12. Apply the Fourier stability test to the difference equation

\[ u_{m+1}^{n+1} = u_m^n + \mu \left[ a_{m-1/2} u_{m-1}^n - (a_{m-1/2} + a_{m+1/2}) u_m^n + a_{m+1/2} u_{m+1}^n \right], \]

where \( a_n = a(nh) \), \( \mu = \frac{\Delta t}{\Delta x^2} \), \( n \geq 0 \), \( 1 \leq m \leq M \) and \( h = \Delta x = \frac{1}{M+1} \). Prove that the local error is \( O(h^4) \). Then, justifying carefully every step of your analysis, show (e.g. by using the eigenvalue technique) that the method is stable for all \( 0 < \mu < \frac{1}{2\Delta x^2} \), where \( \Delta x = \max_{x \in [0,1]} a(x) \).

[Hint: In the second half, use Gershgorin theorem to show that the matrix \( A \) occurring in the relation \( u_{n+1}^{m+1} = Au^n \) satisfies \( \rho(A) \leq 1 \).]

13. A square grid is drawn on the region \((x, t) : 0 \leq x \leq 1, t \geq 0\) in \( \mathbb{R}^2 \), the grid points being \((m\Delta x, n\Delta t)\), \( 0 \leq m \leq M+1, n = 0, 1, 2, \ldots \), where \( \Delta x = \frac{1}{M+1} \) and \( M \) is odd. Let \( u(x, t) \) be an exact solution of the wave equation \( u_{tt} = u_{xx} \) and let the boundary values \( u(x, 0), 0 \leq x \leq 1, u(0, t), t > 0, \) and \( u(1, t), t > 0 \), be given. Further, an approximation to \( \partial u/\partial t \) at \( t = 0 \) allows each of the function values \( u(m\Delta x, \Delta t), m = 1, 2, \ldots, M \), to be estimated to accuracy \( \epsilon \). Then, the difference equation

\[ u_{m+1}^{n+1} = u_{m+1}^n + u_{m-1}^n - u_m^{n+1} \]

is applied to estimate \( u \) at the remaining grid points. Prove that all of the moduli of the errors \( |u_m^n - u(m\Delta x, n\Delta t)| \) are bounded above by \( \frac{\epsilon}{M} \), even when \( M \) is very large.

[Hint: Verify that the local error is zero. For \( n = 1 \) and \( 1 \leq m \leq M \), let the error in \( u(m\Delta x, \Delta t) \) be \( \delta_{nk} \), where \( \delta_{nk} \) is the Kronecker delta and \( k \) is an arbitrary integer in \( (1, 2, \ldots, M) \). Draw a diagram that shows the contribution from this error to \( u_m^n \) for every \( m \) and \( n > 1 \).]

14. A rectangular grid is drawn on \( \mathbb{R}^2 \), with grid spacing \( \Delta x \) in the \( x \)-direction and \( \Delta t \) in the \( t \)-direction. Let the difference equation

\[ u_{m+1}^{n+1} - \frac{2u_m^n + u_{m-1}^{n+1}}{\mu} + b \left( u_m^n - 2u_{m+1}^n + u_{m+1}^{n+1} \right) = c \left( u_{m-1}^{n-1} - u_{m+1}^{n-1} + u_{m+1}^{n+1} \right), \]

where \( \mu = \frac{\Delta x^2}{\Delta t} \), be used to approximate solutions of the wave equation \( u_{tt} = u_{xx} \). Deduce that, with constant \( \mu \), the local error is \( O((\Delta x)^4) \) if and only if the parameters \( a, b \) and \( c \) satisfy \( a = c \) and \( a + b + c = 1 \). Show also that, if these conditions hold, then the Fourier stability condition is achieved for all values of \( \mu \) if and only if the parameters also satisfy \( |b| < 2a \).

[Hint: In the second half, the roots of the characteristic equation satisfy \( x_1 x_2 = 1 \). Then, \( |x_1|, |x_2| \leq 1 \) if \( D \leq 0 \), where \( D \) is the discriminant of the equation.]

15. For a given analytic function \( f \) we consider its truncated Fourier approximation on the interval \([-1, 1]\), i.e.,

\[ f(x) \approx \phi_N(x) = \sum_{n=-N/2}^{N/2} \hat{f}_n e^{i\pi nx}, \quad \text{where} \quad \hat{f}_n = \frac{1}{2} \int_{-1}^{1} f(\tau) e^{-i\pi n \tau} d\tau, \quad n \in \mathbb{Z}. \]
Prove that, given any \( s = 1, 2, \ldots \), we have for all \( n \in \mathbb{Z} \setminus \{0\} \) the equality
\[
\hat{f}_n = \frac{(-1)^{n-1}}{2} \sum_{m=0}^{s-1} \frac{1}{(\pi i n)^{m+1}} \left[ f^{(m)}(1) - f^{(m)}(-1) \right] + \frac{1}{(\pi i n)^s} \hat{f}^{(s)}_n.
\]

16. Unless \( f \) is analytic, the rate of decay of its Fourier harmonics can be very slow, certainly slower than \( O(N^{-1}) \). To explore this, let \( f(x) = |x|^{-1/2} \). Prove that \( \hat{f}_n = g(-n) + g(n) \), where \( g(n) = \int_0^1 e^{i\pi nx^2} \, dx \). Moreover, with the error function \( \text{erf} \) defined as the integral
\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} \, d\tau, \quad z \in \mathbb{C}.
\]
show that its Fourier coefficients are
\[
\hat{f}_n = \frac{\text{erf}(\sqrt{i\pi n})}{2\sqrt{i n}} + \frac{\text{erf}(\sqrt{-i\pi n})}{2\sqrt{-i n}},
\]
and asymptotically for \( |n| \gg 1 \) we have \( \hat{f}_n = O(n^{-1/2}) \). [Hint: For the last identity use without proof the asymptotic estimate \( \text{erf}(\sqrt{x}) = 1 + O(x^{-1}) \) for \( x \in \mathbb{R}, |x| \gg 1 \).]

17. Consider the solution of the two-point boundary value problem
\[
(2 - \cos \pi x)u'' + u = 1, \quad -1 \leq x \leq 1, \quad u(-1) = u(1),
\]
using the spectral method. Plugging the Fourier expansion of \( u \) into this differential equation, show that the \( \hat{u}_n \) obey a three-term recurrence relation. Calculate \( \hat{u}_0 \) separately and using the fact that \( \hat{u}_{-n} = \hat{u}_n \) (why?), prove further that the computation of \( \hat{u}_n \) for \( -N/2+1 \leq n \leq N/2 \) (assuming that \( \hat{u}_n = 0 \) outside this range of \( n \)) reduces to the solution of an \( (N/2) \times (N/2) \) tridiagonal system of algebraic equations.

18. Set
\[
a(x) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{i\pi n x}, \quad (2.1)
\]
the Fourier expansion of \( a \). Explain why \( a \) is periodic with period 2. Further, let \( \tilde{n} \) denote some selected value of \( n \). Evaluate \( \frac{1}{2} \int_{-1}^{1} a(x) e^{-i\pi \tilde{n} x} \, dx \) with \( a(x) \) given by (2.1). Doing so, you have just computed the Fourier coefficient \( \hat{a}_{\tilde{n}} \). Now choose \( a(x) = \cos \pi x \) and compute its corresponding Fourier coefficients. With this, derive an explicit expression for the coefficients in the \( N \)-term truncated Fourier approximation of the solution \( u \) of
\[
\left\{
\begin{aligned}
((\cos \pi x + 2)u_x)_x &= \sin \pi x, \quad x \in [-1, 1] \\
\text{periodic boundary conditions and normalisation condition} &\int_{-1}^{1} u(x) \, dx = 0.
\end{aligned}
\right.
\]

19. Let \( u \) be an analytic function in \([-1, 1]\) that can be extended analytically into the complex plane and possesses a Chebyshev expansion \( u = \sum_{n=0}^{\infty} \tilde{a}_n T_n \). Express \( u' \) in an explicit form as a Chebyshev expansion.

20. The two-point ODE \( u'' + u = 1, u(-1) = u(1) = 0 \), is solved by a Chebyshev method.
\begin{enumerate}
\item[(a)] Show that the odd coefficients are zero and that \( u(x) = \sum_{n=0}^{\infty} \tilde{u}_{2n} T_{2n}(x) \). Express the boundary conditions as a linear condition of the coefficients \( \tilde{u}_{2n} \).
\item[(b)] Express the differential equation as an infinite set of linear algebraic equations in the coefficients \( \tilde{u}_{2n} \).
\item[(c)] Discuss how to truncate the linear system, keeping in mind the exponential convergence of the method and the floating-point precision of your computer.
\item[(d)] While \( u(-1) = u(1) \) we cannot expect a standard spectral method to converge at spectral speed. Why?
\end{enumerate}