30. Let \( A \) be an \( n \times n \) TST matrix such that \( a_{i,i} = \alpha \) and \( a_{i,i+1} = a_{i+1,i} = \beta \). Show that the Jacobi iteration for solving \( Ax = b \) converges if \( 2|\beta| < |\alpha| \). Moreover, prove that if convergence is required for all \( A_n \) with \( n \geq 1 \) then this inequality is necessary as well as sufficient.

31. Let \( A \) be an \( n \times n \) TST matrix with \( a_{k,k} = \alpha \) and \( a_{k,k+1} = a_{k+1,k} = \beta \). Verify that \( \alpha \geq 2|\beta| > 0 \) implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for \( Ax = b \) with the Toeplitz lower-triangular bidiagonal matrix \( Q \).

\[
q_{k,k} = \gamma, \quad q_{k,k-1} = \delta, \quad q_{k,\ell} = 0 \quad \text{otherwise.}
\]

Determine real numbers \( \gamma \) and \( \delta \) such that \( QQ^T \) differs from \( A \) in just the \((1,1)\) coordinate. Prove that with this choice of \( \gamma \) and \( \delta \) the preconditioned conjugate gradient method converges in just two iterations.

32. Let \( A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}, \quad S = \begin{pmatrix} A_1 & O \\ O & A_3 \end{pmatrix}, \)

where \( A_1, A_3 \) are symmetric \( n \times n \) matrices and the rank of the \( n \times n \) matrix \( A_2 \) is \( r \leq n - 1 \). (This is the case for example when \( A \) is a band matrix with the bandwidth \( 2r + 1 \).) We further stipulate that the \((2n) \times (2n) \) matrix \( A \) is positive definite (hence so are \( A_1 \) and \( A_3 \)). Let \( A_1 = Q_1 Q_1^T, \ A_3 = Q_3 Q_3^T \) be Cholesky factorizations and assume that the preconditioner \( Q \) is the lower-triangular Cholesky factor of \( S \) (hence \( QQ^T = S \)).

(a) Prove that

\[
B = Q^{-1}AQ^{-T} = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}, \quad \text{where} \quad F = Q_1^{-1} A_2 Q_3^{-T}.
\]

(b) Assuming that the eigenvalues of \( B \) are \( \lambda_1, \ldots, \lambda_{2n} \), while the eigenvalues of \( FF^T \) are \( \mu_1, \ldots, \mu_n \geq 0 \), prove that, without loss of generality,

\[
\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \ldots, n.
\]

[Hint: For \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \) we have \( \det M = \det(A - BD^{-1}C) \det(D) \).]

(c) Prove that the rank of \( FF^T \) is at most \( r \), thereby deducing that \( B \) has at most \( 2r + 1 \) distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?

33. Let \( A \) be the \( 3 \times 3 \) matrix

\[
A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},
\]

where \( \lambda \) is real and nonzero. Find an explicit expression for \( A^k, k = 1, 2, 3, \ldots \).

The sequence \( x^{(k+1)}, k = 0, 1, 2, \ldots, \) is generated by the power method \( x^{(k+1)} = Ax^{(k)} / \|Ax^{(k)}\| \), where \( x^{(0)} \) is a nonzero vector in \( \mathbb{R}^3 \). Deduce from your expression for \( A^k \) that the second and third components of \( x^{(k+1)} \) tend to zero as \( k \to \infty \). Further, show that this remark implies \( Ax^{(k+1)} - \lambda x^{(k+1)} \to 0 \), so the power method tends to provide a solution to the eigenvalue equation.
34. Let $A$ be a symmetric $2 \times 2$ matrix with distinct eigenvalues and normalized eigenvectors $v_1$ and $v_2$. Given $x^{(0)} \in \mathbb{R}^2$, consider the sequence $x^{(k+1)}$, $k = 0, 1, 2, \ldots$, generated by the Rayleigh quotient iteration:
\[
\begin{aligned}
\lambda_k &= x^{(k)T}Ax^{(k)}/\|x^{(k)}\|^2 \\
y &= (A - \lambda_k I)^{-1}x^{(k)}, 
\end{aligned}
\]
and we set $x^{(k+1)} = y/\|y\|$.
Show that, if $x^{(k)} = (v_1 + \epsilon_k v_2)/(1 + \epsilon_k^2)^{1/2}$, where $|\epsilon_k|$ is small, then $|\epsilon_{k+1}|$ is of magnitude $|\epsilon_k|^3$. In other words, the method enjoys a third order rate of convergence.

35. The symmetric matrix
\[
A = \begin{pmatrix}
  9 & -8 & 2 \\
-8 & 9 & -2 \\
 2 & -2 & 10
\end{pmatrix}
\]
has the eigenvector $v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.
Calculate an orthogonal matrix $\Omega$ by a Householder transformation such that $\Omega v$ is a multiple of the first coordinate vector $e_1$. Then, form the product $\Omega^T A \Omega$. You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of $A$.

36. Show that the vectors $x$, $Ax$, and $A^2 x$ are linearly dependent in the case
\[
A = \begin{pmatrix}
  4 & 5 & 2 & 0 \\
-26 & -14 & 1 & 4 \\
-2 & 2 & 3 & 1 \\
-43 & -8 & 13 & 9
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}.
\]
Hence, calculate two of the eigenvalues of $A$. Obtain by deflation a $2 \times 2$ matrix whose eigenvalues are the remaining eigenvalues of $A$. Then, find the other eigenvalues of $A$.

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix
\[
A = \begin{pmatrix}
  9 & -1 & 2 & 2 \\
-1 & 3 & 4 & 2 \\
 2 & 4 & 14 & -3 \\
 2 & 2 & -3 & 4
\end{pmatrix}.
\]
Your final matrix should be symmetric and should have the same trace as $A$.

38. Let $A$ be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of $A$ that are adjacent to the diagonal are nonzero). Prove that $A$ has $n$ distinct eigenvalues. Prove also that, if $A$ has a zero eigenvalue and a single iteration of the QR algorithm is applied to $A$, then the resultant tridiagonal matrix is deflatable. [Hint: In the first part show that for each eigenvalue $\lambda$ there is a unique solution to $Aw = \lambda w$. In the second part deduce that a diagonal element of $R$ is zero.]

39. Let $A$ be a $2 \times 2$ symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $\{A_k : k = 1, 2, \ldots\}$ be calculated by applying the QR algorithm to $A_0$ (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of $A_k$. Show that, except in the special case when $A$ is already diagonal, the sequence $\{(A_k)_{1,1} : k = 0, 1, \ldots\}$ converges monotonically to the eigenvalue of $A$ of larger modulus. [Hint: The sign of this eigenvalue is the same as the sign of the trace of $A$. Also, for any symmetric matrix $B$, we have $B_{1,1} = e_1^T B e_1$ and $\lambda_{\min} \|x\|^2 \leq x^T B x \leq \lambda_{\max} \|x\|^2$.]

40. Apply a single step of the QR method to the matrix
\[
A = \begin{pmatrix}
  4 & 3 & 0 \\
 3 & 1 & \epsilon \\
 0 & \epsilon & 0
\end{pmatrix},
\]
where $\epsilon > 0$. You should find that the $(2, 3)$ element of the new matrix is $O(\epsilon^3)$ and that the new matrix has exactly the same trace as $A$.

41. (For those who like analysis.) Let $A$ be a real $4 \times 4$ upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices $A_k$, $k = 0, 1, 2, \ldots$, are calculated from $A$ by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$. 

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