Example Sheet 4:
Complex Analysis, Contour Integration and Fourier Transforms

1 Establish the following general methods for calculating residues. [*Note: These are all very useful in practice, and the student is advised to memorise them.*]

(i) If $f(z)$ has a simple pole, then the residue of $f(z)$ at $z = z_0$ is $\lim_{z\to z_0}\{(z-z_0)f(z)\}$.

(ii) If $f(z)$ is analytic, then the residue of $f(z)/(z - z_0)$ at $z = z_0$ is $f(z_0)$.

(iii) If $1/f(z)$ has a simple pole at $z = z_0$, then its residue at $z = z_0$ is $1/f'(z_0)$.

(iv) If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic and non-zero, the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.

(v) If $f(z)$ has a pole of order N at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$\lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left((z-z_0)^N f(z)\right)\right\}.$$

2 Find the poles of the following functions and calculate the residues at each pole:

$$\frac{z+1}{z^2};\qquad \frac{e^{-z}}{z^3};\qquad \frac{\sin^2 z}{z^5};\qquad \cot z;\qquad \frac{z^2}{(1+z^2)^2}.$$

3 (i) State and prove Cauchy's Theorem.

(ii) Suppose that the simple contour C encloses $z = z_0$ in a positive sense and that f is an analytic function. Show that

$$\oint_C (z-z_0)^n\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \text{ is any other integer} \end{cases}$$

and

$$\oint_C \frac{f'(z)\,\mathrm{d}z}{z-z_0} = \oint_C \frac{f(z)\,\mathrm{d}z}{(z-z_0)^2}.$$

4 Suppose that $f(z)$ is analytic in and on the circle $|z - z_0| = r$. Show that for $n \geqslant 0$,

$$\left|f^{(n)}(z_0)\right| \leqslant \frac{n!}{r^n}\max_{|z-z_0|=r}|f(z)|.$$

Hence prove *Liouville's Theorem*: if $f(z)$ is analytic and bounded for all z then f is a constant.

Deduce that any polynomial $p(z)$ of degree at least one has at least one zero. [*Hint: consider* $1/p(z)$.]

5 Describe the method of the calculus of residues.

By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle in the z-plane (where a is real, $a > 1$, and n is a non-negative integer), evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,\mathrm{d}\theta.$$

6 By considering the integral $\oint (z^2+1)^{-1} e^{ikz}\,\mathrm{d}z$ taken around a large semicircle, show that for real positive k,

$$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+1}\,\mathrm{d}x = \pi e^{-k}.$$

What is the value for $k \leqslant 0$?

7 By integrating round a rectangular contour with vertices at $\pm R$ and $i\pi \pm R$, where R is a large real constant, or otherwise, show that $\int_0^\infty \operatorname{sech} x\,\mathrm{d}x = \pi/2$.

8 Verify the following results, where a is a non-zero real constant.

(i) $\displaystyle\int_0^\infty \frac{x^{-a}\,\mathrm{d}x}{x+1} = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$

(ii) $\displaystyle\int_0^\pi \frac{a\,\mathrm{d}\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}} \qquad (a > 1).$

(iii) $\displaystyle\int_0^\infty \frac{x^4}{1+x^8}\,\mathrm{d}x = \frac{\pi}{4}\sqrt{1 - 1/\sqrt{2}}\,.$

[*Although this can be done with the standard semicircle, you might like to consider instead using a sector of a circle of angle* $\pi/4$.]

(iv) $\displaystyle\int_0^\infty \cos(\tfrac{1}{2}ax^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{4|a|}}\,.$

[*Hint: in this case you must use a sector of a circle.*]

(v) $\displaystyle\int_0^\infty \frac{(\log x)^2\,\mathrm{d}x}{1+x^2} = \frac{\pi^3}{8}.$

[*Hint: use a semicircular contour with an appropriate branch cut.*]

9 Sketch possible arrangements of branch cuts for the following, giving the values on either side of each cut:

$$(z^2+1)^{1/2}; \qquad (z^2+1)^{1/3}; \qquad \log\left(\frac{z-i}{z+i}\right)^2.$$

10 Show that the two-dimensional function Φ defined by

$$\Phi(x,y) = \operatorname{Im}\left\{\frac{2}{\pi}\log\tanh z\right\},$$

where $z = x + iy$, satisfies Laplace's equation in $x > 0$; and that $\Phi = 0$ on both $y = 0$ and $y = \frac{\pi}{2}$, while on $x = 0$, $\Phi = 1$ for $0 < y < \frac{\pi}{2}$. Deduce the steady-state temperature distribution in a semi-infinite two-dimensional bar of width L, with the (infinitely) long sides held at zero temperature and the short side held at temperature T_0.

11 Find the function whose Fourier transform is $(1 + k^4)^{-1}$.

12 An overdamped harmonic oscillator ($\gamma > p$) subject to an impulsive force is described by the equation

$$\ddot{x} + 2\gamma\dot{x} + p^2 x = \delta(t)\ .$$

Given that $x = 0$ for $t < 0$, show by Fourier transform methods (and the result of question 9d from Example Sheet 2 last term, or otherwise), that for $t > 0$,

$$x(t) = \frac{1}{\sqrt{\gamma^2 - p^2}}\sinh\left(\sqrt{\gamma^2 - p^2}\,t\right)e^{-\gamma t}$$

13 For $t \geq 0$ the function $u(x,t)$ is defined for all x and satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \lambda\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(x,t) \to 0$ as $|x| \to \infty$, and the initial condition

$$u(x,0) = \begin{cases} -e^{x}\ , & x < 0 \\ 0\ , & x = 0 \\ e^{-x}\ , & x > 0\ . \end{cases}$$

Show using Fourier transform methods that for $t > 0$

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{k e^{-\lambda k^2 t}\sin(kx)}{1 + k^2}\,\mathrm{d}k\ .$$

Suppose instead that $u(x,t)$ is only defined in the half-space $x \geq 0$, that $u(0,t) = 0$ for $t \geq 0$, that $u(x,t) \to 0$ as $x \to \infty$, and that $u(x,0) = e^{-x}$ for $x > 0$. Write down the solution of this modified problem.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at J.B.Gutowski@damtp.cam.ac.uk