**Example Sheet 4**

1. Define the circle of convergence of a power series of a complex variable $z$. Find the radii of convergence of the three series

$$\sum_{n=0}^{\infty} nz^n\,, \quad \sum_{n=0}^{\infty}(\cosh n)z^n \quad \text{and} \quad \sum_{n=0}^{\infty}\left[2^n + (-1)^n\right] z^n\,.$$

Show by example that a power series may or may not converge on its circle of convergence. Hence give an example of a series that is convergent but not absolutely convergent.

2. Calculate the Taylor series of the function $f(z) = \ln(1-z)$ about $z = 0$ and determine its radius of convergence. Now calculate the Taylor series for $f(z)$ about $z = \mathrm{i}$ and determine the new radius of convergence. Comment.

3. The real parts of three analytic functions of $z = x + \mathrm{i}y$ are

$$\sin x \cosh y\,, \qquad \mathrm{e}^{y^2-x^2}\cos 2xy\,, \qquad \frac{x}{x^2+y^2}\,,$$

respectively. Use the Cauchy–Riemann equations to find their imaginary parts and hence deduce the forms of the complex functions.

4. Where are the zeros and singularities of the following complex functions? Give the orders of the zeros, and classify the singularities.

$$\frac{(z-\mathrm{i})^2}{z+1}\,, \quad \frac{1}{1+z} - \frac{1}{1-z}\,, \quad \frac{1}{z^2+\mathrm{i}}\,, \quad \sec^2 \pi z\,, \quad \sin z^{-2}\,, \quad \sinh\frac{z}{z^2-1}\,, \quad \frac{\tanh z}{z}\,.$$

5. Given that $y = x$ is one solution of

$$(1-x^2)y'' - 2xy' + 2y = 0\,,$$

find the general solution.

6. Find power series solutions, about $x = 0$, of the differential equation

$$4xy'' + 2y' + y = 0\,.$$

Sum the series to show that two solutions are $\cos\sqrt{x}$ and $\sin\sqrt{x}$. Are they linearly independent?

7. Identify the singular points of the equation

$$(2z+z^3)y'' - y' - 6zy = 0$$

and determine their nature. Find two linearly independent solutions as power series about $z = 0$. In particular, determine the indicial equation, the recurrence relation, and the radius of convergence of your solutions.

8. Find two linearly independent power series solutions, about $x = 0$, of

$$y'' - xy = 0\,.$$

Discuss the convergence of the series.

9. The Schrödinger equation for a harmonic oscillator of energy $E$ can be written as

$$-\hbar^2\Psi''(x) + mkx^2\Psi(x) = 2mE\Psi(x)$$

for some constants $m$ and $k$. Show how to transform this into the equation

$$w'' + (\lambda - \xi^2)w = 0$$

for a function $w(\xi)$. Write $w = y\exp(-\xi^2/2)$ to obtain Hermite's equation for $y(\xi)$:

$$y'' - 2\xi y' + (\lambda - 1)y = 0\,.$$

Find two power series solutions (about $\xi = 0$) and show that both are convergent for all finite $\xi$ (they behave like $\exp\xi^2$). Show that polynomial solutions are possible for particular values of $\lambda$. Given that $|\Psi|^2$ is the probability density and should be integrable, explain why these are the only acceptable solutions and hence obtain a formula for the energy levels of the harmonic oscillator.

10. Find power series solutions, about $x = 0$, of the following equations:

(a)

$$x(x-1)y'' + 3xy' + y = 0\,,$$

(b)

$$xy'' + (c - x)y' - ay = 0 \qquad (c \notin \mathbf{Z})\,.$$

11. Find solutions of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

in the form of *inverse* power series

$$y = \sum_{n=0}^{\infty} a_n x^{\sigma - n}\,.$$

Show that either $\sigma = \ell$ or $\sigma = -(\ell+1)$, and that one of the solutions is a polynomial in $x$ when $\ell$ is a positive integer.

*This example sheet is available at* `http://www.damtp.cam.ac.uk/user/examples/`
*Please send any comments and corrections to* `M.Wingate@damtp.cam.ac.uk`