11. Let \( a(x) > 0, x \in [0, 1] \), be a given smooth function. We solve the diffusion equation with variable diffusion coefficient, \( u_t = \frac{1}{2}(2 - 5\mu + 6\mu^2)u_m + \frac{2}{3}\mu(2 - 3\mu)(u_{m-1} + u_{m+1}) - \frac{1}{12}\mu(1 - 6\mu)(u_{m-2} + u_{m+2}) \), where \( \mu = \frac{\Delta x}{\Delta t} \), \( n > 0 \), \( 1 \leq m \leq M \) and \( \Delta x = \frac{x}{M+1} \). Prove that the local error is \( O(h^4) \). Then, justifying carefully every step of your analysis, show (e.g. by using the eigenvalue technique) that the method is stable for all \( 0 < \mu < \frac{1}{2\max}\), where \( \max_{x\in[0,1]} a(x) \). [Hint: In the second half, use Gershgorin theorem to show that the matrix \( A \) occurring in the relation \( u^{n+1} = Au^n \) satisfies \( \rho(A) \leq 1 \].

12. Apply the Fourier stability test to the difference equation

\[
\begin{align*}
  u_m^{n+1} &= \frac{1}{2}(2 - 5\mu + 6\mu^2)u_m + \frac{2}{3}\mu(2 - 3\mu)(u_{m-1} + u_{m+1}) - \frac{1}{12}\mu(1 - 6\mu)(u_{m-2} + u_{m+2}),
\end{align*}
\]

where \( m \in \mathbb{Z} \). Deduce that the test is satisfied if and only if \( 0 < \mu < \frac{3}{2} \).

13. A square grid is drawn on the region \( \{(x, t) : 0 \leq x \leq 1, t \geq 0\} \) in \( \mathbb{R}^2 \), the grid points being \( (m\Delta x, n\Delta t), 0 \leq m \leq M + 1, n = 0, 1, 2, \ldots \), where \( \Delta x = \frac{x}{M+1} \) and \( M \) is odd. Let \( u(x, t) \) be an exact solution of the wave equation \( u_{tt} = u_{xx} \) and let the boundary values \( u(x, 0), 0 \leq x \leq 1, \) \( u(0, t), t > 0, \) and \( u(1, t), t > 0, \) be given. Further, an approximation to \( \partial u/\partial t \) at \( t = 0 \) allows each of the function values \( u(m\Delta x, \Delta x), m = 1, 2, \ldots, M, \) to be estimated to accuracy \( \epsilon \). Then, the difference equation

\[
\begin{align*}
  u_m^{n+1} &= u_m^{n+1} + u_m^{n-1} - u_m^{n-1},
\end{align*}
\]

is applied to estimate \( u \) at the remaining grid points. Prove that all of the moduli of the errors \( |u_m^n - u(m\Delta x, n\Delta t)| \) are bounded above by \( \frac{1}{2}M \), even when \( n \) is very large. [Hint: Verify that the local error is zero. For \( n = 1 \) and \( 1 \leq m \leq M \), let the error in \( u(m\Delta x, \Delta x) \) be \( \delta_m \), where \( \delta_m \) is the Kronecker delta and where \( k \) is an arbitrary integer in \( (1, 2, \ldots, M) \). Draw a diagram that shows the contribution from this error to \( u_m^n \) for every \( m \) and \( n \geq 1 \].

Matlab demo: Download the Matlab GUI for Stability of 1D PDEs at http://www.maths.cam.ac.uk/undergrad/course/na/ii/pde_stability/pde_stability.php. Review the stability condition from the lectures Problem 2.28 and test its sharpness empirically using the GUI.

14. A rectangular grid is drawn on \( \mathbb{R}^2 \), with grid spacing \( \Delta x \) in the \( x \)-direction and \( \Delta t \) in the \( t \)-direction. Let the difference equation

\[
\begin{align*}
  u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu \left[ a \left( u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1} \right) + b \left( u_m^{n-1} - 2u_m^n + u_m^{n+1} \right) + c \left( u_{m-1}^{n-1} - 2u_m^n + u_{m+1}^{n-1} \right) \right],
\end{align*}
\]

where \( \mu = \frac{(\Delta x)^2}{\Delta t^2} \), be used to approximate solutions of the wave equation \( u_{tt} = u_{xx} \). Deduce that, with constant \( \mu \), the local error is \( O((\Delta x)^4) \) if and only if the parameters \( a, b, c \) satisfy \( a = c \) and \( a + b + c = 1 \). Show also that, if these conditions hold, then the Fourier stability condition is achieved for all values of \( \mu \) if and only if the parameters also satisfy \( |b| \leq 2a \). [Hint: In the second half, the roots of the characteristic equation satisfy \( x_1, x_2 = 1 \). Then, \( |x_1|, |x_2| \leq 1 \) if \( D \leq 0 \), where \( D \) is the discriminant of the equation.]

15. For a given analytic function \( f \) we consider its truncated Fourier approximation on the interval \([-1, 1]\), i.e.,

\[
  f(x) \approx \phi_N(x) = \sum_{n=-N/2}^{N/2} \hat{f}_n e^{i\pi nx}, \quad \text{where} \quad \hat{f}_n = \frac{1}{2} \int_{-1}^{1} f(\tau) e^{-i\pi n \tau} d\tau, \quad n \in \mathbb{Z}.
\]
16. Unless \( f \) is analytic, the rate of decay of its Fourier harmonics can be very slow, certainly slower than \( O(N^{-1}) \). To explore this, let \( f(x) = |x|^{-1/2} \). Prove that \( \hat{f}_n = g(-n) + g(n) \), where \( g(n) = \int_0^1 e^{inx^2} \, dx \). Moreover, with the error function \( \operatorname{erf} \) defined as the integral
\[
\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt, \quad z \in \mathbb{C},
\]
show that its Fourier coefficients are
\[
\hat{f}_n = \frac{\operatorname{erf}(\sqrt{n} \pi)}{2\sqrt{n}} + \frac{\operatorname{erf}(-\sqrt{n} \pi)}{2\sqrt{n}},
\]
and asymptotically for \( |n| \gg 1 \) we have \( \hat{f}_n = O(n^{-1/2}) \). [Hint: For the last identity use without proof the asymptotic estimate \( \operatorname{erf}(\sqrt{x}) = 1 + O(x^{-1}) \) for \( x \in \mathbb{R}, |x| \gg 1 \).]

17. Consider the solution of the two-point boundary value problem
\[
(2 - \cos \pi x)u'' + u = 1, \quad -1 \leq x \leq 1, \quad u(-1) = u(1),
\]
using the spectral method. Plugging the Fourier expansion of \( u \) into this differential equation, show that the \( \hat{u}_n \) obey a three-term recurrence relation. Calculate \( \hat{u}_0 \) separately and using the fact that \( \hat{u}_{-n} = \hat{u}_n \) (why?), prove further that the computation of \( \hat{u}_n \) for \( N/2 + 1 \leq n \leq N/2 \) (assuming that \( \hat{u}_n = 0 \) outside this range of \( n \)) reduces to the solution of an \((N/2) \times (N/2)\) tridiagonal system of algebraic equations.

18. Set
\[
a(x) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{i\pi nx}, \quad (2.1)
\]
the Fourier expansion of \( a \). Explain why \( a \) is periodic with period 2. Further, let \( \tilde{n} \) denote some selected value of \( n \). Evaluate \( \frac{1}{2} \int_{-1}^{1} a(x) e^{-i\tilde{n}x} \, dx \) with \( a(x) \) given by (2.1). Do so, you have just computed the Fourier coefficient \( \hat{a}_{\tilde{n}} \). Now choose \( a(x) = \cos \pi x \) and compute its corresponding Fourier coefficients. With this, derive an explicit expression for the coefficients in the N-term truncated Fourier approximation of the solution \( u \) of
\[
\left\{
\begin{array}{l}
((\cos \pi x + 2)u_x)_x = \sin \pi x, \quad x \in [-1, 1] \\
\hbox{periodic boundary conditions and normalisation condition } \int_{-1}^{1} u(x) \, dx = 0.
\end{array}
\right.
\]

19. Let \( u \) be an analytic function in \([-1, 1]\) that can be extended analytically into the complex plane and possesses a Chebyshev expansion \( u = \sum_{n=0}^{\infty} \hat{u}_n T_n \). Express \( u' \) in an explicit form as a Chebyshev expansion.

20. The two-point ODE \( u'' + u = 1, \quad u(-1) = u(1) = 0 \), is solved by a Chebyshev method.
   (a) Show that the odd coefficients are zero and that \( u(x) = \sum_{n=0}^{\infty} \hat{u}_{2n} T_{2n}(x) \). Express the boundary conditions as a linear condition of the coefficients \( \hat{u}_{2n} \).
   (b) Express the differential equation as an infinite set of linear algebraic equations in the coefficients \( \hat{u}_{2n} \).
   (c) Discuss how to truncate the linear system, keeping in mind the exponential convergence of the method and the floating-point precision of your computer.
   (d) While \( u(-1) = u(1) \) we cannot expect a standard spectral method to converge at spectral speed. Why?

Matlab demo: Compare your conclusions with the online documentation for solving this ODE at \http://www.maths.cam.ac.uk/undergrad/course/na/ii/chebyshev/chebyshev.php. How are the boundary conditions enforced in practice?