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Example Sheet 2: Elastic Waves and Dispersive Waves

1. *Elastic energy.* Write down the relationship between stress and strain in a linearly elastic solid. Hence express the elastic energy $W = \frac{1}{2}e_{ij}\sigma_{ij}$ in terms of the strain and the Lamé moduli. Show also that $W = \frac{1}{2}(\kappa e_{kk}^2 + 2\mu e'_{ij}e'_{ij})$, where $e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}$ is the traceless part of the strain and $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus. [*Thermodynamic stability implies that elastic deformation should require work rather than release energy, hence that $W \geqslant 0$, hence that $\kappa, \mu \geqslant 0$.*]

2. *Energy and fluxes.* A plane S-wave has displacement $\mathbf{u} = \mathbf{g}(\hat{\mathbf{k}} \cdot \mathbf{x} - c_{\mathrm{S}}t)$, where $\hat{\mathbf{k}} \cdot \mathbf{g} = 0$ and $|\hat{\mathbf{k}}| = 1$. Show that $K = W$ and $\mathbf{I} = (K + W)c_{\mathrm{S}}\hat{\mathbf{k}}$.

Find the time-averaged energy flux vector $\langle \mathbf{I} \rangle$ for (i) a plane harmonic S-wave with $\mathbf{u} = \mathbf{B}\cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$, (ii) a plane harmonic P-wave with $\mathbf{u} = A\hat{\mathbf{k}}\cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$ and *(iii) a linear superposition of the waves in (i) and (ii) for the case that they have the same frequency ω and travel in the same direction $\hat{\mathbf{k}}$. [*Use parallel, but unequal, wavevectors $\mathbf{k}_{\mathrm{S}}$ and $\mathbf{k}_{\mathrm{P}}$.*] Can the separate averaged energy flux *vectors* in (i) and (ii) be added to give the flux vector in (iii)?

3. *Reflection at a fluid–solid interface.* Plane harmonic elastic/sound waves are incident on a plane interface between a homogeneous elastic solid and a homogeneous elastic liquid. Sketch the situation for each of the *possible* combinations of incoming direction (from fluid or solid) and incoming wave type (P, SV or SH), showing the directions and type of the outgoing waves (assuming none are evanescent).

Write down the boundary conditions for a fluid–solid interface and explain why these would provide the right number of conditions for each combination to solve for the unknown amplitudes if needed. Under what conditions would a P-wave (sound wave) incident from the fluid result in both an evanescent wave and a propagating wave in the solid.

4. *Reflection of a SV-wave.* A solid with elastic wavespeeds c_{P} and c_{S} occupies the region $z < 0$ and is bonded to a rigid boundary at $z = 0$. An SV-wave with displacement

$$\mathbf{u} = B(\cos\theta, 0, -\sin\theta)e^{ik(x\sin\theta + z\cos\theta) - i\omega t}$$

is incident from $z < 0$. Find the form and amplitudes of the reflected waves. If $\sin\theta > c_{\mathrm{S}}/c_{\mathrm{P}}$ show that the solution consists of a reflected SV-wave together with an interfacial P-wave.

For $\sin\theta < c_{\mathrm{S}}/c_{\mathrm{P}}$, write down the time-averaged energy flux vector for each wave separately (using results from lectures and/or question 2) and show that their z-components sum to zero. What happens if $\sin\theta > c_{\mathrm{S}}/c_{\mathrm{P}}$?

5*. *Normal modes for an elastic sphere.* A homogeneous elastic sphere of radius a undergoes radially symmetric motion with displacement field $\mathbf{u}(r,t) = (u_r, 0, 0)$ in spherical polar coordinates. Starting from the vector equation of motion, show that $y(r,t) \equiv r(\nabla \cdot \mathbf{u})$ obeys the one-dimensional wave equation $\ddot{y} = c_{\mathrm{P}}^2 y''$. Hence find the solution for $\nabla \cdot \mathbf{u}$ that has frequency ω and is nonsingular at the origin.

By first integrating this solution to obtain the displacement u_r and then imposing a stress-free boundary condition at the surface, show that the eigenfrequencies corresponding to normal-mode 'free' oscillations of the sphere are given by

$$\Omega\cot\Omega = 1 - \frac{c_{\mathrm{P}}^2\Omega^2}{4c_{\mathrm{S}}^2}, \quad \text{where } \Omega = \frac{\omega a}{c_{\mathrm{P}}} .$$

What are the approximate values of ω in the high-frequency limit?

Note: in spherical polar coordinates, $\sigma_{rr} = (\lambda + 2\mu)(\nabla \cdot \mathbf{u}) - 4\mu u_r/r$ and $\nabla \cdot (g(r), 0, 0) = (r^2 g)'/r^2$.

6. *Stoneley waves.* Extend the analysis of Rayleigh waves given in lectures to examine the propagation of surface waves (whose amplitudes decay away from the interface in both directions) at the interface $z = 0$ between a homogeneous elastic solid and a homogeneous elastic fluid.

With a fluid density $\bar{\rho}$ and a fluid sound speed $\bar{c}$ you should find the analogue of Rayleigh's equation as

$$\frac{c^4}{c_{\mathrm{S}}^4}\frac{\bar{\rho}}{\rho}\left(\frac{1-c^2/c_{\mathrm{P}}^2}{1-c^2/\bar{c}^2}\right)^{1/2} = 4\left(1-c^2/c_{\mathrm{P}}^2\right)^{1/2}\left(1-c^2/c_{\mathrm{S}}^2\right)^{1/2} - \left(2-c^2/c_{\mathrm{S}}^2\right)^2 .$$

*Show that this equation has a solution. [*Hint*: recall that $c_{\mathrm{S}} < c_{\mathrm{P}}$, and consider the behaviour of the left- and right-hand sides as $c \to 0$ and $c \to \min(c_{\mathrm{S}}, \bar{c})$.]

7. *SH waves in an elastic layer.* Consider the propagation of SH waves in a planar elastic layer with shear modulus μ and shear wavespeed c_{S}. Suppose that the layer has thickness h, and that the boundaries at $z = 0$ and $z = h$ are both free surfaces. Derive the dispersion relation for modes of the form $u_y = \exp(ikx - i\omega t)f(z)$. Verify that in an average sense (to be made precise), the wave energy flux is equal to the wave energy density multiplied by the group velocity c_g.

8. *Love waves under a* rigid *surface.* An elastic layer of thickness h, shear modulus $\bar{\mu}$ and shear wavespeed $\bar{c}_{\mathrm{S}}$, has a rigid upper boundary, and overlies a uniform elastic half space with shear modulus μ and shear wavespeed c_{S} ($c_{\mathrm{S}} > \bar{c}_{\mathrm{S}}$). Find the dispersion relation for Love waves (SH waves) of frequency ω and wavenumber k in this structure. Determine the cut-off frequency for each mode, and the limiting phase velocity for high-frequency propagation. Sketch graphs of the phase velocity c, frequency ω and group velocity c_g as functions of wavenumber k. [*Hint*: it may be helpful to consider limiting slopes near cut-off and at large k.]

9. *The Klein–Gordon equation.* The transverse displacement $\eta(x,t)$ of a stretched membrane of mass density m and tension T supported by springs with spring constant K and subject to a forcing $f(x,t)$ per unit length, is governed by the Klein–Gordon equation

$$m\frac{\partial^2\eta}{\partial t^2} - T\frac{\partial^2\eta}{\partial x^2} + K\eta = f\ .$$

Show that, for any x_1 and x_2,

$$\frac{d}{dt}\int_{x_1}^{x_2}\left(\tfrac{1}{2}m\dot{\eta}^2 + \tfrac{1}{2}T\eta_x^2 + \tfrac{1}{2}K\eta^2\right)dx = \int_{x_1}^{x_2} f\dot{\eta}\,dx + F(x_1,t) - F(x_2,t),$$

where $F(x,t) = -T\dot{\eta}\eta_x$. Give a physical interpretation to each term.

For an unforced membrane ($f = 0$), find the dispersion relation for harmonic waves and sketch graphs of frequency, phase velocity and group velocity against wavenumber. [*The time-averaged energy flux is again equal to the time averaged energy density times the group velocity* c_g.]

10. *A causal solution where the wavecrests move toward the source.* What is meant by a 'radiation condition'? Show that the solution of

$$\frac{\partial^4\psi}{\partial x^2\partial t^2} - \alpha^2\psi = 0, \quad \alpha > 0,$$

that corresponds to steady propagation into $0 < x < \infty$ of waves generated at the origin by the boundary condition

$$\psi|_{x=0} = ae^{-i\omega t}\ ,$$

is

$$\psi = ae^{-i\omega[t+(\alpha x/\omega^2)]}\ .$$

[*A physical system to which this problem corresponds is that of a vertical tube (x vertical), containing a density-stratified fluid; this acts as a waveguide for internal gravity waves whose wavelength* $2\pi/k$ *is short compared with the dimensions of the tube.*]