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Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Examples' Sheet 4

30. Let A_n be an $n \times n$ TST matrix such that $a_{i,i} = \alpha$ and $a_{i,i+1} = a_{i+1,i} = \beta$. Show that the Jacobi iteration for solving $Ax = b$ converges if $2|\beta| < |\alpha|$. Moreover, prove that if convergence is required for *all* A_n with $n \geq 1$ then this inequality is necessary as well as sufficient.

31. Let A be an $n \times n$ TST matrix with $a_{k,k} = \alpha$ and $a_{k,k+1} = a_{k+1,k} = \beta$. Verify that $\alpha \geq 2|\beta| > 0$ implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for $A\boldsymbol{x} = \boldsymbol{b}$ with the Toeplitz lower-triangular bidiagonal matrix Q,

$$q_{k,k} = \gamma, \qquad q_{k,k-1} = \delta, \qquad q_{k,\ell} = 0 \quad \text{otherwise.}$$

Determine real numbers γ and δ such that QQ^T differs from A in just the $(1,1)$ coordinate. Prove that with this choice of γ and δ the preconditioned conjugate gradient method converges in just two iterations.

32. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}, \quad S = \begin{pmatrix} A_1 & O \\ O & A_3 \end{pmatrix},$$

where A_1, A_3 are symmetric $n \times n$ matrices and the rank of the $n \times n$ matrix A_2 is $r \leq n-1$. (This is the case for example when A is a band matrix with the bandwidth $2r+1$.) We further stipulate that the $(2n) \times (2n)$ matrix A is positive definite (hence so are A_1 and A_3). Let $A_1 = Q_1 Q_1^T$, $A_3 = Q_3 Q_3^T$ be Cholesky factorizations and assume that the preconditioner Q is the lower-triangular Cholesky factor of S (hence $QQ^T = S$).

(a) Prove that

$$B = Q^{-1} A Q^{-T} = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}, \quad \text{where} \quad F = Q_1^{-1} A_2 Q_3^{-T}.$$

(b) Assuming that the eigenvalues of B are $\lambda_1, ..., \lambda_{2n}$, while the eigenvalues of FF^T are $\mu_1, ..., \mu_n \geq 0$, prove that, without loss of generality,

$$\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \ldots, n.$$

[*Hint: For* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, *we have* $\det M = \det(A - BD^{-1}C)\det(D)$.]

(c) Prove that the rank of FF^T is at most r, thereby deducing that B has at most $2r+1$ distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?

33. Let A be the 3×3 matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is real and nonzero. Find an explicit expression for A^k, $k = 1, 2, 3, \ldots$.
The sequence $\boldsymbol{x}^{(k+1)}$, $k = 0, 1, 2, \ldots$, is generated by the power method $\boldsymbol{x}^{(k+1)} = A\boldsymbol{x}^{(k)} / \|A\boldsymbol{x}^{(k)}\|$, where $\boldsymbol{x}^{(0)}$ is a nonzero vector in $\mathbb{R}^3$. Deduce from your expression for A^k that the second and third components of $\boldsymbol{x}^{(k+1)}$ tend to zero as $k \to \infty$. Further, show that this remark implies $A\boldsymbol{x}^{(k+1)} - \lambda \boldsymbol{x}^{(k+1)} \to \boldsymbol{0}$, so the power method tends to provide a solution to the eigenvalue equation.

34. Let A be a symmetric 2×2 matrix with distinct eigenvalues and normalized eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Given $\boldsymbol{x}^{(0)} \in \mathbb{R}^2$, consider the sequence $\boldsymbol{x}^{(k+1)}$, $k = 0, 1, 2, \ldots$, generated by the Rayleigh quotient iteration:

$$\begin{cases} \lambda_k = \boldsymbol{x}^{(k)T} A \boldsymbol{x}^{(k)} / \|\boldsymbol{x}^{(k)}\|^2 \\ \boldsymbol{y} = (A - \lambda_k I)^{-1} \boldsymbol{x}^{(k)}, \quad \text{and we set } \boldsymbol{x}^{(k+1)} = \boldsymbol{y}/\|\boldsymbol{y}\|. \end{cases}$$

Show that, if $\boldsymbol{x}^{(k)} = (\boldsymbol{v}_1 + \epsilon_k \boldsymbol{v}_2)/(1+\epsilon_k^2)^{1/2}$, where $|\epsilon_k|$ is small, then $|\epsilon_{k+1}|$ is of magnitude $|\epsilon_k|^3$. In other words, the method enjoys a *third order* rate of convergence.

35. The symmetric matrix

$$A = \begin{pmatrix} 9 & -8 & 2 \\ -8 & 9 & -2 \\ 2 & -2 & 10 \end{pmatrix} \quad \text{has the eigenvector} \quad \boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Calculate an orthogonal matrix Ω by a Householder transformation such that $\Omega \boldsymbol{v}$ is a multiple of the first coordinate vector $\boldsymbol{e}_1$. Then, form the product $\Omega^T A \Omega$. You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of A.

36. Show that the vectors $\boldsymbol{x}$, $A\boldsymbol{x}$ and $A^2\boldsymbol{x}$ are linearly dependent in the case

$$A = \begin{pmatrix} 4 & 5 & 2 & 0 \\ -26 & -14 & 1 & 4 \\ -2 & 2 & 3 & 1 \\ -43 & -8 & 13 & 9 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}.$$

Hence, calculate two of the eigenvalues of A. Obtain by deflation a 2×2 matrix whose eigenvalues are the remaining eigenvalues of A. Then, find the other eigenvalues of A.

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix

$$A = \begin{pmatrix} 9 & -1 & 2 & 2 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 14 & -3 \\ 2 & 2 & -3 & 4 \end{pmatrix}.$$

Your final matrix should be symmetric and should have the same trace as A.

38. Let A be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of A that are adjacent to the diagonal are nonzero). Prove that A has n distinct eigenvalues. Prove also that, if A has a zero eigenvalue and a single iteration of the QR algorithm is applied to A, then the resultant tridiagonal matrix is deflatable. [*Hint: In the first part show that for each eigenvalue λ there is a unique solution to $A\boldsymbol{w} = \lambda \boldsymbol{w}$. In the second part deduce that a diagonal element of R is zero.*]

39. Let A be a 2×2 symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $\{A_k : k = 1, 2, \ldots\}$ be calculated by applying the QR algorithm to A_0 (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of A_k. Show that, except in the special case when A is already diagonal, the sequence $\{(A_k)_{1,1} : k = 0, 1, \ldots\}$ converges monotonically to the eigenvalue of A of larger modulus. [*Hint: The sign of this eigenvalue is the same as the sign of the trace of A. Also, for any symmetric matrix B, we have $B_{1,1} = \boldsymbol{e}_1^T B \boldsymbol{e}_1$ and $\lambda_{\min} \|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^T B \boldsymbol{x} \leq \lambda_{\max} \|\boldsymbol{x}\|^2$.*]

40. Apply a single step of the QR method to the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix},$$

where $\epsilon > 0$. You should find that the $(2,3)$ element of the new matrix is $\mathcal{O}(\epsilon^3)$ and that the new matrix has exactly the same trace as A.

41. (For those who like analysis). Let A be a real 4×4 upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices A_k, $k = 0, 1, 2, \ldots$, are calculated from A by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$.