Example Sheet 3: Cartesian Tensors

1 The moments of inertia along the principal axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of a rigid body are 1, 2 and 3 respectively. (The off-diagonal components of the inertia tensor are zero in this frame.) An observer has a coordinate basis $\mathbf{e}_1' = \frac{1}{2}(\mathbf{e}_1 + \sqrt{3}\,\mathbf{e}_3)$, $\mathbf{e}_2' = \mathbf{e}_2$ and $\mathbf{e}_3' = \frac{1}{2}(\mathbf{e}_3 - \sqrt{3}\,\mathbf{e}_1)$. Confirm that this is an orthonormal basis, and find the values of the components of the inertia tensor measured by the observer.

In a third frame with axes $\{\mathbf{e}_1'', \mathbf{e}_2'', \mathbf{e}_3''\}$, $\mathbf{e}_1$ has components $\frac{1}{\sqrt{2}}(1, 1, 0)$ relative to these axes and $\mathbf{e}_2$ has components $\frac{1}{\sqrt{3}}(1, -1, 1)$. Find the inertia tensor I_{ij} in this frame, and calculate $I_{ij}I_{ji}$. [*There is a simple way to do this!*]

2 Calculate the inertia tensor of a uniform right circular cylinder of mass M, radius a and height $\sqrt{3}\,a$, about its centre of gravity. Comment on your result.

3 A vector $\mathbf{F}$ is related to vectors $\mathbf{J}$ and $\mathbf{H}$ by a linear relation of the form $F_i = A_{ijk}J_jH_k$. Starting from the transformation laws for the components of vectors, deduce the transformation law for the components of $A = (A_{ijk})$. Hence demonstrate that A is a third rank tensor.

4 (i) Explain why for a 3×3 matrix $A = (a_{ij})$,

$$\epsilon_{ijk} \det A = \epsilon_{lmn} a_{il} a_{jm} a_{kn}.$$

(ii) Show that ϵ_{ijk} is a third rank tensor. Explain why it is isotropic.

(iii) Show that if A is a second rank tensor then $\det A$ is a scalar.

5 Show that any second rank tensor T may be expressed as the sum of a symmetric tensor with zero trace, an isotropic tensor and an anti-symmetric tensor.

The components of T are measured by one observer to be

$$\begin{pmatrix} 3 & -1 & 5 \\ 1 & 0 & 5 \\ 1 & -5 & 3 \end{pmatrix}.$$

Decompose T into three parts as above, and diagonalise the symmetric zero-trace part. Find the sum of the principal values of this part, and comment.

6 Let $A = (a_{ij})$ be an anti-symmetric second rank tensor. Show that for any vector $\mathbf{x}$, $A\mathbf{x}$ may be written as $\boldsymbol{\omega} \times \mathbf{x}$ for some suitable $\boldsymbol{\omega}$ which depends only on A.

Now show that the tensor $B = (b_{ij})$ defined by $b_{ij} = a_{ik}a_{kj}$ is symmetric, and find its principal values in terms of $\boldsymbol{\omega}$.

7 The electrical conductivity σ in a crystal is measured by an observer to have components

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Show that there is one direction in which no current flows, and find that direction.

The rate of energy dissipation per unit volume is given by E_iJ_i where $\mathbf{E}$ is the applied electric field and $\mathbf{J}$ the resulting current density. For a fixed value of $|\mathbf{E}|^2$, find the minimum and maximum possible energy dissipation rates.

8 Show that the fourth rank tensor

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

is isotropic. For the rest of this question you may assume that it is in fact the most general isotropic tensor of rank four.

In an isotropic fluid moving with velocity $\mathbf{v}(\mathbf{x})$, the strain tensor e_{ij} is defined by

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

and the corresponding stress tensor is σ_{ij}. The pressure in the fluid is given by $p = -\frac{1}{3}\sigma_{ii}$. Define the deviatoric stress tensor by $\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}$: show that σ' has zero trace. In most fluids (known as *Newtonian fluids*) the deviatoric stress is linearly related to the strain; show that the relationship between σ and e must then take the form

$$\sigma_{ij} = 2\mu(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk}) - p\delta_{ij}$$

for some scalar μ known as the coefficient of viscosity.

Given that $\mu > 0$, show further that the product $\sigma'_{ij}e_{ij}$ is non-negative (by considering its minimum value in an appropriate frame, or otherwise).

9 The rotationally symmetric crystal lattice of a simple metal has conductivity α parallel to its axis of symmetry and β in any perpendicular direction. Write down the conductivity tensor relative to these axes. A particular crystal is grown in such a way that it takes the form of a long thin wire, the direction of the wire making an angle θ with the symmetry axis of the lattice. By considering a current passing along the wire, show that the wire's resistance is $(L/A)(\alpha^{-1}\cos^2\theta + \beta^{-1}\sin^2\theta)$, where L is the length of the wire and A its cross-sectional area.

10 A second rank tensor is defined in terms of the position vector $\mathbf{x}$ by $T_{ij} = \delta_{ij} + \epsilon_{ijk}x_k$. Calculate the following integrals, where in each case the integration is over the surface of the sphere of unit radius.

$$\iint x_i \, \mathrm{d}S; \qquad \iint T_{ij} \, \mathrm{d}S; \qquad \iint T_{ij}T_{jk} \, \mathrm{d}S.$$

11 Use suffix notation to establish the following vector identities for any scalar field Φ and vector field $\mathbf{F}$:

$$\nabla \times (\nabla\Phi) = \mathbf{0}; \qquad \nabla \times (\Phi\mathbf{F}) = \Phi \, \nabla \times \mathbf{F} + \nabla\Phi \times \mathbf{F};$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0; \qquad \mathbf{F} \times (\nabla \times \mathbf{F}) = \nabla(\tfrac{1}{2}|\mathbf{F}|^2) - (\mathbf{F} \cdot \nabla)\mathbf{F}.$$

12 A conductor carries a steady current density $\mathbf{J} = \mu^{-1} \, \nabla \times \mathbf{B}$ in a magnetic field $\mathbf{B}$, where μ is the permeability. The mechanical force per unit volume acting on the conductor is $\mathbf{J} \times \mathbf{B}$. Show that this force can be written as $\partial S_{ij}/\partial x_j$ in terms of a tensor

$$S_{ij} = \mu^{-1}(B_iB_j - \tfrac{1}{2}B_kB_k\delta_{ij}).$$

[*Note that* $\nabla \cdot \mathbf{B} = 0$.]

Comments on or corrections to this problem sheet are very welcome and may be sent to me at J.B.Gutowski@damtp.cam.ac.uk