Examples Sheet 4

1. Consider the graph for each of the maps $F : \mathbb{R} \to \mathbb{R}$,

$$(a)\, F(x) = -x, \qquad (b)\, F(x) = x - x^3, \qquad (c) F(x) = x + x^3, \qquad (d)\, F(x) = x + x^2.$$

State whether the non-hyperbolic fixed point at the origin is Liapunov stable, asymptotically stable or neither. In case (d) what set of points are attracted to the origin?

2. Find and analyse the successive bifurcations in the one-dimensional map $F(x, \nu) = \nu - \frac{1}{4}x^2$ as ν increases from $-\infty$ to 5.

3. Consider the logistic map F and a quadratic map G on $\mathbb{R}$, as given by

$$F(x, \mu) = \mu x(1 - x), \qquad G(x, \nu) = \nu - \tfrac{1}{4}x^2.$$

Show that for appropriate values of μ and ν, which should be determined, F and G are topologically conjugate, i.e., there exists a map $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ F = G \circ h$. [*Hint*: look for a linear map.] In the light of this result comment on your answer to Q2.

4. Consider the following generalised sawtooth maps on $[0, 1)$: $S_n(x) = nx\,[\text{mod } 1]$. By considering the binary representation, determine explicitly the 3-cycles of S_2 and express them as fractions.

   How many 3-cycles does S_3 have?

5*. Construct an aperiodic orbit of the sawtooth map (S_2 of question 4) that is dense in $[0, 1)$.

6. Let F be a continuous map on $\mathbb{R}$ and let $x_0 < x_1 < x_2 < x_3$ be the members of a 4-cycle with $F(x_n) = x_{n+1[\text{mod}\,4]}$. Prove (formally, using the IVT etc.) that F has a fixed point, a 2-cycle and a 3-cycle. Explain (informally, using a directed graph) why F has at least one 4-cycle in addition to $(x_0, \ldots, x_3)$ and at least two 3-cycles.

7. Let F be a continuous map on $\mathbb{R}$ and let $x_4 < x_2 < x_0 < x_6 < x_1 < x_3 < x_5$ be the members of a 7-cycle with $F(x_n) = x_{n+1[\text{mod}\,7]}$. Show that F has N-cycles for $N = 1, 2, 4$ and all $N \geq 6$.

   [*Hint*: Use a directed graph and work informally. Note that we cannot deduce whether or not F has a 3-cycle or a 5-cycle.]

8. Consider the skewed tent map

$$F(x) = \begin{cases} \mu x & x \in [0, x_m], \\ (\mu/s)(1-x) & x \in [x_m, 1], \end{cases}$$

where $x_m = 1/(1+s)$, and $s \in (0, \infty)$ parametrises the skewness. When is F a map of $[0,1]$ into itself? When is there a nontrivial (orientation-reversing) fixed point x_0, and when is it stable? Sketch F and F^2 together for each of the cases $\mu < 1$, $1 < \mu < s$ and $\mu > 1, s$. For $\mu > 1$ show that F^2 acts like a skewed tent map on two intervals $[x_{-1}, x_0]$ and $[x_0, x_{-2}]$ with x_0 as above, and the remaining x_i and parameters μ' and s' to be determined.

Now consider the case $\mu > 1, s$. Deduce the value $\mu_1(s)$ (with $\mu_1(1) = \sqrt{2}$) such that F^2 has a horseshoe for $\mu \geqslant \mu_1$. Deduce also that F^2 has nontrivial fixed points, that they are unstable, and then that F has 2^k-cycles for all $k = 1, 2, \ldots$ and that F is chaotic. *Sketch in the (μ, s)-plane the regions where $F^n(x)$ is generically attracted to $-\infty$, to 0, to x_0, to one interval, and to more than one interval. **Find values $\mu_k(s)$, with $\mu_k(1) = 2^{2^{-k}}$ such that the chaotic attractor consists of 2^k intervals for $\mu_{k+1} \leqslant \mu < \mu_k$.

9. (i) For the logistic map $F(x,\mu) = \mu x(1-x)$ show that when $2 < \mu < \mu_c$ there is exactly one point for $n = 0$ and exactly two points for each $n \geq 1$ such that

$$F^{n+1}(x,\mu) = 1 - \mu^{-1}, \qquad F^n(x,\mu) \neq 1 - \mu^{-1},$$

where $\mu_c = 3.678\ldots$ is a root of $\mu^4 - 4\mu^3 + 16 = 0$. Show further that the set of all such points as n varies has 0 and 1 as its only points of accumulation.

(ii) What can you say about the domain of attraction of the 2-cycle in $3 < \mu < 1 + \sqrt{6}$?

10*. In the bifurcation diagram for the logistic map there are several smooth dark tracks running through the chaotic part of the diagram. What are they? [*Hint* Think about $F(\frac{1}{2}, \mu)$.]

These tracks all intersect at the tip of the "white wedge" at $\mu \approx 3.68$. Can you obtain a more precise value of μ?

11*. The bifurcation diagram for the logistic map $F(x,\mu) = \mu x(1-x)$ shows a broad window around $\mu \approx 3.83$, because of the creation of a stable 3-cycle. Investigate the appearance of the 3-cycle as μ increases using the following approach.

A 3-cycle may be represented by

$$x_n = \alpha + \beta\omega^n + \beta^*\omega^{*n}$$

where $\omega = e^{2\pi i/3}$ and * denotes complex conjugate. Show from the logistic map that

$$\begin{aligned} \alpha &= \mu\alpha(1-\alpha) - 2\mu|\beta|^2 \\ \mu\beta^{*2} &= [\mu(1-2\alpha) - \omega]\beta \\ \mu\beta^2 &= [\mu(1-2\alpha) - \omega^*]\beta^*. \end{aligned}$$

Hence by eliminating β show that

$$9\mu^2\alpha^2 - (9\mu^2 + 3\mu)\alpha + 2(\mu^2 + \mu + 1) = 0.$$

Deduce that the 3-cycle appears at $\mu = 1 + \sqrt{8} \approx 3.8284$.