Examples Sheet 2

1. Use the Lyapunov function $V = x^2 + y^2$ to show that the origin is asymptotically stable for the system
   \[ \dot{x} = -2x - y^2, \quad \dot{y} = -y - x^2, \]
   and that the region $x^2 + y^2 \leq 1$ is included in its domain of stability.

2. Determine the values of $k$ for which $V = x^2 + ky^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin (how small need not be determined) for the system
   \[ \dot{x} = -x + y - x^2 - y^2 + xy^2, \quad \dot{y} = -y + xy - y^2 - x^2 y. \]

   What information about the domain of stability of the origin can be deduced when $k = 1$?
   
   [Hint: $\dot{V}$ has a homogeneous quadratic factor when $k = 1$.]

3. Sketch the phase portrait of the ideal pendulum
   \[ \dot{\theta} = p, \quad \dot{p} = -\sin \theta. \]
   (The cylindrical topology of phase space is represented by $[-\pi, \pi) \times \mathbb{R}$ with periodic edges.) By considering the function $H(\theta, p) = \frac{1}{2}p^2 + 1 - \cos \theta$ explain informally why all of the periodic orbits are Lyapunov stable. Show that the invariant set $\Sigma = \{(\theta, p) : H(\theta, p) = 2\}$ consists of one fixed point and two homoclinic orbits. Are these orbits Lyapunov stable?

   Consider the damped pendulum $\ddot{\theta} + k \dot{\theta} + \sin \theta = 0$, with $k > 0$. Find the fixed points. Use La Salle’s Invariance Principle to show that $(0, 0)$ is asymptotically stable. What can you say about the domain of stability?

4. Consider the linear system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$, where all of the eigenvalues $\lambda_i$ of $A$ are distinct and have negative real parts, and correspond to left eigenvectors $e_i^L$. Consider the function $V = \sum_{i=1}^{n} v_i |e_i^L \cdot x|^2$ where the $v_i$s are any set of strictly positive constants. Show that $\dot{V} < 0$ for $x \neq 0$. Deduce that $x = 0$ is asymptotically stable.

   Note: left eigenvectors go on the left of the matrix, so $(e_i^L)^T A = \lambda_i e_i^L$, and right eigenvectors are the usual eigenvectors, so $Ae_i^R = \lambda_i e_i^R$. You may assume without loss of generality that the sets of eigenvectors are normalised so that $e_i^L \cdot e_j^R = \delta_{ij}$.

   [Hint: First let $a_i(t) = e_i^L \cdot x(t)$ and calculate $\dot{a}_i$ and $\ddot{a}_i$.]

5. Consider the system in $\mathbb{R}^2$
   \[ \dot{x} = Ax - |x|^2 x, \]
   where $A$ is a constant real matrix with complex eigenvalues $p \pm iq$ ($q > 0$). Use the divergence test (Dulac with $\phi = 1$) and the Poincaré–Bendixson theorem to prove that there are no periodic orbits for $p < 0$ and there is at least one for $p > 0$. [Hint: Consider contours of the function $V$ from Q4.] *Show that there is only one periodic orbit for $p > 0$. [Hint: $\dot{\theta}$ is independent of $|x|$.]
6. Prove that
\[
\dot{x} = x - y - (x^2 + \frac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \frac{1}{2}y^2)y,
\]
has a periodic solution. [Hint: polar coordinates.]

7. Show using Dulac’s criterion with a suitable weighting function, that there are no periodic orbits in \( x \geq 0, \ y \geq 0 \) for the population model
\[
\dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3).
\]
Find and analyse the fixed points and sketch the phase portrait (positive quadrant only). If \( x \) and \( y \) are the population densities of two species, what final outcome does your sketch suggest?

8. Suppose a differential equation in \( \mathbb{R}^2 \) has only three fixed points, two of which are sinks and the other a saddle. Sketch examples, where possible, of flows with a periodic orbit such that the set of fixed points enclosed by it: (a) is empty; (b) contains just one sink; (c) contains just the saddle; (d) contains one sink and the saddle; (e) contains both sinks; (f) contains all three fixed points. Where impossible, justify your answer.

9. Consider the system
\[
\dot{x} = 2x + x^2 - y^2, \quad \dot{y} = -2y + x^2 - y^2.
\]
Use the Poincaré index test to show that there are no periodic orbits. *Make a sensible change of variables and sketch the phase plane.

10. Consider the system
\[
\dot{x} = -x - y + \frac{3}{2}\alpha xy^2 + x^3, \quad \dot{y} = \alpha^{-1}x - y + \frac{1}{2}x^2y + \alpha y^3,
\]
where \( \alpha \) is a positive constant. Show that the origin is asymptotically stable by finding a Lyapunov function \( V(x) = x^2 + cy^2 \) for an obvious choice of the constant \( c \). Find the domain of asymptotic stability for the origin. What happens outside this region? *Show that the boundary of the region is a periodic orbit only when \( \alpha < 16 \). [Hint: parameterise the boundary and look at the dynamics purely on this boundary.]

11. Show that \( \dot{x} + ax + x^2 = 0 \) conserves \( V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}x^3 \) where \( p = \dot{x} \). Sketch the phase plane for \( a > 0 \), and describe the different sorts of orbits in the system. Show that when \( k > 0 \) each solution of \( \dot{x} + k\dot{x} + ax + x^2 = 0 \) converges to one of two fixed points or diverges to infinity. Compute the linear stability of each fixed point. Draw a sketch showing the sets of points whose orbits converge to each of the fixed points.

12. The Lorenz equations are
\[
\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,
\]
where \( r, \sigma \) and \( b \) are positive constants. For \( 0 < r < 1 \) show that the origin is globally asymptotically stable by considering a function \( V_1(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2 \) for a suitable choice of the constants \( \alpha, \beta \) and \( \gamma \). For \( r \geq 1 \) show, by considering the function \( V_2(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2 \), that all trajectories eventually enter and then remain within a bounded region of phase space.