Mathematical Tripos Part IA: Introductory Examples Sheet

The material on this sheet is not particularly related to any course. The intention is to give you something to do until the lecture courses have progressed far enough for you to tackle the first proper examples sheet. Some of the questions have hard bits, so don't expect to do everything!

1 Sketch the graph of the function $y(x)$ given by

$$y(x) = \frac{x-3}{(x+1)(x-2)},$$

indicating the positions of the turning points.

Prove that there is a range of values which y cannot take if x is real.

2 Find dy/dx when $\tan y = [(x-1)/(2-x)]^{\frac{1}{2}}$, where $1 < x < 2$. Hence integrate $(x-1)^{-\frac{1}{2}}(2-x)^{-\frac{1}{2}}$ with respect to x $(1 < x < 2)$.

Guess the integral with respect to t of $(t-a)^{-\frac{1}{2}}(b-t)^{-\frac{1}{2}}$ $(a < t < b)$, and verify your guess directly. Do this integral also by writing $(t-a)(b-t)$ in the form $A^2 - (t-B)^2$ and using the new variable θ defined by $(t-B) = A\sin\theta$.

3 A curve is given parametrically by

$$x(\theta) = a(\theta - \sin\theta), \quad y(\theta) = a(1-\cos\theta),$$

where a is a constant. Show that the gradient of the curve is $\cot\frac{1}{2}\theta$. Sketch the curve in the x-y plane, explaining carefully how you discovered its main features.

Show that the arc-length s from the origin to the point P corresponding to θ (with $0 < \theta < 2\pi$) is given by $s = 4a(1-\cos\frac{1}{2}\theta)$. Show also that this can be written in the form $s = 4a(1-\sin\psi)$, where ψ is the angle from the x-axis to the tangent to the curve at P.

4 Let

$$I_n = \int_0^\infty \operatorname{sech}^n u \; du.$$

By integrating by parts, show that for $n > 0$

$$\int_0^\infty (\operatorname{sech}^{n+2} u \;\sinh u)\sinh u \; du = (n+1)^{-1} I_n$$

and deduce that $(n+1)I_{n+2} = nI_n$. Find the value of I_6.
[NB: $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\operatorname{sech} x = (\cosh x)^{-1}$. You will need to use the identity $\sinh^2 u = \cosh^2 u - 1$.]

5 Show that $\int_0^{2\pi} e^{in\theta} d\theta = 0$ where n is a non-zero integer. Express $\cos\theta$ in terms of $e^{i\theta}$.

Write down the coefficient of the term which is independent of x in binomial expansion of $(x + x^{-1})^{2n}$, where n is a positive integer. By taking $x = e^{i\theta}$ and using the results of the first paragraph, evaluate

$$\int_0^{2\pi} \cos^{2n}\theta \; d\theta \qquad \text{and} \qquad \int_0^{2\pi} \cos^{2n} 2\theta \; d\theta.$$

6 For any fixed number q, let

$$A_n = \frac{q^n}{n!}\int_0^\pi x^n(\pi - x)^n \sin x \, \mathrm{d}x.$$

Show that $A_{n+2} = (4n+6)qA_{n+1} - (q\pi)^2 A_n$ and hence show that A_n is an integer for all n if q and $q\pi$ are integers.

By considering $\sum\limits_0^\infty A_n$, or otherwise, show also that $A_n \to 0$ as $n \to \infty$.

Deduce that π is irrational.

7 A uniform circular wire is bent into the shape of a circular arc of radius a which subtends an angle 2α at the centre of the circle. The distance, d, of the centre of mass of the wire from the centre of the circle may be written as $d = f(\alpha)$. By cutting the arc into two similar arcs, or otherwise, show that

$$f(\alpha) = f(\alpha/2)\cos(\alpha/2). \qquad (*)$$

Show that $f(\alpha) = (A\sin\alpha)/\alpha$ satisfies this equation. Assume that this is the correct form for $f(\alpha)$. By considering the case when α is very small, show that $d = (a\sin\alpha)/\alpha$. Show that the corresponding result for a lamina in the shape of a sector of a circle is $d = (2a\sin\alpha)/(3\alpha)$.

Try solving the functional equation $(*)$ be considering the function $g(x)$ defined by $g(x) = \sin x/f(x)$.

8 Let

$$f(\theta) = \sum_{k=1}^{n} k^{-1}\sin k\theta.$$

By considering the real part of $\sum_{k=1}^n \exp(ik\theta)$, show that, for $\theta \neq 2N\pi$ (where N is an integer),

$$f'(\theta) = \frac{\cos(\frac{1}{2}(n+1)\theta)\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)} = \frac{\sin m\theta}{2\sin(\frac{1}{2}\theta)} - \frac{1}{2},$$

where $m = n + \frac{1}{2}$.

Show further that, for $0 < \theta_1 < \theta_2 < 2\pi$,

$$f(\theta_1) - f(\theta_2) = \frac{\theta_2 - \theta_1}{2} + \frac{\cos m\theta_2}{2m\sin(\frac{1}{2}\theta_2)} - \frac{\cos m\theta_1}{2m\sin(\frac{1}{2}\theta_1)} + \frac{1}{m}\int_{\theta_1}^{\theta_2} \frac{\cos m\theta\cos(\frac{1}{2}\theta)}{4\sin^2(\frac{1}{2}\theta)} d\theta$$

and deduce that, for $0 < \theta < 2\pi$,

$$\sum_{k=1}^{\infty} k^{-1}\sin k\theta = \frac{\pi}{2} - \frac{\theta}{2}.$$

What is the value of this sum when $2\pi < \theta < 4\pi$? Can the value of this sum in the limit $\theta \to 0$ be inferred from the above results?