Example Sheet 4: Nonlinear Waves

Parts of the old Tripos questions 6 and 9 overlap with earlier questions on this sheet. Where this is the case, you may simply quote the earlier results rather than rederive them.

1. Shock formation. At time \( t = 0 \) the velocity \( u(x, t) \) in a one-dimensional simple wave, propagating in the positive \( x \) direction through a perfect gas, has the form \( u = u_m \sin kx \), where \( u_m \) and \( k \) are positive constants. Find the time \( t^* \) at which shocks form. Sketch \( u(x) \) at times \( t = 0, t = \frac{1}{2} t^* \) and \( t = t^* \). Show that in the time interval \((0, t^*)\) a single wave-crest (i.e. a local maximum of \( u(x, t) \)) travels a distance

\[
\frac{1}{k} \left( \frac{2c_0}{(\gamma + 1)u_m} + 1 \right).
\]

Comment: When \( k = 2\pi \times (1\text{kHz})/c_0, c_0 = 340\text{ms}^{-1}, \gamma = 1.4, \) and \( u_m = 0.05\text{ms}^{-1} \) (equivalent to 120dB, the pain threshold for the ear), the distance is about 320m.

2. Shock formation. A perfect gas, initially at rest, occupies the region to the right of a piston whose position is \( X(t) = \frac{1}{2}at^2 \) for \( t > 0 \). Find the time and position where a shock first forms.

3. Blood flow. An artery is modelled as a long straight tube with elastic walls and cross-sectional area \( A(x, t) \), which contains incompressible, inviscid blood of density \( \rho \). On the assumption that the fluid velocity \( U \) and pressure \( p \) do not vary across the artery, conservation of mass and momentum imply that

\[
\frac{A}{t} + (uA)_x = 0 \quad \text{and} \quad \rho u_t + \rho uu_x = -p_x.
\]

The area \( A \) is related to the fluid pressure \( p \) by an elastic ‘tube law’ of the form \( p = P(A) \), where \( P(A) \) is some given, strictly increasing function. Find the Riemann invariants and their corresponding propagation speeds.

Now suppose that

\[
P(A) = p_0 + \frac{\kappa c_0^2}{2\kappa} \left( \frac{A}{A_0} \right)^{2\kappa},
\]

where \( p_0, A_0, c_0 \) and \( \kappa \) are positive constants. For \( t < 0 \) the artery has uniform area \( A_0 \) and there is no flow. Blood is then pumped into the artery \((x > 0)\) with velocity \( U(t) \) at \( x = 0 \), where

\[
U(t) = \begin{cases} U_0 \frac{t}{t_1} \left( \frac{2 - t}{t_1} \right) & (0 \leq t \leq 2t_1), \\ 0 & (t > 2t_1), \end{cases}
\]

and \( U_0(1 - \kappa) < c_0 \). Calculate the time and place at which a ‘shock’ first forms.

Comment: In an adult human, typical values are \( A_0 = 5 \times 10^{-4} \text{m}^2, U_0 = 1.2 \text{ms}^{-1}, \kappa = 1, c_0 = 5 \text{m} \text{s}^{-1}, p_0 = 10^4 \text{N} \text{m}^{-2}, \rho = 10^3 \text{kg} \text{m}^{-3}, t_1 = 0.35 \text{s} \). Do you expect shocks to form?

4. A general expansion fan. A piston confines an inviscid compressible fluid (not necessarily a perfect gas) to the right-hand half, \( x > 0 \), of an infinite tube. The fluid is initially at rest, \( u = 0 \), with uniform density \( \rho_0 \) and sound speed \( c_0 \). For \( t > 0 \) the piston moves with constant speed \( V \) away from the fluid. Assuming that the fluid can keep up with the piston, show that there is a region \( R_2 \) in the \((x, t)\)-plane, in which the local sound speed \( c \) takes a constant value \( c_2 \), which differs from the value \( c_0 \) in the undisturbed region \( R_0 \). Find an equation that determines \( c_2 \) in terms of \( V \) and the function \( c(\rho) \). Deduce the condition on \( V \) for the fluid to keep up with the piston.

Show by \textit{reductio ad absurdum}, or otherwise, that all the \( C_+ \) characteristics lying outside both \( R_2 \) and \( R_0 \) must pass through the origin. Deduce that for \( t > 0 \)

\[
u + c = \begin{cases} c_2 - V, & -Vt \leq x \leq (c_2 - V)t \\
xt^{-1}, & (c_2 - V)t \leq x \leq c_0t \\
c_0, & x \geq c_0t \end{cases}
\]

Sketch the forms of \( u \) and \( c \) as functions of \( x \) at two different times.
5. **Expansion fan and escape velocity.** Consider the situation in question 4 for the case of a perfect gas with specific-heat ratio $\gamma$. Find the equations in regions $R_0$, $R_1$ and $R_2$ of
   (i) the $C_-$ characteristic that originates at $x = \xi$ and $t = 0$
   (ii) the trajectory of the gas particle which is at $x = \xi$ when $t = 0$.

Sketch the $C_+$ and $C_-$ characteristics and the particle trajectories in the $(x, t)$-plane. Hence explain what happens when $V > 2(\gamma - 1)^{-1}c_0$.

6. **Two expansion fans (Tripos 75425).** A perfect gas, with constant specific heats in the ratio $\gamma$, is initially at rest with uniform sound speed $c_0$. It is confined by two pistons to the region $0 < x < 2\ell$ of a long cylindrical tube. At time $t = 0$, both pistons are set into impulsive motion away from the gas with constant velocities $u = -V < 0$ and $u = U > 0$.

   (i) For $0 \leq t \leq \ell/c_0$ show that in the part $x \leq \ell$ of the tube (which cannot have been reached by any signal from the piston initially at $x = 2\ell$), every $C_+$ characteristic is a straight line. Show that the fluid velocity $u$ takes the value
   \[
   u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right) \quad \text{for} \quad \left( c_0 - \frac{\gamma + 1}{2} V \right) t < x < c_0 t .
   \]

   Give the corresponding value of $c$. Find the shape of the $C_-$ characteristics when $u$ and $c$ take these values.

   (ii) Deduce that, when $t > \ell/c_0$, the equation
   \[
   u = \frac{2}{\gamma + 1} \left( \frac{x}{t} - c_0 \right)
   \]
   is satisfied only in the smaller interval
   \[
   \left( c_0 - \frac{\gamma + 1}{2} V \right) t < x < \frac{\ell}{\gamma - 1} \left( (\gamma + 1) \left( c_0 t / \ell \right)^{(3-\gamma)/(\gamma+1)} - 2 \left( c_0 t / \ell \right) \right) .
   \]

   (iii) For a case with $V/c_0$ about $\frac{1}{2}$ and $U/c_0$ about $\frac{1}{4}$, give a rough sketch indicating four areas of the $(x, t)$ plane throughout each of which $u$ takes a different constant value, to be specified.

7. **A piston-generated shock.** A piston moves with constant positive velocity $u_1$ into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure $p_0$, and is moving with constant velocity $u_1$ in the region between the piston and the shock, throughout which region the density and pressure also take constant values $\rho_1, p_1$ which are determined by
   \[
   \frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta} , \quad \frac{1}{\beta^2} + \frac{\gamma + 1}{2\beta^2} = \frac{c_0^2}{\gamma^2 u_1^2} ,
   \]
   where $\beta$ is the shock strength defined as $(p_1 - p_0)/p_0 > 0$, and $p_0$ and $c_0$ are the density and sound speed of the undisturbed gas. Show also that the shock speed $V = c_0(1 + \gamma + 1/\gamma)1/2$.

8. **Traffic flow.** Assume that the speed of cars down a long straight (one-way) road is a known, monotonically decreasing function $u(\rho)$ of the local density $\rho$ of traffic. The flux of cars is thus given by $q(\rho) = \rho u$. From conservation of cars deduce that $\rho$ is constant on characteristics $dx/dt = c(\rho)$, where $c = dq/d\rho$. Deduce also that if a shock develops between regions of density $\rho_1$ and $\rho_2$ then it propagates with speed $[q(\rho_1) - q(\rho_2)]/[(\rho_1 - \rho_2)$.

Consider the case $u(\rho) = U(1 - \rho/\rho_0)$ where $U$ is (10%) faster than the speed limit and $\rho_0$ is the density of a nose-to-tail traffic jam. Sketch the functions $q(\rho)$ and $c(\rho)$. Explain why shocks only form when light traffic is behind heavy traffic, and why the shocks can travel either forwards or backwards depending on the density of traffic.

A queue of cars with density $\rho_0$ is waiting in $-L < x < 0$ behind a red traffic light at $x = 0$. There are no other cars on the road. The light turns green at $t = 0$. Find the time $T$ when the last car starts to move, and determine the velocity of the last car for $t > T$. [Hint: The solution involves both a shock and an expansion fan.]
A method to generate shock waves in a ‘shock tube’ (Tripos 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at \( x = 0 \). The gas in \( x > 0 \) has pressure \( p_1 \), density \( \rho_1 \) and specific heat ratio \( \gamma_1 \); the corresponding values for the gas in \( x < 0 \) are \( p_2, \rho_2, \gamma_2 \) where \( p_2 > p_1 \). At \( t = 0 \) the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed \( V \), use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

\[
p = p_2 \quad \text{for} \quad x < -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} t ,
\]

\[
p = p_1 \quad \text{for} \quad x > Ut ,
\]

\[
p = p_m \quad \text{for} \quad -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} - \frac{\gamma_2 + 1}{2} V t < x < Ut ,
\]

where \( p_m \) is as yet unknown, and the shock velocity, \( U \), is a constant to be found in terms of \( p_m, p_1, \rho_1, \gamma_1 \).

Show that \( V \) is related to \( p_m \) by the following two equations:

\[
V = (p_m - p_1) \left(\frac{1}{2} \rho_1 [(\gamma_1 + 1)p_m + (\gamma_1 - 1)p_1]\right)^{-1/2} ,
\]

\[
V = \frac{2}{\gamma_2 - 1} \left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} \left[ 1 - \left(\frac{p_m}{p_2}\right)^{(\gamma_2 - 1)/2\gamma_2} \right] ,
\]

and hence show that there is a unique solution for \( p_m \) and \( V \).