Waves: D24d **Example Sheet 4: Dispersive Waves and Ray Theory** Lent 2018

*As usual, a $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** necessarily harder than unstarred questions.*

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1. *Finite-depth capillary–gravity waves.* The dispersion relation for water waves of wavenumber k, including the effects of surface tension, is

$$\omega^2 = k\left(g + \frac{Tk^2}{\rho}\right)\tanh kh\ .$$

Show that for sufficiently large k the group and phase velocities c_g and c become proportional to $k^{1/2}$ and independent of g and h, and that $c_g \sim \frac{3}{2}c$. What is 'sufficiently large'?

In ripple-tank experiments it is desired to keep c_g and c as constant as possible for smallish values of kh. By expanding ω^2 about $k = 0$, determine approximately what value of h, h_0 say, should be used. $*$Show also that for $h > h_0$ there must exist a minimum value of the group velocity at some finite non-zero value of k.

Comment: For water, a typical value of $T/\rho g$ would be $7.5\,\mathrm{mm}^2$, so that $h_0 = 4.7\,\mathrm{mm}$.

2. *Stationary phase (Tripos 77126).* A hypothetical physical system permits one-dimensional wave propagation in the x-direction according to the equation

$$\frac{\partial\psi}{\partial t} - \beta\frac{\partial^3\psi}{\partial x^3} = 0 \qquad (\beta > 0,\quad \text{constant})\ . \tag{1}$$

Write down the corresponding dispersion relation and sketch graphs of frequency, phase velocity and group velocity as functions of wave number. Determine whether the shortest or the longest waves are found at the front of a dispersing wave packet arising from a localised initial disturbance. Do the wave crests move faster or slower than the wave packet as a whole?

Assume that ψ is real. An initial disturbance is given in the form of a Fourier integral,

$$\psi(x, 0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk\ .$$

Write down the corresponding solution $\psi(x, t)$ of (1). Use the method of stationary phase to obtain an approximation to this solution for large t with $V = x/t$ held constant, where $V > 0$.

What can you say about ψ in the same large-time limit if (i) $V < 0$, $*$(ii) $V = 0$?

3. *Stationary phase.* Find a combination of gravity waves (neglecting surface tension) on deep water travelling in the directions x increasing and x decreasing that satisfies the conditions

$$\zeta = \zeta_0\cos kx, \quad \frac{\partial\zeta}{\partial t} = 0\,,$$

at time $t = 0$, where ζ is the upward displacement of the free surface at $z = 0$.

In a deep and very long channel parallel to the x-axis, the water surface is distorted by the action of air jets into a shape

$$\zeta = -\zeta_0 e^{-(x/a)^2}\ ,$$

independent of y, and then released from rest at time $t = 0$. Obtain the subsequent shape of the free surface as a sum of two Fourier integrals. Use the method of stationary phase to obtain their approximate value when x and t are both large and positive.

4. *Trapped internal gravity waves.* Two semi-infinite layers of fluid with uniform densities $\rho_0 - \Delta\rho$ and $\rho_0 + \Delta\rho$ are separated by a layer of fluid in $-H \leqslant z \leqslant H$ where $\rho(z) = \rho_0 - (z/H)\Delta\rho$. Using the *exact* equation(s) for a linearised perturbation, explain why the vertical velocity w and its derivative $\partial w/\partial z$ should be expected to be continuous at $z = \pm H$.

   Consider plane harmonic waves trapped by the stratification, with

$$w = \mathrm{e}^{\mathrm{i}\kappa\cos\theta x + \mathrm{i}\kappa\sin\theta y - \mathrm{i}\Omega t} f(z)$$

   where $f(z)$ is an even function of z and $f(z) \to 0$ as $z \to \pm\infty$. Obtain the approximate dispersion relation

$$\left(\frac{N_0^2}{\Omega^2} - 1\right)^{1/2} \tan\left[\left(\frac{N_0^2}{\Omega^2} - 1\right)^{1/2} \kappa H\right] = 1\,, \qquad (2)$$

   on the assumptions that

$$\Delta\rho \ll \rho_0\ , \quad \frac{\Delta\rho}{\rho_0 H} \ll \left(\frac{N_0^2}{\Omega^2} - 1\right)^{1/2} \kappa\ ,$$

   where N_0 is the constant value which the Brunt-Väisälä frequency approximately takes in the middle layer.

   Comment on the form of (2) in the double limit $N_0 \to \infty$, $H \to 0$ with $N_0^2 H$ held constant.

5. *Rays in a slowly varying medium.* Derive the ray-tracing equations for wave propagation through a slowly varying medium.

   (i) Show that for a time-independent medium the frequency ω is constant at a ‘ray point’ moving with the group velocity.

   (ii) If the properties of the medium are also independent of x and y, deduce Snell’s law that along a ray

$$\sin\alpha \propto c\ ,$$

   where α is the angle between the wavenumber $\mathbf{k}$ and the z-axis, and $c(\mathbf{k})$ is the local phase speed.

   (iii) For what type of dispersion relation is the direction of the ray parallel to $\mathbf{k}$?

   Consider the dispersion relation $\omega = A|\mathbf{k}|z$, where A is a constant. Show that each ray is the arc of a circle. Show also that a wave packet moving towards the plane $z = 0$ takes an infinite time to reach it.

6. *Wave breaking.* Ocean surface waves propagate obliquely from $x = \infty$ towards a straight beach at $x = 0$ where they break and are dissipated. The water depth $h(x)$ is a slowly varying, increasing function of x, with $h(0) = 0$ and $h(x) \to \infty$ as $x \to \infty$, and the waves approximately satisfy the local dispersion relation

$$\Omega^2 = g\kappa\tanh\kappa h\ ,$$

   where $\kappa^2 = k_1^2 + k_2^2$ for the surface wavenumber (k_1, k_2). Show that the shorewards component of the wavenumber increases in magnitude, with $k_1 \sim -\omega[gh(x)]^{-1/2}$ as $x \to 0$.

   $*$ The amplitude A of the waves varies in such a way that the shorewards energy flux, proportional to $(\mathbf{c}_g \cdot \widehat{\mathbf{x}})A^2$, is constant (from conservation of wave action). Show that if the waves break in a region where $\kappa h \ll 1$ and when the maximum wave slope $A\kappa$ reaches a critical value S_c, then the point x_b at which they break is given by

$$A^2(\infty)\frac{-k_1(\infty)}{\kappa(\infty)}\frac{\omega g}{[gh(x_b)]^{3/2}} = 2S_c^2\ .$$

$*$7. *The wave-crest pattern near a shore line (Tripos 87327).* Surface waves on water have a dispersion relation $\omega = \Omega(\kappa; x, y)$ where $\kappa^2 = k_1^2 + k_2^2$, (x, y) are coordinates in the plane of the surface, and the medium is slowly varying in the (x, y) coordinates.

Use the ray-tracing equations to show that ω is constant on rays, $dy/dx = k_2/k_1$. Show also that the wave crests at any instant are given by $dy/dx = -k_1/k_2$.

The wave motion takes place over a sloping beach with the unperturbed water depth specified by $h(x) = \alpha x^{1/2}$, with α a small positive constant. The dispersion relation for such waves is given by

$$\Omega^2 = g\kappa \tanh \kappa h.$$

Far from the shore-line $x = 0$, the waves are plane, have frequency ω, and have angle Φ between the crests and the shore-line. As the waves propagate towards the shore they become non-planar. Obtain the parametric equations

$$x = \frac{\lambda^2 g^2}{\alpha^2 \omega^4} \tanh^2 \lambda \,,$$

$$y - y_0 = \frac{g^2}{\alpha^2 \omega^4} \int_0^\lambda \frac{(1 - \tanh^2 \xi \sin^2 \Phi)^{1/2}}{\tanh \xi \sin \Phi} \frac{d}{d\xi}(\xi^2 \tanh^2 \xi)\, d\xi \,,$$

for the wave crest that passes through the shore-line at $y = y_0$. [*Hint: consider* $\lambda = \kappa h$.] Show that near the shore-line the equation of the wave crest can be written

$$(y - y_0)^4 \approx \left(\frac{4}{3 \sin \Phi}\right)^4 \frac{g^2}{\alpha^2 \omega^4} x^3 \,.$$

8. *The steady wave pattern generated by a duck swimming on a (pseudo-)fluid.* Consider a duck swimming steadily with velocity $(U, 0)$ on a deep homogeneous 'pseudo-fluid'. In the duck's frame, the dispersion relation is found to be

$$\Omega(\mathbf{k}) = \lambda |\mathbf{k}|^p - U k_1 \,, \qquad \mathbf{k} = (k_1, k_2)\,,$$

where λ and p are constants, and $0 < p < 1$. (For a 'real' fluid $p = \frac{1}{2}$ and $\lambda = g^{1/2}$.) The duck generates a steady wave pattern. By writing $(k_1, k_2) = |\mathbf{k}|(\cos \beta, \sin \beta)$, show that the waves satisfy

$$|\mathbf{k}| = \left(\frac{\lambda}{U \cos \beta}\right)^{1/(1-p)} \,,$$

and that the group velocity of these waves can be expressed as

$$\mathbf{c}_g = U(p \cos^2 \beta - 1,\, p \sin \beta \cos \beta)\,.$$

Deduce that the waves occupy a wedge of semi-angle $\sin^{-1}[p/(2-p)]$ about the negative x_1-axis. Find equation(s) describing the wave crests.

$*$ Sketch, or plot numerically, the wave-crest pattern for the case $p = \frac{2}{3}$.

9. *Reflection and absorption of internal gravity waves.* Two-dimensional internal gravity waves on a slowly varying shear flow in the atmosphere satisfy the dispersion relation

$$\omega = \gamma z k + \frac{N k}{(k^2 + m^2)^{1/2}}\,,$$

where γ and N are positive constants, and $\mathbf{k} = (k, 0, m)$. Show that, as a wave packet moves, ω and k remain constant, while

$$m(t) = m_0 - \gamma k t\,,$$

where m_0 is a constant. If $k, m_0 > 0$, find the vertical motion $z(t)$ of a wave packet generated at the origin. By considering the values of $\mathrm{d}x/\mathrm{d}t$ (and m) near

$$z = 0 \text{ (twice)}, \quad z = -\frac{N}{\gamma}\left(\frac{1}{k} - \frac{1}{(k^2 + m_0^2)^{1/2}}\right) \quad \text{and} \quad z = \frac{N}{\gamma (k^2 + m_0^2)^{1/2}},$$

or otherwise, sketch the ray path, $*$ and the orientation of the crests at points along the path.