30. Let \( A_n \) be an \( n \times n \) TST matrix such that \( a_{i,i} = \alpha \) and \( a_{i,i+1} = a_{i+1,i} = \beta \). Show that the Jacobi iteration for solving \( Ax = b \) converges if \( 2|\beta| < |\alpha| \). Moreover, prove that if convergence is required for all \( A_n \) with \( n \geq 1 \) then this inequality is necessary as well as sufficient.

31. Let \( A \) be an \( n \times n \) TST matrix with \( a_{k,k} = \alpha \) and \( a_{k,k+1} = a_{k+1,k} = \beta \). Verify that \( \alpha \geq 2|\beta| > 0 \) implies that the matrix is positive definite. Now, we precondition the conjugate gradient method for \( Ax = b \) with the Toeplitz lower-triangular bidiagonal matrix \( Q \).

Determine real numbers \( \gamma \) and \( \delta \) such that \( QQ^T \) differs from \( A \) in just the \((1,1)\) coordinate. Prove that with this choice of \( \gamma \) and \( \delta \) the preconditioned conjugate gradient method converges in just two iterations.

32. Let

\[
A = \begin{pmatrix}
A_1 & A_2 \\
A_2^T & A_3
\end{pmatrix}, \quad S = \begin{pmatrix}
A_1 & O \\
O & A_3
\end{pmatrix},
\]

where \( A_1, A_3 \) are symmetric \( n \times n \) matrices and the rank of the \( n \times n \) matrix \( A_2 \) is \( r \leq n - 1 \). We further stipulate that the \((2n) \times (2n)\) matrix \( A \) is positive definite. Let \( A_1 = Q_1 Q_1^T, A_3 = Q_3 Q_3^T \) be Cholesky factorizations and assume that the preconditioner \( Q \) is the lower-triangular Cholesky factor of \( S \) (hence \( QQ^T = S \)). Prove that

\[
B = Q^{-1} A Q^{-T} = \begin{pmatrix}
I & F \\
F^T & I
\end{pmatrix}, \quad \text{where} \quad F = Q_1^{-1} A_2 Q_3^{-T}.
\]

Supposing that the eigenvalues of \( B \) are \( \lambda_1, \ldots, \lambda_{2n} \), while the eigenvalues of \( FF^T \) are \( \mu_1, \ldots, \mu_n \geq 0 \), prove that, without loss of generality,

\[
\lambda_k = 1 - \sqrt{\mu_k}, \quad \lambda_{n+k} = 1 + \sqrt{\mu_k}, \quad k = 1, 2, \ldots, n.
\]

Prove that the rank of \( FF^T \) is at most \( r \) thereby deducing that \( B \) has at most \( 2r + 1 \) distinct eigenvalues. What does this tell you about the number of steps before the preconditioned conjugate gradient method terminates in exact arithmetic?

Matlab demo: Download the Matlab GUI for Preconditioning of Conjugate Gradient from http://www.damtp.cam.ac.uk/user/naweb/ii/pressure/precond.precond.php. Setup an example for a system matrix \( A \) of the type just discussed and use the GUI to compute the eigenvalues of \( A^T A \) and of the preconditioned matrix. How does the number of iterations the CG method needs changes? How robust is the CG method to perturbations of \( A \) or \( b \) by a small random matrix or vector respectively?

33. Let \( A \) be the \( 3 \times 3 \) matrix

\[
A = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{pmatrix},
\]

where \( \lambda \) is real and nonzero. Find an explicit expression for \( A^k, k = 1, 2, 3, \ldots \).

The sequence \( x^{(k+1)}, k = 0, 1, 2, \ldots, \) is generated by the power method \( x^{(k+1)} = A x^{(k)}/\|A x^{(k)}\| \), where \( x^{(0)} \) is a nonzero vector in \( \mathbb{R}^3 \). Deduce from your expression for \( A^k \) that the second and third components of \( x^{(k+1)} \) tend to zero as \( k \to \infty \). Further, show that this remark implies \( A x^{(k+1)} = \lambda x^{(k+1)} \to 0 \), so the power method tends to provide a solution to the eigenvalue equation.

Matlab demo: Reproduce your findings using the Matlab GUI for Computing eigenvalues and eigenvectors from http://www.damtp.cam.ac.uk/user/naweb/ii/eigenstuff/eigenstuff.php. How does the situation change when you change one of the \( \lambda \)-entries in \( A \) to another value?

34. Let \( A \) be a symmetric \( 2 \times 2 \) matrix with distinct eigenvalues and normalized eigenvectors \( v_1 \) and \( v_2 \).

Given \( x^{(0)} \in \mathbb{R}^2 \), the sequence \( x^{(k+1)}, k = 0, 1, 2, \ldots, \) is generated in the following way. The Rayleigh
38. Let $A = \frac{x^{(k)}}{\|x^{(k)}\|^2}$ be an estimate for an eigenvalue of $A$, the vector norm being Euclidean. Then, inverse iteration gives

$$y = (A - \lambda_k I)^{-1}x^{(k)},$$

and we set $x^{(k+1)} = y/\|y\|$. Show that, if $x^{(k)} = (v_1 + \epsilon_k v_2)/(1 + \epsilon_k^2)^{1/2}$, where $|\epsilon_k|$ is small, then $|\epsilon_{k+1}|$ is of magnitude $|\epsilon_k|^3$. In other words, the method enjoys a third order rate of convergence.

35. The symmetric matrix

$$A = \begin{pmatrix} 9 & -8 & 2 \\ -8 & 9 & -2 \\ 2 & -2 & 10 \end{pmatrix}$$

has the eigenvector $v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Calculate an orthogonal matrix $\Omega$ by a Householder transformation such that $\Omega x$ is a multiple of the first coordinate vector $e_1$. Then, form the product $\Omega^T A \Omega$. You should find that this matrix is suitable for deflation. Hence, identify all the eigenvalues and eigenvectors of $A$.

36. Show that the vectors $x$, $Ax$ and $A^2x$ are linearly dependent in the case

$$A = \begin{pmatrix} 4 & 5 & 2 & 0 \\ -26 & -14 & 1 & 4 \\ -2 & 2 & 3 & 1 \\ -43 & -8 & 13 & 9 \end{pmatrix}$$

and $x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}$.

Hence, calculate two of the eigenvalues of $A$. Obtain by deflation a $2 \times 2$ matrix whose eigenvalues are the remaining eigenvalues of $A$. Then, find the other eigenvalues of $A$.

37. Use Householder transformations to generate a tridiagonal matrix that is similar to the matrix

$$A = \begin{pmatrix} 9 & -1 & 2 & 2 \\ -1 & 3 & 4 & 2 \\ 2 & 4 & 14 & -3 \\ 2 & 2 & -3 & 4 \end{pmatrix}.$$  

Your final matrix should be symmetric and should have the same trace as $A$.

38. Let $A$ be an $n \times n$ symmetric tridiagonal matrix that is not deflatable (i.e., all the elements of $A$ that are adjacent to the diagonal are nonzero). Prove that $A$ has $n$ distinct eigenvalues. Prove also that, if $A$ has a zero eigenvalue and a single iteration of the QR algorithm is applied to $A$, then the resultant tridiagonal matrix is deflatable. [Hint: In the first part show that for each eigenvalue $\lambda$ there is a unique solution to $A w = \lambda w$. In the second part deduce that a diagonal element of $R$ is zero.]

39. Let $A$ be a $2 \times 2$ symmetric matrix whose trace does not vanish, let $A_0 = A$, and let the sequence of matrices $\{A_k : k = 1, 2, \ldots\}$ be calculated by applying the QR algorithm to $A_0$ (without any origin shifts). Express the matrix element $(A_{k+1})_{1,1}$ in terms of the elements of $A_k$. Show that, except in the special case when $A$ is already diagonal, the sequence $\{(A_k)_{1,1} : k = 0, 1, \ldots\}$ converges monotonically to the eigenvalue of $A$ of larger modulus. [Hint: The sign of this eigenvalue is the same as the sign of the trace of $A$. Also, for any symmetric matrix $B$, we have $B_{1,1} = e_1^T B e_1$ and $\lambda_{\min} \|x\|^2 \leq x^T B x \leq \lambda_{\max} \|x\|^2$.]

40. Apply a single step of the QR method to the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix},$$

where $\epsilon > 0$. You should find that the $(2,3)$ element of the new matrix is $O(\epsilon^3)$ and that the new matrix has exactly the same trace as $A$.

Matlab demo: Download the Matlab GUI for Visual QR from [http://www.damtp.cam.ac.uk/user/naweb/ii/qr_hex/qr_hex.php](http://www.damtp.cam.ac.uk/user/naweb/ii/qr_hex/qr_hex.php) and let the QR method run for the matrix $A$ above. Try it with your own choice of a square matrices $A$ and see what the QR method is doing to the entries of $A$. What happens if you choose a symmetric matrix $A$, what if $A$ is upper Hessenberg?

41. (For those who like analysis.) Let $A$ be a real $4 \times 4$ upper Hessenberg matrix whose eigenvalues all have nonzero imaginary parts, where the moduli of the two complex pairs of eigenvalues are different. Prove that, if the matrices $A_k$, $k = 0, 1, 2, \ldots$, are calculated from $A$ by the QR algorithm, then the subdiagonal elements $(A_k)_{2,1}$ and $(A_k)_{4,3}$ stay bounded away from zero, but $(A_k)_{3,2}$ converges to zero as $k \to \infty$.  

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