**Examples Sheet 3**

1. Use the 'energy-balance' method to find (approximately) the limit cycle in the nearly Hamiltonian system

$$\ddot{x} - \epsilon(x-3)(x+1)\dot{x} + x = 0,$$

where $0 < \epsilon \ll 1$. Use the divergence of the flow to compute the Floquet multiplier of the limit cycle, and hence determine its stability.

How does the multiplier relate to the value of $d\Delta H/dH_0$ on the limit cycle?

2. Describe the behaviour of the system

$$\dot{x} = 2y + \tfrac{1}{8}\epsilon\alpha x, \quad \dot{y} = -2x + \epsilon y^3(x^2 - \tfrac{1}{2}),$$

when $\epsilon = 0$ and $\alpha$ is a constant. Investigate whether there are any limit cycles for $0 < \epsilon \ll 1$. What are the possible periodic orbits when $\alpha = 2$ and which are stable? What, if anything, can you say about the system when $\alpha > \frac{9}{4}$ and when $\alpha = \frac{9}{4}$? [*Hint*: $8\cos^2 u \sin^4 u = \sin^2 2u(1 - \cos 2u)$.]

3. Use the divergence of the flow to find approximations to the Floquet multiplier for the periodic orbit of the van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

when (a) $0 < \mu \ll 1$ and (b) $\mu \gg 1$.

[*Hint for (b)*: Use the same variables as in lectures, and arguments similar to those for the period.]

4. (a) Sketch the bifurcation diagram for the system

$$\dot{x} = f(x) = x(\mu + x - 2)(\mu + 2x - x^2),$$

showing the position and stability of the fixed points in the $(\mu, x)$-plane. State the type of bifurcation at each bifurcation point.

(b) Repeat part (a) for the system

$$\dot{x} = x(9 - \mu x)(\mu + 2x - x^2)[(x+2)^2 + \tfrac{1}{25}(\mu - 3)^2 - 1]$$

Discuss the effect on the bifurcations of adding a small positive constant $\epsilon$ to the right-hand side.

5. Find the critical value $a_c$ at which there is a bifurcation at the origin in the system

$$\dot{x} = y - x - 2x^2, \quad \dot{y} = ax - y - 2y^2.$$

Change the variables to $u = x + y$, $v = x - y$, $\mu = a - a_c$ and explain why this might be advantageous. Hence find the extended centre manifold and the evolution equation on it, each correct to third order. Deduce the nature of the bifurcation.

To classify the bifurcation correctly, what orders (of the evolution equation and the extended centre manifold) did we actually need?

6. Compute the extended centre manifold near $x = y = \mu = 0$ to sufficiently high order to identify the bifurcation type for each of the systems

$$(a) \qquad \dot{x} = \mu + y - 3x^2 + xy, \quad \dot{y} = -3y + y^2 - x^2,$$

$$(b) \qquad \dot{x} = -2x + y - x^2, \quad \dot{y} = \mu + x(y - x).$$

For each case sketch the bifurcation diagram and sketch the $(x, y)$-phase portrait of the local dynamics very close to the origin for $0 < \mu \ll 1$.

[*Hint:* In (b) you can either make a change of variables to make the centre manifold tangent to one of the coordinate axes, or – more simply – having determined the eigenvector associated with the zero eigenvalue you can approximate the centre manifold as $x(y) = a_{10}y + a_{01}\mu + a_{20}y^2 + \ldots$ choosing $a_{10}$ so that the manifold is tangent to the eigenvector.]

7. Consider the system

$$\dot{x} = x(1 - y - 4x^2), \quad \dot{y} = y(\mu - y - x^2).$$

Show that the fixed point $(0, \mu)$ has a bifurcation at $\mu = 1$, while the fixed points $(\pm\frac{1}{2}, 0)$ both have bifurcations at $\mu = \frac{1}{4}$. By finding the first approximation to the extended centre manifold in each case, construct the relevant evolution equation and determine the type of bifurcation. [*Hint*: Use an appropriate change of the origin.] Sketch the bifurcation diagrams.

8. Investigate the linear stability at the origin for the system

$$\dot{x} = \mu x + \omega(x + y) - x(4x^2 + y^2), \quad \dot{y} = \mu y - \omega(2x + y) - y(4x^2 + y^2),$$

as the parameter $\mu$ is varied. What kind of bifurcation does this suggest? Find a transformation that reduces the *linear* problem to canonical form, and then rewrite the transformed system in polar coordinates. Deduce that the bifurcation is supercritical. *Obtain the same result using the Poincaré–Bendixson theorem.

9. Show that when $u$, $v = O(\epsilon)$, $\mu = O(\epsilon^2)$ and $\epsilon \ll 1$, the system

$$\dot{u} = \mu u - v + u^2 - v^2 - u(u + v)^2, \quad \dot{v} = u + \mu v - uv - v(u^2 - uv + v^2)$$

can be transformed to the system

$$\dot{x} = \mu x - y - x(x^2 + y^2) + O(\epsilon^4), \quad \dot{y} = x + \mu y - y(x^2 + y^2) + O(\epsilon^4)$$

by the nonlinear transformation $u = x + xy$, $v = y$. Deduce the nature of the bifurcation at $\mu = 0$.

10. $*$ By considering the behaviour of $\dot{r}$ and $\dot{\theta}$ in the system

$$\dot{r} = r(\mu + 2r^2 - r^4), \qquad \dot{\theta} = 1 - \nu r^2 \cos\theta\ ,$$

find the conditions on $\mu$ and $\nu$ under which there are zero, one and two periodic orbits. For $\nu = 0$, deduce the stability of these orbits and show the results in the $(\mu, r)$ plane. Describe the types of bifurcation that occur as $\mu$ and $\nu$ are varied.