result for \( x_0 < 0 \)? [Later in the course, we will analyse the effects of nonlinearity.]

2. **Reflection and transmission.** An interface at \( x = 0 \) separates fluid of density \( \rho_0 \) and sound speed \( c_0 \) in \( x < 0 \) from fluid of density \( \rho_1 \) and sound speed \( c_1 \) in \( x > 0 \). A plane harmonic sound wave is incident from \( x < 0 \) with wavevector \( k = (k, 0, 0) \) and amplitude \( A \) (of its pressure perturbation). What is the frequency \( \omega \) and the wavevector \( k' \) of the transmitted sound wave in \( x > 0 \)?

Write down the form of the pressure perturbation in \( x < 0 \) and \( x > 0 \), find the corresponding velocity potential and state the interfacial boundary conditions. Hence find the amplitudes of the reflected and transmitted waves.

Assume wlog that \( A = 1 \). Verify that the time-averaged acoustic energy flux is conserved. When is all the energy flux transmitted? How much is reflected if \( \rho_0 \gg \rho_1 \) and \( c_0 \approx c_1 \)?

3. **Evanescent waves near an interface.** Find solutions to the wave equation of the form

\[
\phi(x,y,t) = \exp(ikx - i\omega t)f(y),
\]

for the case \( k > \omega/c_0 > 0 \). Hence find the solution in \( y \geq 0 \) in which there is no disturbance as \( y \to \infty \) and waves are forced by the inhomogenous boundary condition

\[
v = \text{Re} [v_0 \exp(ikx - i\omega t)] \quad \text{on} \quad y = 0.
\]

Here \( \nabla \phi = (u, v, 0) \) and \( v_0 \) is a real constant. Over what lengthscale do the waves decay away from the boundary?

Calculate the time-averaged acoustic energy flux \( \langle I \rangle \) and verify that:

(a) the energy flux perpendicular to the boundary \( y = 0 \) satisfies \( \langle I_y \rangle = 0 \);

(b) the energy flux parallel to the boundary satisfies \( \langle I_x \rangle = c \langle E \rangle \) at any position \( y \), where \( E \) is the acoustic energy density and \( c = \omega/k \) is the phase velocity in the \( x \)-direction. [Since \( c < c_0 \), the disturbance and its energy travel subsonically along the boundary.]

*Assuming that surface tension and gravity are negligible, determine whether a non-zero solution can exist in which evanescent sound waves propagate along both sides of an (unforced) interface between two fluids with different physical properties in \( y < 0 \) and \( y > 0 \),

4. **Acoustic waveguide.** Find solutions to the wave equation of the form (2) for a region \( 0 < y < h \) with a rigid boundary at \( y = 0 \) and a free boundary at \( y = h \). (Take \( \omega > 0 \), but make no \textit{a priori} assumption about \( k \).) Show that a wave can propagate in the \( x \)-direction only if \( \omega \) exceeds a critical value \( \omega_c \). What happens if a disturbance is generated at \( x = 0 \) with frequency \( \omega < \omega_c \)?
5. **Spherical waves and radiation.** Explain why the general spherically symmetric solution \( \phi(r,t) \) to the wave equation can be written as

\[
\phi = \frac{-1}{4\pi \rho_0} \left( \frac{q(t-r/c_0)}{r} + \frac{Q(t+r/c_0)}{r} \right),
\]

where \( q \) and \( Q \) are arbitrary functions. Assume from now on that there are only outgoing waves. Calculate the radial velocity \( u_r \) and the pressure perturbation \( \tilde{p} \).

(a) By considering the volume flux through a sphere of radius \( \epsilon \) as \( \epsilon \to 0 \), show that \( q(t) \) is the mass flux out of \( r = 0 \). Show also that \( \phi \) actually satisfies

\[
\nabla^2 \phi - c_0^{-2} \frac{\partial^2 \phi}{\partial t^2} = q(t)\delta(x)/\rho_0,
\]

where \( \delta \) is the Dirac delta function. (Hint: integrate (4) over \( r \leq \epsilon \) and let \( \epsilon \to 0 \).) [The notation in (3) is standard and motivated by the meaning of \( q \). In some of the detailed calculations below, you may prefer to write \( q(t)/4\pi \rho_0 = f(t) \) for brevity.]

(b) Show that in the far-field, i.e. for ‘large’ \( r \), the kinetic energy density \( K \), the potential energy density \( W \), and the acoustic-energy flux \( I = \tilde{p}u \), approximately satisfy the same equations, \( K = W \) and \( I = (K+W)c_0 \), as in a plane wave. Similarly, show that the total power radiated across a ‘large’ sphere of radius \( R \) is approximately

\[
(q(t-R/c_0))^2/4\pi \rho_0 c_0.
\]

[The solution (3) with \( Q = 0 \) is called a point source, or an acoustic monopole of strength \( \dot{q}(t) \).]

What does ‘large \( r \)’ mean for a time-harmonic source with \( q(t) = \text{Re}(q_0 e^{i\omega t}) \)?

(c) For the same time-harmonic source, show that \( \langle K \rangle/\langle W \rangle \sim (c_0/\omega)^2 \) as \( r \to 0 \) and find both \( I_r \) and \( \langle I_r \rangle \) in the same limit. Comment on these results. Compare (1) with (5) for the case \( A \ll (c_0/\omega)^2 \). (What does this condition mean physically?) [This is one of the principles behind the ‘horn loudspeaker’.]

6. **Harmonic series.** Explain why (3) describes a possible acoustic disturbance in a conical tube of any cross-sectional shape. Model an oboe (with all the finger-holes closed) as a small-angle conical tube of length \( \ell \): at the narrow end the cross-sectional area is effectively zero and \( \tilde{p} \) is finite; the larger end is open and \( \tilde{p} \) may be assumed to be zero. [This is a good approximation only if the radius of the larger end is much less than \( c_0/\omega \).] Show that the instrument has a set of normal modes (i.e. standing-wave solutions of the form \( R(r)e^{-i\omega t} \)) with frequencies

\[
\omega \ell/c_0 = n\pi \quad (n \in \mathbb{Z}).
\]

If, instead, the larger end is closed, so that the radial velocity is zero there, show that the corresponding normal-mode frequencies are the solutions of

\[
\omega \ell/c_0 = \tan(\omega \ell/c_0).
\]

Find approximate solutions of (‡) in the high-frequency limit. [The set of frequencies (†) forms a musical ‘harmonic series’, while the set (‡) does not.]
7. An oscillating bubble (Tripos 93124). A bubble makes small spherically symmetric oscillations in a compressible inviscid fluid. When the radius $a(t)$ is perturbed slightly from its mean value $a_0$, the internal dynamics of the bubble are such that the bubble exerts a perturbation pressure $-\beta(a - a_0)$ on the fluid, where $\beta$ is a constant. Derive the linearised equation of motion for the oscillations

$$\rho_0 a_0 \ddot{a} + \frac{\beta a_0}{c_0} a + \beta(a - a_0) = 0,$$

where $\rho_0$ is the undisturbed density of the fluid and $c_0$ is the sound speed (you may quote results from question 5). What is the mechanism of energy loss from the oscillations represented by the ‘damping’ term in this ODE for $a$?

8*. Images. Explain briefly how the method of images can be used to find the sound field produced by a point source placed either near a plane rigid boundary, or in the corner between two plane rigid boundaries at right-angles (i.e. find an image system that satisfies the boundary conditions).

For each case, use the results of 5(b) to write down an approximation to the time-averaged total power radiated by a time-harmonic point source if the distance of the source to the boundaries is much less than a wavelength. Will a whistle sound louder if blown near a wall?

9*. Source with boundaries. A point source (monopole) is placed at $x = x_0$ inside a straight semi-infinite tube aligned along the positive $x$-axis with a closed end at $x = 0$ and cross-sectional area $A$. By integrating (4) with $\delta(x)$ replaced by $\delta(x - x_0)$, and using the boundary conditions, show that the cross-sectional average potential

$$\bar{\phi}(x, t) = \frac{1}{A} \int \int \phi \, dy \, dz$$

satisfies

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} = \frac{q(t)}{\rho_0 A} \delta(x - x_0).$$

[Hint: Recall $\delta(x) = \delta(x)\delta(y)\delta(z).$] What conditions should be imposed on $\bar{\phi}$ and $\partial \bar{\phi}/\partial x$ at $x = 0$, $x = x_0$ and as $x \to \infty$?

For a time-harmonic source, $q(t) = q_0 e^{i\omega t}$ (real part understood), show that

$$\bar{\phi} = \frac{ic_0 q_0}{\omega \rho_0 A} \exp\left\{i\omega \left[ t - (x - x_0)/c_0 \right] \right\} \frac{1 + i \tan(\omega x_0/c_0)}{1 + i \tan(\omega x_0/c_0)}$$

in $x > x_0$.

If $A \ll (c_0/\omega)^2$, why it is reasonable to assume that the sound field is almost one-dimensional (i.e. $\phi(x, t) \approx \bar{\phi}(x, t)$) except near $x = x_0$? Making this assumption, show that if $x_0 \ll c_0/\omega$ (what does this mean physically?) then the time-averaged power radiated across a section in $x > x_0$ is the same as the time-average of (1) for this $q(t)$ and a large factor $4\pi c_0^2/\omega^2 A$ bigger than (5).