Mathematical Tripos Part II D26a

Dynamical Systems Michaelmas 2017

Example Sheet 1 J.R. Lister

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1. (a) Given any fixed $t_1 < 0 < t_2$, show that the initial-value problem $\dot{x} = x^{2/3}$, $x(0) = 0$ has a solution that satisfies $x(t) = 0$ iff $t \in [t_1, t_2]$. The solution to the IVP is clearly not unique. Why can this occur here?

(b) Solve the differential equation $\dot{x} = 1 + x^2$ with $x(0) = x_0$. Why is the solution unique? Show that for all values of x_0 the solution blows up in finite time both forwards and backwards.

2. By considering only the signs of $\dot{x}$ and $\dot{y}$, i.e. *without* linearizing about the fixed points, sketch the phase portrait for each of the systems in $\mathbb{R}^2$

$$\text{(i)}\quad \dot{x} = x^2 - 1, \quad \dot{y} = y^2 - 1, \qquad \text{(ii)}\quad \dot{x} = y^2 - 1, \quad \dot{y} = x^2 - 1.$$

[*Hint*: one of these systems is Hamiltonian.]

3. Use polar coordinates to sketch the phase portrait of

$$\dot{x} = -y(x^2 + y^2), \quad \dot{y} = x(x^2 + y^2).$$

Find the flow $\boldsymbol{\phi}_t(r_0, \theta_0)$. Identify the orbit $\mathcal{O}(\mathbf{x}_0)$ and the ω-limit set $\omega(\mathbf{x}_0)$.

*Discuss the stability of the solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ and of the orbit $\mathcal{O}(\mathbf{x}_0)$ with respect to small perturbations of the initial value $\mathbf{x}_0$. (Stability has not yet been formally defined, so think about possible definitions in your discussion.)

4. Consider the system in $\mathbb{R}^3$

$$\dot{x} = yz, \quad \dot{y} = -zx, \quad \dot{z} = -z^3.$$

Solve the equations by first transforming the (x, y)-plane to polar coordinates. Then determine $\boldsymbol{\phi}_t(\mathbf{x}_0)$. Hence show that the ω-limit set for the orbit of (x_0, y_0, z_0) with $z_0 > 0$ is a circle in the plane $z = 0$. Find the flow and ω-limit set if $\dot{z} = -z$ instead of $-z^3$.

*Discuss the stability of all fixed points in both cases.

5. Sketch the phase portrait of the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = x^2 - 4.$$

The solution with intial condition $(-4, 0)$ is $\mathbf{x}(t) = (2 - 6\,\mathrm{sech}^2\, t, 12\,\mathrm{sech}^2\, t \tanh t)$. Show this solution on your phase portrait. What are $\omega(-4, 0)$, $\omega(-3, 0)$ and $\omega(-5, 0)$?

6. Sketch the phase portraits of the following linear systems, classifying the fixed point at the origin, and finding the eigenvectors only when it helps the sketching:

$$
\begin{array}{llllll}
(a) & \dot{x} = -2x - 3y, & \dot{y} = 8x + 8y, & (e) & \dot{x} = 2x + y, & \dot{y} = 2y, \\
(b) & \dot{x} = x, & \dot{y} = -3x - y, & (f) & \dot{x} = 4x + 2y, & \dot{y} = 2x + y, \\
(c) & \dot{x} = 7x - 2y, & \dot{y} = 5x + 5y, & (g) & \dot{x} = 5x, & \dot{y} = 5y. \\
(d) & \dot{x} = 7x + 5y, & \dot{y} = -10x - 7y. & & &
\end{array}
$$

For which systems is the origin a hyperbolic fixed point? For which is it a sink? Which systems are Hamiltonian?

7. Find and classify all fixed points of the system

$$\dot{x} = 2x - x^3 - 3xy^2, \quad \dot{y} = y - y^3 - x^2 y.$$

[*Hint*: the system is invariant under $x \mapsto -x$ and $y \mapsto -y$; some thought will save yourself some work!] Sketch the phase portrait.

8. Consider the system

$$\dot{x} = -x + \frac{y}{\log\sqrt{x^2+y^2}}, \quad \dot{y} = -y - \frac{x}{\log\sqrt{x^2+y^2}}.$$

Show that the origin is a stable focus, in the sense that $\theta \to \infty$ as $t \to \infty$, even though the linearized system at the origin is a stable node. [Note that the nonlinear terms are not $O(|\mathbf{x}|^2)$ here.]

9*. Sketch the phase portraits near the origin for the following non-hyperbolic fixed points.

$$
\begin{array}{llllll}
(a) & \dot{x} = x^2, & \dot{y} = y, & (b) & \dot{x} = x^2 + xy, & \dot{y} = \frac{1}{2}y^2 + xy, \\
(c) & \dot{x} = y, & \dot{y} = x^2, & (d) & \dot{x} = y, & \dot{y} = -x^3 + 4xy.
\end{array}
$$

[*Hints*: Consider the signs of $\dot{x}$, $\dot{y}$. Are the axes trajectories? Is there symmetry? One system is Hamiltonian. Note that case (d) has two exact solutions of the form $y = a_{\pm}x^2$.]

10. Calculate the stable and unstable manifolds of the origin to third order (i.e. up to and including the cubic terms) for the system

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2.$$

Sketch the phase portrait for $|\mathbf{x}| \ll 1$ showing the slight curvature of the two manifolds. Find and classify the other fixed point, and sketch the phase portrait on the scale $|\mathbf{x}| = O(1)$.

11. Calculate the stable and unstable manifolds of the origin up to fourth order for the system

$$\dot{x} = x + y^2, \quad \dot{y} = -y + 4x^3 + xy.$$

12**. Consider the non-autonomous differential equation $\dot{y} = y^2 - t$ (which Liouville proved cannot be solved in terms of solutions of algebraic equations or integrals thereof). Convert it to an autonomous system and sketch its portrait in the (t, y) plane.

Show that solutions that enter the region $y^2 < t$ cannot leave it. By changing variable from y to $u = y/\sqrt{t}$ and considering $\dot{u}$ for $t \gg 1$, explain why all forwards solutions (i.e. t increasing) tend to $u = \pm 1$ or $u = \infty$.

Argue that all backwards solutions, and all forwards solutions with $u \to \infty$, enter regions where $\dot{y} > \frac{1}{2}y^2$ and hence blow up in finite time. Explain why there is a unique value $y(0)$ such that $y(t)$ remains finite and positive for all finite t as t increases and show this solution on your phase portrait.