Examples Sheet 2

1. Use the Lyapunov function \( V = x^2 + y^2 \) to show that the origin is asymptotically stable for the system

\[
\dot{x} = -2x - y^2, \quad \dot{y} = -y - x^2,
\]

and that the region \( x^2 + y^2 \leq 1 \) is included in its domain of stability.

2. Determine the values of \( k \) for which \( V = x^2 + ky^2 \) is a Lyapunov function in a sufficiently small neighbourhood of the origin (how small need not be determined) for the system

\[
\dot{x} = -x + y - x^2 - y^2 + xy^2, \quad \dot{y} = -y + xy - y^2 - x^2 y.
\]

What information about the domain of stability of the origin can be deduced when \( k = 1 \)?

[Hint: \( \dot{V} \) has a homogeneous quadratic factor when \( k = 1 \).]

3. Sketch the phase portrait of the ideal pendulum

\[
\dot{\theta} = p, \quad \dot{p} = -\sin \theta.
\]

(The cylindrical topology of phase space is represented by \([-\pi, \pi) \times \mathbb{R}\) with periodic edges.) By considering the function \( H(\theta, p) = \frac{1}{2}p^2 + 1 - \cos \theta \) explain informally why all of the periodic orbits are Lyapunov stable. Show that the invariant set \( \Sigma = \{ (\theta, p) : H(\theta, p) = 2 \} \) consists of one fixed point and two homoclinic orbits. Are these orbits Lyapunov stable?

Consider the damped pendulum \( \ddot{\theta} + k\dot{\theta} + \sin \theta = 0 \), with \( k > 0 \). Find the fixed points. Use La Salle’s Invariance Principle to show that \((0, 0)\) is asymptotically stable. What can you say about the domain of stability?

4. Consider the linear system \( \dot{x} = Ax \) with \( x \in \mathbb{R}^n \), where all of the eigenvalues \( \lambda_i \) of \( A \) are distinct and have negative real parts, and correspond to left eigenvectors \( e^L_i \). Consider the function \( V = \sum_{i=1}^{n} v_i |e^L_i \cdot x|^2 \) where the \( v_i \)s are any set of strictly positive constants. Show that \( \dot{V} < 0 \) for \( x \neq 0 \). Deduce that \( x = 0 \) is asymptotically stable.

Note: left eigenvectors go on the left of the matrix, so \( (e_i^L)^T A = \lambda_i (e_i^L)^T \), and right eigenvectors are the usual eigenvectors, so \( Ae_i^R = \lambda_i e_i^R \). You may assume without loss of generality that the sets of eigenvectors are normalised so that \( e_i^L \cdot e_j^R = \delta_{ij} \).

[Hint: First let \( a_i(t) = e_i^L \cdot x(t) \) and calculate \( \dot{a_i} \) and \( \ddot{a_i} \).]

5. Consider the system in \( \mathbb{R}^2 \)

\[
\dot{x} = Ax - |x|^2 x,
\]

where \( A \) is a constant real matrix with complex eigenvalues \( p \pm iq \) \((q > 0)\). Use the divergence test (Dulac with \( \phi = 1 \)) and the Poincaré–Bendixson theorem to prove that there are no periodic orbits for \( p < 0 \) and there is at least one for \( p > 0 \).

[Hint: Consider contours of the function \( V \) from Q4.] *Show that there is only one periodic orbit for \( p > 0 \).

[Hint: \( \dot{\theta} \) is independent of \( |x| \).]
6. Prove that

\[ \dot{x} = x - y - (x^2 + \frac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \frac{1}{2}y^2)y, \]

has a periodic solution. [Hint: polar coordinates.]

7. Show using Dulac’s criterion with a suitable weighting function, that there are no periodic orbits in \( x \geq 0, \ y \geq 0 \) for the population model

\[ \dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3). \]

Find and analyse the fixed points and sketch the phase portrait (positive quadrant only). If \( x \) and \( y \) are the population densities of two species, what final outcome does your sketch suggest?

8. Suppose a differential equation in \( \mathbb{R}^2 \) has only three fixed points, two of which are sinks and the other a saddle. Sketch examples, where possible, of flows with a periodic orbit such that the set of fixed points enclosed by it: (a) is empty; (b) contains just one sink; (c) contains just the saddle; (d) contains one sink and the saddle; (e) contains both sinks; (f) contains all three fixed points. Where impossible, justify your answer.

9. Consider the system

\[ \dot{x} = 2x + x^2 - y^2, \quad \dot{y} = -2y + x^2 - y^2. \]

Use the Poincaré index test to show that there are no periodic orbits. *Make a sensible change of variables and sketch the phase plane.

10. Consider the system

\[ \dot{x} = -x - y + \frac{3}{2}axy^2 + x^3, \quad \dot{y} = \alpha^{-1}x - y + \frac{1}{2}x^2y + \alpha y^3, \]

where \( \alpha \) is a positive constant. Show that the origin is asymptotically stable by finding a Lyapunov function \( V(x) = x^2 + cy^2 \) for an obvious choice of the constant \( c \). Find the domain of asymptotic stability for the origin. What happens outside this region? *Show that the boundary of the region is a periodic orbit only when \( \alpha < 16 \). [Hint: parameterise the boundary and look at the dynamics purely on this boundary.]

11. Show that \( \dot{x} + ax + x^2 = 0 \) conserves \( V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}x^3 \) where \( p = \dot{x} \). Sketch the phase plane for \( a > 0 \), and describe the different sorts of orbits in the system. Show that when \( k > 0 \) each solution of \( \dot{x} + k\dot{x} + ax + x^2 = 0 \) converges to one of two fixed points or diverges to infinity. Compute the linear stability of each fixed point. Draw a sketch showing the sets of points whose orbits converge to each of the fixed points.

12. The Lorenz equations are

\[ \dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz, \]

where \( r, \sigma \) and \( b \) are positive constants. For \( 0 < r < 1 \) show that the origin is globally asymptotically stable by considering a function \( V_1(x, y, z) = ax^2 + \beta y^2 + \gamma z^2 \) for a suitable choice of the constants \( a, \beta \) and \( \gamma \). For \( r \geq 1 \) show, by considering the function \( V_2(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \), that all trajectories eventually enter and then remain within a bounded region of phase space.