

Example Sheet 2: Elastic Waves and Dispersive Waves

1. *Elastic energy.* Write down the relationship between stress and strain in a linearly elastic solid. Hence express the elastic energy $W = \frac{1}{2}e_{ij}\sigma_{ij}$ in terms of the strain and the Lamé moduli. Show also that $W = \frac{1}{2}(\kappa e_{kk}^2 + 2\mu e'_{ij}e'_{ij})$, where $e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}$ is the traceless part of the strain and $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus. [*Thermodynamic stability implies that elastic deformation should require work rather than release energy, hence that $W \geq 0$, hence that $\kappa, \mu \geq 0$.*]

2. *Energy and fluxes.* A plane S-wave has displacement $\mathbf{u} = \mathbf{g}(\hat{\mathbf{k}} \cdot \mathbf{x} - c_s t)$, where $\hat{\mathbf{k}} \cdot \mathbf{g} = 0$ and $|\hat{\mathbf{k}}| = 1$. Show that $K = W$ and $\mathbf{I} = (K + W)c_s \hat{\mathbf{k}}$.

Find the time-averaged energy flux vector $\langle \mathbf{I} \rangle$ for (i) a plane harmonic S-wave with $\mathbf{u} = \mathbf{B} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$, (ii) a plane harmonic P-wave with $\mathbf{u} = A \hat{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$ and (iii) a linear superposition of the waves in (i) and (ii) for the case that they have the same frequency ω and travel in the same direction $\hat{\mathbf{k}}$. [*Use parallel, but unequal, wavevectors \mathbf{k}_s and \mathbf{k}_p .*] Can the separate averaged energy flux vectors in (i) and (ii) be added to give the flux vector in (iii)?

3. *Reflection at a fluid–solid interface.* Plane harmonic elastic/sound waves are incident on a plane interface between a homogeneous elastic solid and a homogeneous elastic liquid. Sketch the situation for each of the possible combinations of incoming direction (from fluid or solid) and incoming wave type (*P*, *SV* or *SH*), showing the directions and type of the outgoing waves (assuming none are evanescent).

Write down the boundary conditions for a fluid–solid interface and explain why these would provide the right number of conditions for each combination to solve for the unknown amplitudes if needed. Under what conditions would a P-wave (sound wave) incident from the fluid result in both an evanescent wave and a propagating wave in the solid.

4. *Reflection of a SV-wave.* A solid with elastic wavespeeds c_p and c_s occupies the region $z < 0$ and is bonded to a rigid boundary at $z = 0$. An SV-wave with displacement

$$\mathbf{u} = B(\cos \theta, 0, -\sin \theta)e^{ik(x \sin \theta + z \cos \theta) - i\omega t}$$

is incident from $z < 0$. Find the form and amplitudes of the reflected waves. If $\sin \theta > c_s/c_p$ show that the solution consists of a reflected SV-wave together with an interfacial P-wave.

For $\sin \theta < c_s/c_p$, write down the time-averaged energy flux vector for each wave separately (using results from lectures and/or question 2) and show that their z -components sum to zero. What happens if $\sin \theta > c_s/c_p$?

5*. *Normal modes for an elastic sphere.* A homogeneous elastic sphere of radius a undergoes radially symmetric motion with displacement field $\mathbf{u}(r, t) = (u_r, 0, 0)$ in spherical polar coordinates. Starting from the vector equation of motion, show that $y(r, t) \equiv r(\nabla \cdot \mathbf{u})$ obeys the one-dimensional wave equation $\ddot{y} = c_p^2 y''$. Hence find the solution for $\nabla \cdot \mathbf{u}$ that has frequency ω and is nonsingular at the origin.

By first integrating this solution to obtain the displacement u_r and then imposing a stress-free boundary condition at the surface, show that the eigenfrequencies corresponding to normal-mode ‘free’ oscillations of the sphere are given by

$$\Omega \cot \Omega = 1 - \frac{c_p^2 \Omega^2}{4c_s^2}, \quad \text{where } \Omega = \frac{\omega a}{c_p}.$$

What are the approximate values of ω in the high-frequency limit?

Note: in spherical polar coordinates, $\sigma_{rr} = (\lambda + 2\mu)(\nabla \cdot \mathbf{u}) - 4\mu u_r/r$ and $\nabla \cdot (g(r), 0, 0) = (r^2 g)' / r^2$.

6. *Stoneley waves.* Extend the analysis of Rayleigh waves given in lectures to examine the propagation of surface waves (whose amplitudes decay away from the interface in both directions) at the interface $z = 0$ between a homogeneous elastic solid and a homogeneous elastic fluid.

With a fluid density $\bar{\rho}$ and a fluid sound speed \bar{c} you should find the analogue of Rayleigh's equation as

$$\frac{c^4 \bar{\rho}}{c_s^4 \rho} \left(\frac{1 - c^2/c_P^2}{1 - c^2/\bar{c}^2} \right)^{1/2} = 4(1 - c^2/c_P^2)^{1/2} (1 - c^2/c_s^2)^{1/2} - (2 - c^2/c_s^2)^2.$$

*Show that this equation has a solution. [*Hint:* recall that $c_s < c_P$, and consider the behaviour of the left- and right-hand sides as $c \rightarrow 0$ and $c \rightarrow \min(c_s, \bar{c})$.]

7. *SH waves in an elastic layer.* Consider the propagation of SH waves in a planar elastic layer with shear modulus μ and shear wavespeed c_s . Suppose that the layer has thickness h , and that the boundaries at $z = 0$ and $z = h$ are both free surfaces. Derive the dispersion relation for modes of the form $u_y = \exp(ikx - i\omega t)f(z)$. Verify that in an average sense (to be made precise), the wave energy flux is equal to the wave energy density multiplied by the group velocity c_g .

8. *Love waves under a rigid surface.* An elastic layer of thickness h , shear modulus $\bar{\mu}$ and shear wavespeed \bar{c}_s , has a rigid upper boundary, and overlies a uniform elastic half space with shear modulus μ and shear wavespeed c_s ($c_s > \bar{c}_s$). Find the dispersion relation for Love waves (SH waves) of frequency ω and wavenumber k in this structure. Determine the cut-off frequency for each mode, and the limiting phase velocity for high-frequency propagation. Sketch graphs of the phase velocity c , frequency ω and group velocity c_g as functions of wavenumber k . [*Hint:* it may be helpful to consider limiting slopes near cut-off and at large k .]

9. *The Klein–Gordon equation.* The transverse displacement $\eta(x, t)$ of a stretched membrane of mass density m and tension T supported by springs with spring constant K and subject to a forcing $f(x, t)$ per unit length, is governed by the Klein–Gordon equation

$$m \frac{\partial^2 \eta}{\partial t^2} - T \frac{\partial^2 \eta}{\partial x^2} + K\eta = f.$$

Show that, for any x_1 and x_2 ,

$$\frac{d}{dt} \int_{x_1}^{x_2} \left(\frac{1}{2} m \dot{\eta}^2 + \frac{1}{2} T \eta_x^2 + \frac{1}{2} K \eta^2 \right) dx = \int_{x_1}^{x_2} f \dot{\eta} dx + F(x_1, t) - F(x_2, t),$$

where $F(x, t) = -T \dot{\eta} \eta_x$. Give a physical interpretation to each term.

For an unforced membrane ($f = 0$), find the dispersion relation for harmonic waves and sketch graphs of frequency, phase velocity and group velocity against wavenumber. [*The time-averaged energy flux is again equal to the time averaged energy density times the group velocity c_g .*]

10. *A causal solution where the wavecrests move toward the source.* What is meant by a 'radiation condition'? Show that the solution of

$$\frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \alpha^2 \psi = 0, \quad \alpha > 0,$$

that corresponds to steady propagation into $0 < x < \infty$ of waves generated at the origin by the boundary condition

$$\psi|_{x=0} = a e^{-i\omega t},$$

is

$$\psi = a e^{-i\omega[t + (\alpha x/\omega^2)]}.$$

[*A physical system to which this problem corresponds is that of a vertical tube (x vertical), containing a density-stratified fluid; this acts as a waveguide for internal gravity waves whose wavelength $2\pi/k$ is short compared with the dimensions of the tube.*]