8 The Peano kernel theorem

8.1 The theorem

Our point of departure is the Taylor formula with an integral remainder term,
\[
f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^k}{k!}f^{(k)}(a) + \frac{1}{k!} \int_a^x (x-\theta)^k f^{(k+1)}(\theta) \, d\theta, \quad (8.1)
\]
which can be verified by integration by parts. Suppose that we are given an approximant (e.g. to a function, a derivative, an integral etc.) whose error vanishes for all \( f \in \mathbb{P}_k[x] \). The Taylor formula produces an expression for the error that depends on \( f^{(k+1)} \). This is the basis for the Peano kernel theorem.

Formally, let \( L(f) \) be an error of an approximant. Thus, \( L \) maps \( C[a, b] \), say, to \( \mathbb{R} \). We assume that it is linear, i.e. \( L(af + \beta g) = \alpha L(f) + \beta L(g) \) \( \forall \alpha, \beta \in \mathbb{R} \), and that \( L(f) = 0 \) for all \( f \in \mathbb{P}_k[x] \). Thus, (8.1) implies
\[
L(f) = \frac{1}{k!}L \left\{ \int_a^x (x-\theta)^k f^{(k+1)}(\theta) \, d\theta \right\}, \quad a \leq x \leq b.
\]
To make the range of integration independent of \( x \), we introduce the notation
\[
(x-\theta)^k_{\pm} := \begin{cases} (x-\theta)^k, & x \geq \theta, \\ 0, & x \leq \theta, \end{cases} \quad \text{whence} \quad L(f) = \frac{1}{k!}L \left\{ \int_a^b (x-\theta)^k_{\pm} f^{(k+1)}(\theta) \, d\theta \right\}.
\]
Let \( K(\theta) := L[(x-\theta)^k_{\pm}] \) for \( x \in [a, b] \). [Note: \( K \) is independent of \( f \).] Suppose that it is allowed to exchange the order of action of \( L \) and \( f \). Because of the linearity of \( L \), we then have
\[
L(f) = \frac{1}{k!} \int_a^b K(\theta) f^{(k+1)}(\theta) \, d\theta. \quad (8.2)
\]

The **Peano kernel theorem** Let \( L \) be a linear functional (a linear mapping from a space of functions to \( \mathbb{R} \)) such that \( L(f) = 0 \) for all \( f \in \mathbb{P}_k[x] \). Provided that \( f \in C^{k+1}[a, b] \) and the above exchange of \( L \) with the integration sign is valid, the formula (8.2) is true. \( \square \)

8.2 An example and few useful formulæ

Let \( L(f) := f'(0) - \left[ -\frac{3}{2} f(0) + 2 f(1) - \frac{1}{2} f(2) \right] \) this corresponds to approximating
\[
f'(0) \approx -\frac{3}{2} f(0) + 2 f(1) - \frac{1}{2} f(2).
\]
Then \( L(f) = 0 \) for \( f \in \mathbb{P}_2[x] \) (verify by trying \( f(x) = 1, x, x^2 \) and invoking linearity). Thus, for \( f \in C^3[0, 2] \) we have
\[
L(f) = \frac{1}{2} \int_0^2 K(\theta) f'''(\theta) \, d\theta.
\]
To evaluate the Peano kernel $K$, we fix $\theta$. Letting $g(x) := (x - \theta)^2$, we have

$$K(\theta) = L(g) = g'(0) - \left[ -\frac{3}{2}g(0) + 2g(1) - \frac{1}{2}g(2) \right]$$

$$= 2(0 - \theta) + \left[ -\frac{3}{2}(0 - \theta)^2 + 2(1 - \theta)^2 - \frac{1}{2}(2 - \theta)^2 \right]$$

$$= \begin{cases} -2\theta + \frac{3}{2}\theta^2 + (2\theta - \frac{3}{2}\theta^2) = 0, & \theta \leq 0, \\ -2(1 - \theta)^2 + \frac{3}{2}(2 - \theta)^2 = 2\theta - \frac{3}{2}\theta^2, & 0 \leq \theta \leq 1, \\ \frac{1}{2}(2 - \theta)^2, & 1 \leq \theta \leq 2, \\ 0, & \theta \geq 2. \end{cases}$$

[Note: It is obvious that $K(\theta) = 0$ for $\theta \notin [0, 2]$, since then it acts on a quadratic polynomial.]

**Back to the general case...** Typically, $L$ involves differentiation and integration. Since

$$\frac{d}{dx} (x - \theta)^k = k(x - \theta)^{k-1}, \quad \int_0^x (t - \theta)^{k-1} \, dt = \frac{1}{k} \left[ (x - \theta)^{k+1} - (a - \theta)^{k+1} \right],$$

the exchange of $L$ with integration is justified in these cases.

**Theorem** Suppose that $K$ doesn’t change sign in $(a, b)$ and that $f \in C^{k+1}[a, b]$. Then

$$L(f) = \frac{1}{k!} \left[ \int_a^b K(\theta) \, d\theta \right] f^{(k+1)}(\xi) \text{ for some } \xi \in (a, b).$$

**Proof.** Let (perversely!) $K \leq 0$. Then

$$L(f) \leq \frac{1}{k!} \int_a^b K(\theta) \min_{x \in [a, b]} f^{(k+1)}(x) \, d\theta = \frac{1}{k!} \left( \int_a^b K(\theta) \, d\theta \right) \min_{x \in [a, b]} f^{(k+1)}(x).$$

Likewise $L(f) \geq \frac{1}{k!} \left[ \int_a^b K(\theta) \, d\theta \right] \max_{x \in [a, b]} f^{(k+1)}(x)$, consequently

$$\min_{x \in [a, b]} f^{(k+1)}(x) \leq \frac{L[f]}{\frac{1}{k!} \int_a^b K(\theta) \, d\theta} \leq \max_{x \in [a, b]} f^{(k+1)}(x)$$

and the required result follows from the mean value theorem. Similar analysis pertains to the case $K \geq 0$. \qed

**Back to our example** We have $K \geq 0$ and $\int_a^b K(\theta) \, d\theta = \frac{3}{2}$. Consequently $L(f) = \frac{1}{k!} \times \frac{3}{2} f''(\xi) = \frac{1}{2} f''(\xi)$ for some $\xi \in (0, 2)$. We deduce in particular that $|L[f]| \leq \frac{1}{2} \|f''\|_{\infty}$, where $\|g\|_{\infty} := \max_{x \in [0, 2]} |g(x)|$ the $\infty$-norm.

Likewise, generalising the definition of the $\infty$-norm to an arbitrary interval $[a, b]$, we can easily deduce from

$$\left| \int_a^b f(x)g(x) \, dx \right| \leq \|g\|_{\infty} \int_a^b |f(x)| \, dx,$$

that $|L(f)| \leq \frac{1}{k!} \int_a^b |K(\theta)| \|f^{(k+1)}\|_{\infty}$ and that $|L(f)| \leq \frac{1}{k!} \|K\|_\infty \int_a^b f^{(k+1)}(x) \, dx$. This is valid also when $K$ changes sign. Moreover, letting $\|f\|_2 := \left[ \int_a^b |f(x)|^2 \, dx \right]^{1/2}$ the 2-norm the Cauchy–Schwarz inequality $\left| \int_a^b f(x)g(x) \, dx \right| \leq \|f\|_2 \|g\|_2$ implies that $|L(f)| \leq \frac{1}{k!} \|K\|_2 \|f^{(k+1)}\|_2$.