Mathematical Tripos Part II: Michaelmas Term 2021
Numerical Analysis – Examples’ Sheet 4

30. Let $A_n$ be an $n \times n$ TST matrix such that $a_{i,i} = \alpha$ and $a_{i,i+1} = a_{i+1,i} = \beta$. Show that the Jacobi iteration for solving $Ax = b$ converges if $2|\beta| < |\alpha|$. Moreover, prove that if convergence is required for all $A_n$ with $n \geq 1$ then this inequality is necessary as well as sufficient.

31. Let $A$ be an $n \times n$ TST matrix with $a_{k,k} = \alpha$ and $a_{k,k+1} = a_{k+1,k} = \beta$. Verify that $|\alpha| \geq 2|\beta| > 0$ implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for $Ax = b$ with the Toeplitz lower-triangular bidiagonal matrix $Q$. Determine real numbers $\gamma$ and $\delta$ such that $QQ^T$ differs from $A$ in just the $(1,1)$ coordinate. Prove that with this choice of $\gamma$ and $\delta$ the preconditioned conjugate gradient method converges in just two iterations.

32. Let $A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}$, $S = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}$, where $A_1, A_3$ are symmetric $n \times n$ matrices and the rank of the $n \times n$ matrix $A_2$ is $r \leq n - 1$. (This is the case for example when $A$ is a band matrix with the bandwidth $2r + 1$.) We further stipulate that the $(2n) \times (2n)$ matrix $A$ is positive definite (hence so are $A_1$ and $A_3$). Let $A_1 = Q_1Q_1^T$, $A_3 = Q_3Q_3^T$ be Cholesky factorizations and assume that the preconditioner $Q$ is the lower-triangular Cholesky factor of $S$ (hence $QQ^T = S$).

(a) Prove that

$$B = Q^{-1}AQ^{-T} = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix},$$

where $F = Q_1^{-1}A_2Q_3^{-T}$.

(b) Assuming that the eigenvalues of $B$ are $\lambda_1, \ldots, \lambda_{2n}$, while the eigenvalues of $FF^T$ are $\mu_1, \ldots, \mu_n \geq 0$, prove that, without loss of generality,

$$\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \ldots, n.$$

[Hint: For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have $\det M = \det(A - BD^{-1}C) \det(D)$.]

(c) Prove that the rank of $FF^T$ is at most $r$, thereby deducing that $B$ has at most $2r + 1$ distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?

33. Let $A$ be the $3 \times 3$ matrix

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda$ is real and nonzero. Find an explicit expression for $A^k$, $k = 1, 2, 3, \ldots$. The sequence $x^{(k+1)}$, $k = 0, 1, 2, \ldots$, is generated by the power method $x^{(k+1)} = Ax^{(k)} \|Ax^{(k)}\|$, where $x^{(0)}$ is a nonzero vector in $\mathbb{R}^3$. Deduce from your expression for $A^k$ that the second and third components of $x^{(k+1)}$ tend to zero as $k \to \infty$. Further, show that this remark implies $Ax^{(k+1)} - \lambda x^{(k+1)} \to 0$, so the power method tends to provide a solution to the eigenvalue equation.
34. Let $A$ be a symmetric $2 \times 2$ matrix with distinct eigenvalues and normalized eigenvectors $v_1$ and $v_2$. Given $x^{(0)} \in \mathbb{R}^2$, consider the sequence $x^{(k+1)}$, $k = 0, 1, 2, \ldots$, generated by the Rayleigh quotient iteration:
\[
\begin{cases}
\lambda_k = x^{(k)T}Ax^{(k)}/\|x^{(k)}\|^2 \\
y = (A - \lambda_k I)^{-1}x^{(k)}, \quad \text{and we set } x^{(k+1)} = y/\|y\|.
\end{cases}
\]
Show that, if $x^{(k)} = (v_1 + \epsilon v_2)/(1 + \epsilon^2)^{1/2}$, where $|\epsilon|$ is small, then $|\epsilon_{k+1}|$ is of magnitude $|\epsilon_k|^3$. In other words, the method enjoys a third order rate of convergence.

35. The symmetric matrix
\[
A = \begin{pmatrix} 9 & -8 & 2 \\
-8 & 9 & -2 \\
2 & -2 & 10 \end{pmatrix}
\]
has the eigenvector $v = \begin{pmatrix} 2 \\
-2 \\
1 \end{pmatrix}$.
Calculate an orthogonal matrix $\Omega$ by a Householder transformation such that $\Omega v$ is a multiple of the first coordinate vector $e_1$. Then, form the product $\Omega^T A \Omega$. You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of $A$.

36. Show that the vectors $x$, $Ax$, and $A^2 x$ are linearly dependent in the case
\[
A = \begin{pmatrix} 4 & 5 & 2 & 0 \\
-26 & -14 & 1 & 4 \\
-2 & 2 & 3 & 1 \\
-43 & -8 & 13 & 9 \end{pmatrix}
\quad \text{and} \quad x = \begin{pmatrix} 1 \\
0 \\
1 \\
5 \end{pmatrix}.
\]
Hence, calculate two of the eigenvalues of $A$. Obtain by deflation a $2 \times 2$ matrix whose eigenvalues are the remaining eigenvalues of $A$. Then, find the other eigenvalues of $A$.

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix
\[
A = \begin{pmatrix} 9 & -1 & 2 & 2 \\
-1 & 3 & 4 & 2 \\
2 & 4 & 14 & -3 \\
2 & 2 & -3 & 4 \end{pmatrix}.
\]
Your final matrix should be symmetric and should have the same trace as $A$.

38. Let $A$ be an $n \times n$ symmetric tridiagonal matrix that is not deflectable (i.e., all the elements of $A$ that are adjacent to the diagonal are nonzero). Prove that $A$ has $n$ distinct eigenvalues. Prove also that, if $A$ has a zero eigenvalue and a single iteration of the QR algorithm is applied to $A$, then the resultant tridiagonal matrix is deflectable. [Hint: In the first part show that for each eigenvalue $\lambda$ there is a unique solution to $Aw = \lambda w$. In the second part deduce that a diagonal element of $R$ is zero.]

39. Let $A$ be a $2 \times 2$ symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $(A_k : k = 1, 2, \ldots)$ be calculated by applying the QR algorithm to $A_0$ (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of $A_k$. Show that, except in the special case when $A$ is already diagonal, the sequence $(A_{k+1})_{1,1}$ converges monotonically to the eigenvalue of $A$ of larger modulus. [Hint: The sign of this eigenvalue is the same as the sign of the trace of $A$. Also, for any symmetric matrix $B$, we have $B_{1,1} = \epsilon^T B e_1$ and $\lambda_{\min} \|x\|^2 \leq x^T B x \leq \lambda_{\max} \|x\|^2$.]

40. Apply a single step of the QR method to the matrix
\[
A = \begin{pmatrix} 4 & 3 & 0 \\
3 & 1 & \epsilon \\
0 & \epsilon & 0 \end{pmatrix},
\]
where $\epsilon > 0$. You should find that the $(2, 3)$ element of the new matrix is $O(\epsilon^3)$ and that the new matrix has exactly the same trace as $A$.

41. (For those who like analysis.) Let $A$ be a real $4 \times 4$ upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices $A_k$, $k = 0, 1, 2, \ldots$, are calculated from $A$ by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{5,2}$ converges to zero as $k \to \infty$. 
