Examples Sheet 2

1. Use the Lyapunov function $V = x^2 + y^2$ to show that the origin is asymptotically stable for the system

$$\dot{x} = -2x - y^2, \quad \dot{y} = -y - x^2,$$

and that the region $x^2 + y^2 \leqslant 1$ is included in its domain of stability.

2. Determine the values of k for which $V = x^2 + ky^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin (how small need not be determined) for the system

$$\dot{x} = -x + y - x^2 - y^2 + xy^2, \quad \dot{y} = -y + xy - y^2 - x^2 y.$$

What information about the domain of stability of the origin can be deduced when $k = 1$?
[*Hint*: $\dot{V}$ has a homogeneous quadratic factor when $k = 1$.]

3. Sketch the phase portrait of the ideal pendulum

$$\dot{\theta} = p, \quad \dot{p} = -\sin\theta.$$

(The cylindrical topology of phase space is represented by $[-\pi, \pi) \times \mathbb{R}$ with periodic edges.) By considering the function $H(\theta, p) = \frac{1}{2}p^2 + 1 - \cos\theta$ explain informally why all of the periodic orbits are Lyapunov stable. Show that the invariant set $\Sigma = \{(\theta, p) : H(\theta, p) = 2\}$ consists of one fixed point and two homoclinic orbits. Are these orbits Lyapunov stable? Is Σ Lyapunov stable?

Consider the damped pendulum $\ddot{\theta} + k\dot{\theta} + \sin\theta = 0$, with $k > 0$. Find the fixed points. Use La Salle's Invariance Principle to show that $(0, 0)$ is asymptotically stable. What can you say about the domain of stability?

4. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^n$, where all of the eigenvalues λ_i of A are distinct and have negative real parts, and correspond to left eigenvectors $\mathbf{e}_i^L$. Consider the function $V = \sum_{i=1}^n v_i |\mathbf{e}_i^L \cdot \mathbf{x}|^2$ where the v_is are any set of strictly positive constants. Show that $\dot{V} < 0$ for $\mathbf{x} \neq \mathbf{0}$. Deduce that $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

Note: left eigenvectors go on the left of the matrix, so $(\mathbf{e}_i^L)^\intercal A = \lambda_i (\mathbf{e}_i^L)^\intercal$, and right eigenvectors are the usual eigenvectors,so $A\mathbf{e}_i^R = \lambda_i \mathbf{e}_i^R$. You may assume without loss of generality that the sets of eigenvectors are normalised so that $\mathbf{e}_i^L \cdot \mathbf{e}_j^R = \delta_{ij}$.

[Hint: First let $a_i(t) = \mathbf{e}_i^L \cdot \mathbf{x}(t)$ and calculate $\dot{a}_i$ and $\dot{\overline{a_i}}$.]

5. Consider the system in $\mathbb{R}^2$

$$\dot{\mathbf{x}} = A\mathbf{x} - |\mathbf{x}|^2 \mathbf{x},$$

where A is a constant real matrix with complex eigenvalues $p \pm iq$ ($q > 0$). Use the divergence test (Dulac with $\phi = 1$) and the Poincaré–Bendixson theorem to prove that there are no periodic orbits for $p < 0$ and there is at least one for $p > 0$. [*Hint*: Consider contours of the function V from Q4.] *Show that there is only one periodic orbit for $p > 0$. [*Hint*: $\dot{\theta}$ is independent of $|\mathbf{x}|$.]

6. Prove that

$$\dot{x} = x - y - (x^2 + \tfrac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \tfrac{1}{2}y^2)y,$$

has a periodic solution. [*Hint*: polar coordinates.]

7. Show using Dulac's criterion with a suitable weighting function, that there are no periodic orbits in $x \geqslant 0$, $y \geqslant 0$ for the population model

$$\dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3).$$

Find and analyse the fixed points and sketch the phase portrait (positive quadrant only). If x and y are the population densities of two species, what final outcome does your sketch suggest?

8. Suppose a differential equation in $\mathbb{R}^2$ has only three fixed points, two of which are sinks and the other a saddle. Sketch examples, where possible, of flows with a periodic orbit such that the set of fixed points enclosed by it: (a) is empty; (b) contains just one sink; (c) contains just the saddle; (d) contains one sink and the saddle; (e) contains both sinks; (f) contains all three fixed points. Where impossible, justify your answer.

9. Consider the system

$$\dot{x} = 2x + x^2 - y^2, \quad \dot{y} = -2y + x^2 - y^2.$$

Use the Poincaré index test to show that there are no periodic orbits. *Make a sensible change of variables and sketch the phase plane.

10. Consider the system

$$\dot{x} = -x - y + \tfrac{3}{2}\alpha x y^2 + x^3, \quad \dot{y} = \alpha^{-1}x - y + \tfrac{1}{2}x^2 y + \alpha y^3,$$

where α is a positive constant. Show that the origin is asymptotically stable by finding a Lyapunov function $V(\mathbf{x}) = x^2 + cy^2$ for an obvious choice of the constant c. Find the domain of asymptotic stability for the origin. What happens outside this region? *Show that the boundary of the region is a periodic orbit only when $\alpha < 16$. [*Hint*: parameterise the boundary and look at the dynamics purely on this boundary.]

11. Show that $\ddot{x} + ax + x^2 = 0$ conserves $V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}x^3$ where $p = \dot{x}$. Sketch the phase plane for $a > 0$, and describe the different sorts of orbits in the system. Show that when $k > 0$ each solution of $\ddot{x} + k\dot{x} + ax + x^2 = 0$ converges to one of two fixed points or diverges to infinity. Compute the linear stability of each fixed point. Draw a sketch showing the sets of points whose orbits converge to each of the fixed points.

12. The Lorenz equations are

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,$$

where r, σ and b are positive constants. For $0 < r < 1$ show that the origin is globally asymptotically stable by considering a function $V_1(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2$ for a suitable choice of the constants α, β and γ. For $r \geqslant 1$ show, by considering the function $V_2(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$, that all trajectories eventually enter and then remain within a bounded region of phase space.