Example Sheet 4: Nonlinear Waves

Parts of the old Tripos questions 6 and 9 overlap with earlier questions on this sheet. Where this is the case, you may simply quote the earlier results rather than rederive them.

1. Shock formation. At time $t = 0$ the velocity $u(x, t)$ in a one-dimensional simple wave, propagating in the positive $x$ direction through a perfect gas, has the form $u = u_m \sin kx$, where $u_m$ and $k$ are positive constants. Find the time $t^*$ at which shocks form. Sketch $u(x)$ at times $t = 0$, $t = \frac{1}{2} t^*$ and $t = t^*$. Show that in the time interval $(0, t^*)$ a single wave-crest (i.e. a local maximum of $u(x, t)$) travels a distance

$$\frac{1}{k} \left( \frac{2c_0}{(\gamma + 1)u_m} + 1 \right).$$

Comment: When $k = 2\pi \times (1\text{kHz})/c_0$, $c_0 = 340\text{ms}^{-1}$, $\gamma = 1.4$, and $u_m = 0.05\text{ms}^{-1}$ (equivalent to $120\text{dB}$, the pain threshold for the ear), the distance is about $320\text{m}$.

2. Shock formation. A perfect gas, initially at rest, occupies the region to the right of a piston whose position is $X(t) = \frac{1}{2}at^2$ for $t > 0$. Find the time and position where a shock first forms.

3. Blood flow. An artery is modelled as a long straight tube with elastic walls and cross-sectional area $A(x, t)$, which contains incompressible, inviscid blood of density $\rho$. On the assumption that the fluid velocity $u$ and pressure $p$ do not vary across the artery, conservation of mass and momentum imply that

$$A_t + (uA)_x = 0 \quad \text{and} \quad \rho u_t + \rho uu_x = -p_x.$$

The area $A$ is related to the fluid pressure $p$ by an elastic ‘tube law’ of the form $p = P(A)$, where $P(A)$ is some given, strictly increasing function. Find the Riemann invariants and their corresponding propagation speeds.

Now suppose that

$$P(A) = p_0 + \frac{\rho c_0^2}{2\kappa} \left( \frac{A}{A_0} \right)^{2\kappa},$$

where $p_0$, $A_0$, $c_0$ and $\kappa$ are positive constants. For $t < 0$ the artery has uniform area $A_0$ and there is no flow. Blood is then pumped into the artery $(x > 0)$ with velocity $U(t)$ at $x = 0$, where

$$U(t) = \begin{cases} U_0 \frac{t}{t_1} \left( 2 - \frac{t}{t_1} \right) & (0 \leq t \leq 2t_1), \\ 0 & (t > 2t_1), \end{cases}$$

and $U_0(1 - \kappa) < c_0$. Calculate the time and place at which a ‘shock’ first forms.

Comment: In an adult human, typical values are $A_0 = 5 \times 10^{-4}\text{m}^2$, $U_0 = 1.2\text{m}\text{s}^{-1}$, $\kappa = 1$, $c_0 = 5\text{m}\text{s}^{-1}$, $p_0 = 10^4\text{N}\text{m}^{-2}$, $\rho = 10^3\text{kg}\text{m}^{-3}$, $t_1 = 0.35\text{s}$. Do you expect shocks to form?

4. A general expansion fan. A piston confines an inviscid compressible fluid (not necessarily a perfect gas) to the right-hand half, $x > 0$, of an infinite tube. The fluid is initially at rest, $u = 0$, with uniform density $\rho_0$ and sound speed $c_0$. For $t > 0$ the piston moves with constant speed $V$ away from the fluid. Assuming that the fluid can keep up with the piston, show that there is a region $R_2$ in the $(x, t)$-plane, in which the local sound speed $c$ takes a constant value $c_2$, which differs from the value $c_0$ in the undisturbed region $R_0$. Find an equation that determines $c_2$ in terms of $V$ and the function $c(\rho)$. Deduce the condition on $V$ for the fluid to keep up with the piston.

Show by reductio ad absurdum, or otherwise, that all the $C_+$ characteristics lying outside both $R_2$ and $R_0$ must pass through the origin. Deduce that for $t > 0$

$$u + c = \begin{cases} c_2 - V, & -Vt \leq x \leq (c_2 - V)t \\
xt^{-1}, & (c_2 - V)t \leq x \leq c_0t \\
c_0, & x \geq c_0t \end{cases}$$

Sketch the forms of $u$ and $c$ as functions of $x$ at two different times.
5. **Expansion fan and escape velocity.** Consider the situation in question 4 for the case of a perfect gas with specific-heat ratio $\gamma$. Find the equations in regions $R_0$, $R_1$ and $R_2$ of
   (i) the $C_-$ characteristic that originates at $x = \xi$ and $t = 0$
   (ii) the trajectory of the gas particle which is at $x = \xi$ when $t = 0$.
Sketch the $C_+$ and $C_-$ characteristics and the particle trajectories in the $(x,t)$-plane. Hence explain what happens when $V > 2(\gamma - 1)^{-1}c_0$.

6. **Two expansion fans (Tripos 75425).** A perfect gas, with constant specific heats in the ratio $\gamma$, is initially at rest with uniform sound speed $c_0$. It is confined by two pistons to the region $0 < x < 2\ell$ of a long cylindrical tube. At time $t = 0$, both pistons are set into impulsive motion away from the gas with constant velocities $u = -V < 0$ and $u = U > 0$.
   (i) For $0 \leq t \leq c_0/t$ show that in the part $x \leq \ell$ (which cannot have been reached by any signal from the piston initially at $x = 2\ell$), every $C_+$ characteristic is a straight line. Show that the fluid velocity $u$ takes the value
   \[
   u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right) \quad \text{for} \quad \left( c_0 - \frac{\gamma + 1}{2} V \right)t < x < c_0t.
   \]
   Give the corresponding value of $c$. Find the shape of the $C_-$ characteristics when $u$ and $c$ take these values. 
   (ii) Deduce that, when $t > \ell/c_0$, the equation
   \[
   u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right)
   \]
   is satisfied only in the smaller interval
   \[
   \left( c_0 - \frac{\gamma + 1}{2} V \right)t < x < \frac{\ell}{t - 1} \left( (\gamma + 1) \left( \frac{ct}{\ell} \right)^{(3-\gamma)/(\gamma+1)} - 2 \left( \frac{ct}{\ell} \right) \right).
   \]
   (iii) For a case with $V/c_0$ about $\frac{1}{2}$ and $U/c_0$ about $\frac{1}{4}$, give a rough sketch indicating four areas of the $(x,t)$ plane throughout each of which $u$ takes a different constant value, to be specified.

7. **A piston-generated shock.** A piston moves with constant positive velocity $u_1$ into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest and the trajectory of the gas particle which is at $x = \xi$ when $t = 0$.
   (i) For $0 < x < 2\ell$, where
   \[
   \frac{p_1}{p_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \quad \frac{1}{\beta^2} + \frac{\gamma + 1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2},
   \]
   where $\beta$ is the shock strength defined as $(p_1 - p_0)/p_0 > 0$, and $p_0$ and $c_0$ are the density and sound speed of the undisturbed gas. Show also that the shock speed $V = c_0(1 + \frac{\gamma + 1}{2\gamma} \beta)^{1/2}$.

8. **Traffic flow.** Assume that the speed of cars down a long straight (one-way) road is a known, monotonically decreasing function $u(\rho)$ of the local density $\rho$ of traffic. The flux of cars is thus given by $q(\rho) = \rho u$. From conservation of cars deduce that $\rho$ is constant on characteristics $dx/dt = c(\rho)$, where $c = dq/d\rho$. Deduce also that if a shock develops between regions of density $\rho_1$ and $\rho_2$ then it propagates with speed $\left[ q(\rho_1) - q(\rho_2) \right]/(\rho_1 - \rho_2)$.
   Consider the case $u(\rho) = U(1 - \rho/\rho_0)$ where $U$ is (10%) faster than the speed limit and $\rho_0$ is the density of a nose-to-tail traffic jam. Sketch the functions $q(\rho)$ and $c(\rho)$. Explain why shocks only form when light traffic is behind heavy traffic, and why the shocks can travel either forwards or backwards depending on the density of traffic.
   A queue of cars with density $\rho_0$ is waiting in $-L < x < 0$ behind a red traffic light at $x = 0$. There are no other cars on the road. The light turns green at $t = 0$. Find the time $T$ when the last car starts to move, and determine the velocity of the last car for $t > T$. [Hint: The solution involves both a shock and an expansion fan.]
9.* A method to generate shock waves in a ‘shock tube’ (Tripos 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at $x = 0$. The gas in $x > 0$ has pressure $p_1$, density $\rho_1$ and specific heat ratio $\gamma_1$; the corresponding values for the gas in $x < 0$ are $p_2$, $\rho_2$, $\gamma_2$ where $p_2 > p_1$. At $t = 0$ the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed $V$, use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

\[ p = p_2 \text{ for } x < -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2}t, \]
\[ p = p_1 \text{ for } x > Ut, \]
\[ p = p_m \text{ for } -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} - \frac{\gamma_2 + 1}{2}V < x < Ut, \]

where $p_m$ is as yet unknown, and the shock velocity, $U$, is a constant to be found in terms of $p_m$, $p_1$, $\rho_1$, $\gamma_1$.

Show that $V$ is related to $p_m$ by the following two equations:

\[ V = (p_m - p_1) \left(\frac{1}{2} \rho_1 [\gamma_1 + 1)p_m + (\gamma_1 - 1)p_1]\right)^{-1/2}, \]
\[ V = \frac{2}{\gamma_2 - 1} \left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} \left[1 - \left(\frac{p_m}{p_2}\right)^{(\gamma_2 - 1)/2\gamma_2}\right], \]

and hence show that there is a unique solution for $p_m$ and $V$. 