§4 FLOWS WITH A FREE SURFACE

Such flows are conspicuous all around us; they include water waves, river flow including flow over a weir, and ocean tides, storm surges and tsunamis. For instance we shall see how the height of a weir precisely controls the flow rate over it — a matter of some importance in hydraulic engineering. Again, if an earthquake generates a tsunami or 'tidal wave' near Japan, at one side of the Pacific, it is fairly simple to estimate how long the wave takes to reach California, or southern Chile.

(We shall find that the answer is roughly half a day to California and a day to southern Chile. Such waves travel at speeds just over 200 metres per second or 400 knots, not far short of the airspeeds of subsonic passenger jet aircraft.)

In all these flows, the shape of the free surface, more precisely the water–air interface, is a crucial part of the flow dynamics. Because the water has inertia about 3 powers of 10 greater than air, such flows can often be well modelled by taking the pressure at the free surface to be constant,

$$p_{(\text{free surface})} = p_{\text{atm}} = \text{constant}\,,$$

as we did in the problem of manometer oscillations. The surface is 'free' in the sense of being free to move, albeit only in ways permitted by the dynamics.

*It is obvious to any observant person when you *can't* take $p_{(\text{free surface})} = p_{\text{atm}} = \text{constant}$, namely when there is enough wind to generate waves on the water surface. Exactly how turbulent airflow interacts with a wavy water surface is a complicated and — this might surprise you — largely unsolved research problem.*

§4.1 Governing equations and boundary conditions

Assume that the gravity acceleration is uniform, so that the gravitational potential $\Phi = gz$, where $g =$ constant. Restrict attention to flows that were, or could have been, started from rest, hence irrotational, hence representable through a velocity potential, $\mathbf{u} = \nabla\phi$ with

$$\nabla^2\phi = 0\ .$$

This is the only 'governing equation'; everything else is a matter of boundary conditions. At the free surface there are *two* boundary conditions: the pressure condition, and the kinematic boundary condition derived in §1.7.

As in §1.7, denote the elevation of the free surface by $z = \zeta(x, y, t)$. The kinematic boundary condition derived there is

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} - w = 0 \qquad \text{on} \quad z = \zeta(x, y, t)\ , (*)$$

signifying that fluid on the free surface stays on the free surface.

The condition on the pressure field is, as usual, most conveniently expressed via the time-dependent, irrotational form of Bernoulli's theorem (§3.3), with $\Phi = gz$ and $\frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}|\nabla\phi|^2$:

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = \tilde{H}(t) \ ,$$

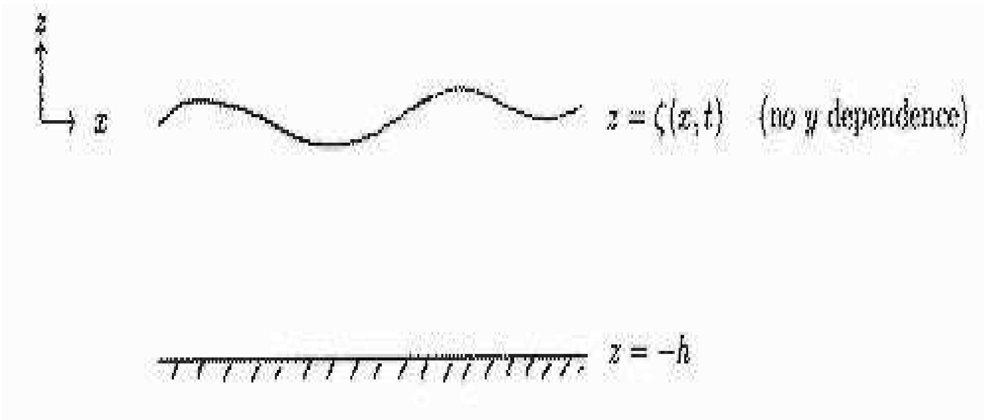
a function of time t alone. Apply this at the free surface $z = \zeta$, ignoring surface tension so that pressure p in the water, i.e., the p appearing in Bernoulli's equation above, is equal to p_{atm} at $z = \zeta$:

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = f(t) \quad \text{on} \quad z = \zeta(x, y, t) \ , (**)$$

where $f(t) = \tilde{H}(t) - \rho^{-1}p_{\text{atm}}$, a function of t alone.

§4.2 Small-amplitude water waves

Consider a layer of water, of depth h when undisturbed, assumed inviscid:



Full problem:

$$\begin{aligned}
\nabla^2\phi &= 0 \quad \text{in} \quad -h \le z \le \zeta(x,\, t) \\
\frac{\partial\phi}{\partial z} &= 0 \quad \text{on} \quad z = -h \\
\frac{\partial\zeta}{\partial t} &+ \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = \zeta \ , \quad \text{from } (*) \quad \text{with } \ u = \partial\phi/\partial x \\
\frac{\partial\phi}{\partial t} &+ \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = f(t) \quad \text{on} \quad z = \zeta \ , \quad \text{from } (**)
\end{aligned}$$

Linearize:

Linearize: 1) neglect quadratic terms such as $\dfrac{\partial\phi}{\partial x}\dfrac{\partial\zeta}{\partial x}$ and $\frac{1}{2}|\nabla\phi|^2$;
2) use Taylor series to express the boundary condition at $z+\zeta$ in terms of quantities at $z=0$,

e.g.

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\zeta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \zeta\left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0} + \dots\,,$$

in which the last term can be neglected because it is again quadratic (i.e., of the second order in small disturbance quantities). Hence the linearized equations and boundary conditions are:

$$\begin{aligned}
\nabla^2\phi &= 0 \quad \text{in} \quad -h\le z\le 0\\
\frac{\partial\phi}{\partial z} &= 0 \quad \text{on} \quad z=h\\
\frac{\partial\zeta}{\partial t} & -\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z=0\\
\frac{\partial\phi}{\partial t} & +g\zeta = f(t) \quad \text{on} \quad z=0
\end{aligned}$$

Now seek solutions of the form $\phi = Re\left(\hat\phi(z)e^{ikx-i\sigma t}\right)$ and $\zeta = Re\left(\hat\zeta e^{ikx-i\sigma t}\right)$ where k and σ are constants, and $\hat\zeta$ also. We shall always take k to be real and nonzero, so that the solution describes a wavy free surface extending from $x=-\infty$ to $x=\infty$. Notice that, with such a solution, $f(t)=0$ in the last-displayed boundary condition; for otherwise $f(t)$ would have to depend on x, which is a contradiction.

Notice furthermore that, with the form of solution just assumed, we can now replace $\partial/\partial x$ by ik, $\partial^2/\partial x^2$ by $-k^2$, and ∇^2 by $-k^2+\partial^2/\partial z^2$.

*Alternatively, if you prefer: *Fourier-transform* the linearized problem with respect to x to get ordinary differential equations (ODEs) in t with constant coefficients, implying a time dependence of the form $e^{-i\sigma t}$. (Note, σ could be a negative or positive constant — or even complex in a case to be noted later: gravity g can be negative!)*

The problem now becomes

$$\frac{d^2\hat\phi}{dz^2} - k^2\hat\phi = 0 \quad \text{in} \quad -h\le z\le 0$$

with

$$\frac{d\hat\phi}{dz} = 0 \quad \text{on} \quad z=-h\,,$$

$$i\sigma\hat{\zeta} + \frac{d\hat{\phi}}{dz} = 0 \quad \text{on} \quad z = 0 \ ,$$

$$-i\sigma\hat{\phi} + g\hat{\zeta} = 0 \quad \text{on} \quad z = 0 \ .$$

The first of the above is an ordinary differential equation with constant coefficients 1 and $-k^2$. Its general solution can be written as a linear combination of real exponential functions, or equivalently as $A\cosh\{k(z-z_0)\}$, where A and z_0 are arbitrary constants. We can satisfy the bottom boundary condition $d\hat{\phi}/dz = 0$ at $z = -h$ if we choose $z_0 = -h$, i.e.,

$$\hat{\phi} = A\cosh k(z+h) \ .$$

Then from the first (the kinematic) boundary condition on $z = 0$ we see that

$$i\sigma\hat{\zeta} + kA\sinh kh = 0$$

and from the second (pressure) (*)

$$g\hat{\zeta} - i\sigma A\cosh kh = 0 \ .$$

Regarding these as a system of two linear algebraic equations for the two unknown constants $\hat{\zeta}$ and A, we see that this system has nontrivial solutions if and only if its determinant vanishes, i.e. if and only if

$$\begin{vmatrix} i\sigma & k\sinh kh \\ g & -i\sigma\cosh kh \end{vmatrix} = 0 \ ,$$

i.e. if and only if

$$\sigma^2 = gk\tanh kh \ .$$

This transcendental equation, relating the two constants σ and k, is usually called the *dispersion relation.*

Because the shape of the free surface is described by $Re\left(\hat{\zeta}e^{ikx-i\sigma t}\right)$, successive wavecrests are spaced apart at a distance $2\pi/k$: this distance $2\pi/k$ is called the wavelength. The dispersion relation says that the crests all travel with speed $c = \sigma/k$; and we have

$$c^2 = gk^{-1}\tanh kh \ .$$

This last is also, sometimes, called 'the dispersion relation'. Notice that both σ and c vary as k varies — *unlike* sound waves, whose phase speed is independent of wavelength.

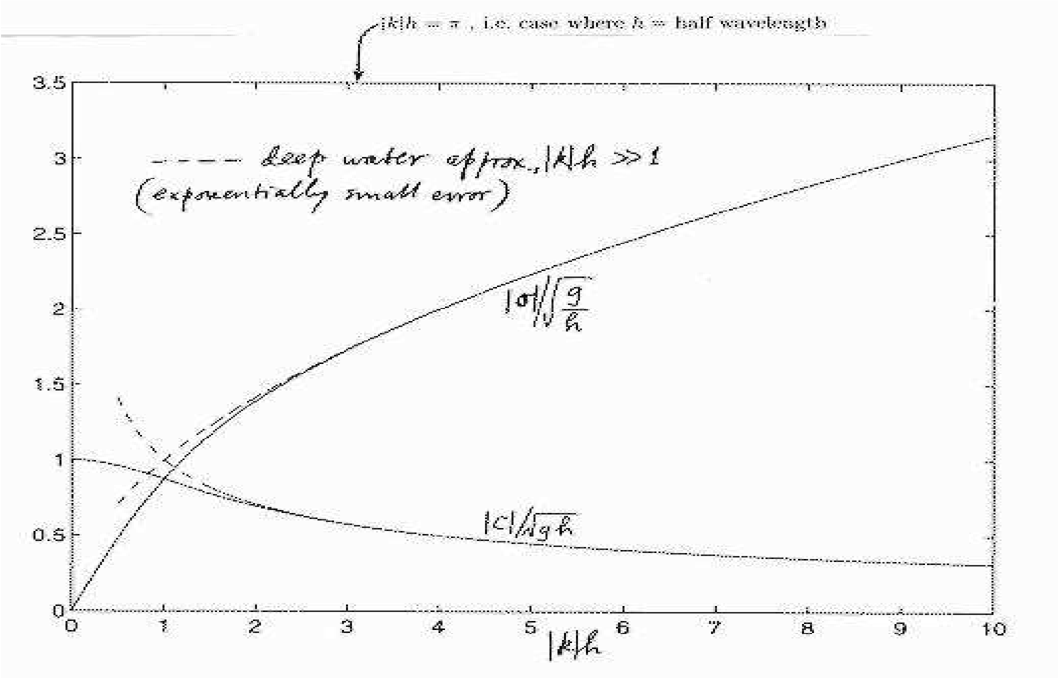
*Waves for which c varies as k varies are called 'dispersive'. The idea is that different Fourier components, i.e. of solutions with different values of k, will travel with different speeds and therefore disperse to different regions in space if we wait long enough. You can see this happen with the waves generated by dropping a stone into a pond. Of course this idea does not really make sense without further explanation, because a Fourier component — a single solution of the kind we are considering — already occupies the whole of an infinite space. To analyse such 'dispersion', we would need to consider Fourier *superpositions*, i.e. sums or integrals over k; this leads to the idea of 'group velocity', the speed c_g at which a finite group of wavecrests will travel. The value of c_g, which can be shown to be equal to $\partial\sigma/\partial k$, is usually unequal to c. To distinguish it from c_g, we usually call c the 'phase velocity'.*

Special cases

1) 'Deep water' case: $|kh| \gg 1$ (i.e., wavelength$/2\pi \ll h$) implies, with exponentially small error, that $|\tanh kh| = 1$ hence $\sigma^2 = g|k|$.

Note for instance that $\tanh\pi = 0.9963$; so, even if h is only half a wavelength, i.e. $h = \pi/|k|$, this 'deep-water approximation' $\sigma^2 = g|k|$ is correct to better than half a percent. The formulae $\sigma = \pm(g|k|)^{1/2}$ and $c = \sigma/k = \pm(g/|k|)^{1/2}$ are therefore correct to better than a quarter percent.

The following graphs show $|\sigma|$ and $|c|$ and the deep-water approximations to them (dashed):



Numerical magnitudes: an interesting example is ocean swell (small-amplitude, low-frequency waves generated by distant storms, the waves responsible causing big 'breakers' on surf beaches like Malibu). The deep-water and small-amplitude approximations are excellent approximations throughout most of the waves' journey across the open ocean, though not near the beach. The ocean is about 5 km deep away from continental margins; and typical periods $\sim 15\,\mathrm{s}$, $\Rightarrow$ $|\sigma| \sim 2\pi/15\,\mathrm{s} \simeq 0.4\,\mathrm{s}^{-1}$, $\Rightarrow$ $|k|^{-1} = g/\sigma^2 \simeq 63\,\mathrm{m}$, $\ll 5\,\mathrm{km}$. The wavelength $2\pi/|k|$ is nearly 400 m, and $|c| \simeq 25\,\mathrm{m\,s^{-1}}$.

($g = 10\,\mathrm{m\,s^{-2}}$ to 2%.)

*In a famous paper,[1]Snodgrass, F. E., Groves, G. W., Hasselmann, K. F., Miller, G. R., Munk, W. H., Powers, W. H., 1966: Propagation of ocean swell across the Pacific. Phil. Trans. Roy. Soc. Lond., **A 259**, 431–497. one of the classics of geophysics, Professor Walter Munk and co-workers showed that ocean swell generated by storms in the 'roaring fifties' of the Southern Ocean, between Tasmania and Antarctica, can propagate all the way across the Pacific to Alaska in close accordance with the above theory. This was done by setting up measuring stations in New Zealand, in Alaska, and on some intervening Pacific islands. It is far from obvious that the modelling assumptions of the theory — especially irrotationality and linearization — should not give significant cumulative errors when waves propagate over such large distances.*

2) 'Shallow water' or 'long wave' case: $|kh| \ll 1$ (i.e., wavelength$/2\pi \gg h$) $\Rightarrow$ $\tanh kh \simeq kh$ $\Rightarrow$ $\sigma^2 = gk^2h$ and $c^2 = gh$; so $c = \pm(gh)^{1/2}$.

Notice also that, as shown by the graphs above, $|c\,|$ increases monotonically as $kh \to 0$. So its asymptotic value $|c\,| = (gh)^{1/2}$ is maximal: the longest waves are fastest.

E.g. flood waves in river: say $h = 2\,\mathrm{m}$ $\Rightarrow$ $|c| = (gh)^{1/2} \simeq 4.5\,\mathrm{m\,s^{-1}}$ or 16 km per hour.

The magnitude $|\partial\sigma/\partial k|$ of the group velocity also tends to the same (maximal) value $(gh)^{1/2}$; long waves are nondispersive, like sound waves. So if an earthquake generates waves on one side of the Pacific, then it is the longest waves that are the first to arrive on the other side. These are the much-feared tsunamis or 'tidal waves'.

Tides, tsunamis and storm surges in shallow seas: say $h = 50\,\mathrm{m}$ $\Rightarrow$ $|c| = (gh)^{1/2} \simeq 22\,\mathrm{m\,s^{-1}}$ or 80 km per hour; e.g. for propagation of storm surges around North Sea coast, as in the 1953 event, the corresponding time delay is 12 hours from Aberdeen to the Dutch coast. *The waves have a tendency to be guided along coasts with the coast on their right; this guiding effect is *not* described by our simple theory, but depends on the Earth's rotation, as was shown by Lord Kelvin.*

Trans-Pacific propagation of tsunamis: The longest wavelengths travel the fastest; and the longest wavelengths, for this purpose, are certainly far greater than 2π times the typical ocean depth, 5 km. So this (maximal) wave speed is $|c| = (gh)^{1/2} \sim (10\,\mathrm{m\,s^{-2}}\times 5000\,\mathrm{m})^{1/2} = 224\,\mathrm{m\,s^{-1}}$. I.e., as stated earlier, these waves travel almost as fast as a subsonic passenger jet aircraft.

Velocity field

¿From the first of (∗) on p. 57, we have $A = -i\sigma\hat{\zeta}/\big(k\,\sinh\,kh\big)$, hence

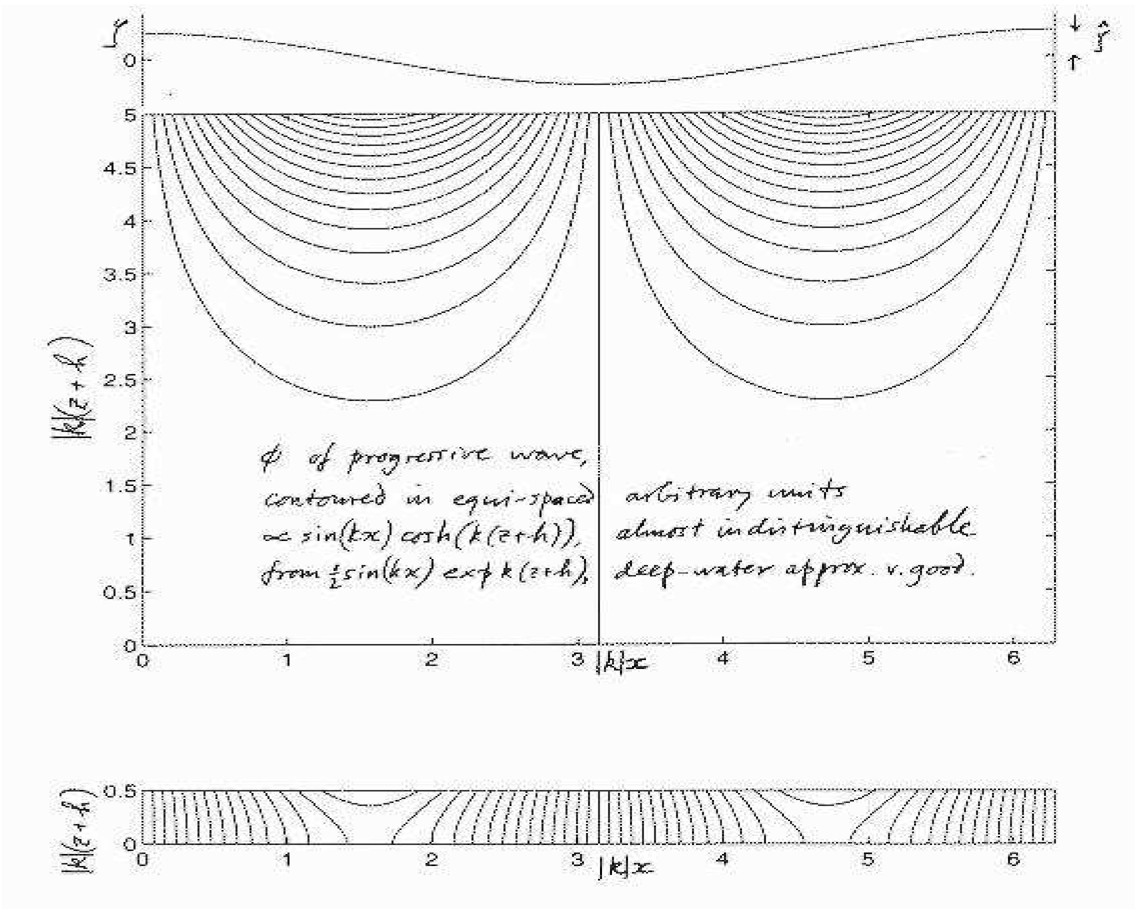
$$\phi(x,z,t) \;=\; Re\left\{-\frac{i\sigma\hat{\zeta}}{k\sinh kh}e^{i(kx-\sigma t)}\cosh k(z+h)\right\}$$

[1]†

for the velocity potential, hence

$$\mathbf{u} \;=\; \nabla\phi \;=\; Re\left\{\frac{\sigma\hat{\zeta}}{\sinh kh}e^{i(kx-\sigma t)}\Big(\cosh k(z+h), \;\; -i\sinh k(z+h)\Big)\right\}$$

for the velocity field. The following plots show ζ and ϕ in arbitrary units for the two cases $|k|h = 5$ and $|k|h = 0.5$; ϕ is contoured at equal contour intervals. They are snapshots at those instants t for which $\sigma t = 0,\, 2\pi,\, 4\pi, \ldots$. The number $\hat{\zeta}$ (generally a complex number) is taken to be real and positive here, giving a cosine function for the surface undulation $\zeta(x,\, t)$ at $\sigma t = 0,\, 2\pi,\, 4\pi, \ldots$. If the wave is propagating from left to right ($c > 0$, sgn σ = sgn k) then ϕ has positive values on the left and negative on the right, giving upward velocities $\partial\phi/\partial z$ on the left and downward on the right. Such velocities, a quarter wavelength out of phase with the surface displacement, are exactly those required to make the surface undulation $\zeta(x,\, t)$ propagate toward the right:



The second case $|k|h = 0.5$ has a velocity field $\mathbf{u} = \nabla\phi$ whose directions are close to the horizontal (contours of ϕ nearly vertical) in a substantial part of the domain. In this respect the second case is beginning to resemble the shallow-water limit, $|k|h \ll 1$, in which $\mathbf{u}$ is close to the horizontal almost everywhere, because $\cosh k(z+h) \gg |\sinh k(z+h)|$.

Particle paths

We solve for the paths of particles in the fluid below the free surface, using the above expression for $\mathbf{u}(x,t)$, which is

$$\mathbf{u} \;=\; \nabla\phi \;=\; Re\left\{\frac{\sigma\hat{\zeta}}{\sinh kh}e^{i(kx-\sigma t)}\Big(\cosh k(z+h),\;\; -\,i\sinh k(z+h)\Big)\right\}\;.$$

Let the position of the particle whose undisturbed position is $\mathbf{X}_0$ be $\mathbf{X}(t)$, say, with $d\mathbf{X}/dt = \mathbf{u}\big(\mathbf{X}(t),\, t\big)$; the components of $\mathbf{X}_0$ are (X_0, Z_0), those of $\mathbf{X}(t)$ are $\big(X(t),\, Z(t)\big)$, and those of $\mathbf{u}$ are $(u,\, w)$. Continue to assume small-amplitude, time-harmonic motion. Assume that the displacement $\mathbf{X} - \mathbf{X}_0$ is small, such that the position at which the velocity is evaluated may be linearized about the fixed point $\mathbf{X}_0$. More precisely, we assume that products of small quantities can be neglected in the Taylor expansion $\mathbf{u}\big(\mathbf{X}(t),\, t\big) = \mathbf{u}\big(\mathbf{X}_0,\, t\big) + (\mathbf{X}-\mathbf{X}_0).\nabla\mathbf{u}\big|_{\mathbf{x}=\mathbf{X}_0} + O\big(|\mathbf{X}-\mathbf{X}_0|^2\big) \simeq \mathbf{u}\big(\mathbf{X}_0,\, t\big)$. Thus we need only solve

$$\frac{dX}{dt} = u(X_0, Z_0, t)\;, \qquad\qquad \frac{dZ}{dt} = w(X_0, Z_0,\, t)\;.$$

The right-hand sides $\propto \exp(-i\sigma t)$, so solutions $=$ right-hand sides divided by $-i\sigma$, plus constants of integration. We can take the constants to be zero w.l.o.g.; this amounts to defining the undisturbed position $\mathbf{X}_0$ as the time-averaged position.

The undisturbed position can also be defined by considering the wave motion to have been generated from rest, for instance by applying a fluctuating pressure disturbance at the free surface. The undisturbed position can then be defined as the starting position according to linearized theory. Such problems can be solved e.g. using Laplace transforms, and the results show that (according to linearized theory) the starting position is the same as the time-averaged position.

Using the above expression for $\mathbf{u} = (u, w)$, dividing the quantity in braces by $-i\sigma$, we now have

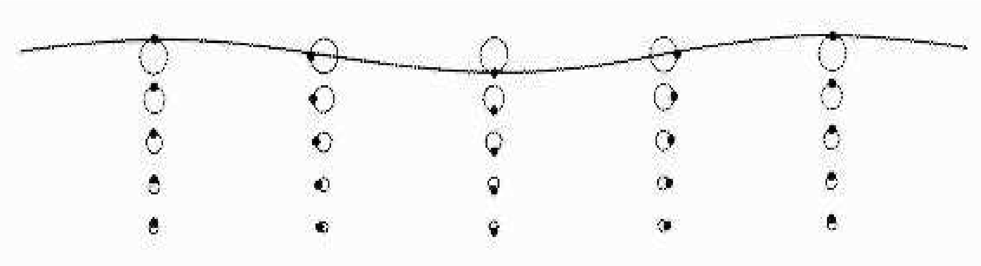
$$(X, Z) = (X_0, Z_0) + Re\left\{\frac{i\hat{\zeta}}{\sinh kh}e^{i(kX_0-\sigma t)}\Big(\cosh k(Z_0+h),\; -i\sinh k(Z_0+h)\Big)\right\}\;.$$

This says that the particle paths are ellipses with their major axes horizontal, their minor axes vertical, and their aspect ratios $|\tanh k(Z_0+h)|$.

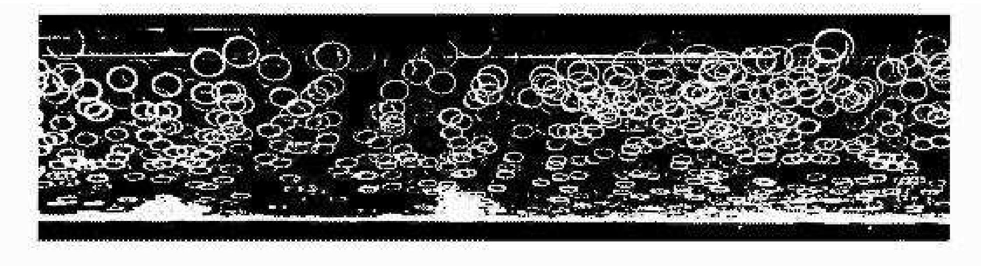
The vertical displacements are in phase with wave crests; and, putting $Z_0 = 0$ in the second or vertical component of the above expression, we have simply $Z - Z_0 = Z = Re\big\{\hat{\zeta}\, e^{i(kX_0-\sigma t)}\big\}$. This nicely checks the self-consistency of the whole theory, from §1.7 onward (derivation of the kinematic boundary condition for an impermeable moving boundary).

Particles move clockwise around the ellipses if $c > 0$ (rightward propagating waves), and anticlockwise if $c < 0$ (leftward propagating waves).

Here is a picture from Lighthill's book 'Waves in Fluids', showing instantaneous particle positions and particle paths in a rightward-propagating wave in deep water; the ellipses become circles:



Here is a laboratory time-exposure photograph, from Van Dyke's book, showing the particle paths in real waves in a case where the wavelength is about 4.7 times the depth h. So $\tanh|k|h = \tanh(2\pi/4.7) \simeq 0.87$, and the aspect ratios $|\tanh k(Z_0 + h)|$ of the ellipses vary from 0 at the bottom, $Z_0 = -h$, to about 0.87 at the top, $Z_0 = 0$:



If you iteratively correct the whole solution to the next order in small quantities, i.e. to the order of squares and quadratic products, then you find that (according to potential-flow theory, for motion generated from rest) the ellipses don't quite close up. This can be seen in the laboratory photograph if you look carefully. The time exposure is for slightly more than one wave period $2\pi/\sigma$. Particles have a small but systematic drift in the direction of propagation. It is called the 'Stokes drift', having been pointed out in a paper published in 1847 by Sir George Gabriel Stokes, two years before becoming Lucasian Professor at Cambridge.

Free-surface modes in a container

We now solve the surface wave problem in a different geometry, relevant to understanding the behaviour of harbours and lakes and to industrial problems involving fluid containment and associated hazards. It may be important to know the frequencies of oscillation of the modes of free surface oscillation within such a container. The problem is even more closely analogous to the manometer-oscillation problem, and can be viewed as another illustration of the general theory of small oscillations, in which the motion can be represented as a sum of *normal modes*, in any one of which all the particles of the system oscillate synchronously, with e.g. all particles coming to rest at the same instant in each half cycle.

Take the case of a rectangular container partly filled with fluid to depth h:

$$0 < x < a, \quad 0 < y < b, \quad -h < z < \zeta(x, y, t) \qquad \text{(height of free surface)}$$

The undisturbed configuration is $\zeta(x, y, t) = 0$ with the fluid at rest everywhere. The linearized equations and boundary conditions will be valid if ζ is sufficiently small; they are

$$\nabla^2\phi = 0 \quad (0 < x < a, \quad 0 < y < b, \quad -h < z < 0);$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = -h\ ; \qquad \frac{\partial\phi}{\partial x} = 0 \quad \text{on} \quad x = 0,\, a\ ; \qquad \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0,\, b\ ;$$

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0\ ;$$

$$\frac{\partial\phi}{\partial t} + g\zeta = f(t) \quad \text{on} \quad z = 0\ .$$

We seek solutions of *normal-mode form*

$$\phi = Re\big(\hat{\phi}(x, y, z)e^{-i\sigma t}\big)\ , \qquad \zeta = Re\big(\hat{\zeta}(x, y)e^{-i\sigma t}\big)\ .$$

These will be normal modes in the classical sense if $\hat{\phi} \propto$ a real-valued function, which we shall find to be the case. For a nontrivial solution we must once again have $f(t) = 0$ in the last boundary condition; and the boundary conditions at $z = 0$ become simply

$$i\sigma\hat{\zeta} + \left.\frac{\partial\hat{\phi}}{\partial z}\right|_{z=0} = 0 \qquad \text{and} \qquad -i\sigma\hat{\phi}\Big|_{z=0} + g\,\hat{\zeta} = 0\ .$$

Eliminating $\hat{\zeta}$ gives

$$\left.\frac{\partial\hat{\phi}}{\partial z}\right|_{z=0} = \left.\frac{\sigma^2}{g}\hat{\phi}\right|_{z=0} .(*)$$

We solve the problem by separation of variables. To satisfy the boundary conditions in x and y, try

$$\hat{\phi} = \hat{\phi}_{mn}(z)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad \text{where } m, n \text{ are integers.}$$

This solves $\nabla^2\hat{\phi} = 0$ provided that $\hat{\phi}_{mn}(z)$ satisfies

$$\frac{d^2\hat{\phi}_{mn}}{dz^2} - k_{mn}^2\,\hat{\phi}_{mn} = 0 \quad \text{where} \quad k_{mn} = \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^{1/2} > 0\ .$$

The boundary conditions in z will be satisfied if

$$\left.\frac{d\hat{\phi}_{mn}}{dz}\right|_{z=-h} = 0\ ,$$

$$\left.\frac{d\hat{\phi}_{mn}}{dz}\right|_{z=0} = \frac{\sigma^2}{g}\hat{\phi}_{mn}(0)\ ,$$

implying in turn that

$$\hat{\phi}_{mn}(z) \;\propto\; \cosh\{k_{mn}(z+h)\}$$

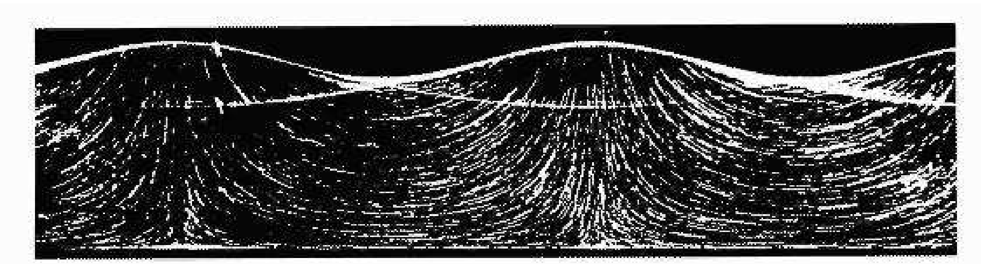
a nontrivial solution provided that one or both of the integers m and n are nonzero so that $k_{mn} \neq 0$, and provided also that $(*)$ is satisfied. This requires $\sigma = \sigma_{mn}$ where

$$\sigma_{mn}^2 = g\,k_{mn}\tanh(k_{mn}h)\ .$$

This mode of oscillation is a normal mode in the classical sense because $\cosh\{k_{mn}(z+h)\}$ is a real-valued function: all the particles oscillate synchronously, in the strict sense already mentioned.

It is a straightforward exercise to rewrite the normal-mode solution as a superposition of progressive waves of the kind we first analyzed, by replacing each cosine function by its complex exponential counterpart, $\cos(\mu) = \frac{1}{2}(e^{i\mu} + e^{-i\mu})$. For m, n both nonzero, these progressive propagate travel obliquely in pairs of opposite directions and have the structure $\exp(\pm ik_x x \pm ik_y y - i\sigma t)$, where the horizontal wavenumber vectors $\mathbf{k} = (\pm k_x,\ \pm k_y)$ all have the same magnitude $|\mathbf{k}| = k_{mn}$. That is why the dispersion relation has the same form as before, $\sigma^2 = g\,k_{mn}\tanh(k_{mn}h)$. Thus one can picture the normal mode in terms of progressive waves reflecting off the walls of the container — a bit like billiard balls bouncing around a billiard table in closed paths. Each pair of counterpropagating progressive waves adds up to a two-dimensional 'standing wave'.

For a simple illustration, take the case $h = \infty$, $m = 2$, $n = 0$, $\Rightarrow$ $k_{mn} = 2\pi/a = k$, say, a purely two-dimensional standing wave in which the value of b is irrelevant. Then $\hat{\phi} = \hat{\phi}(x,z) \propto \exp(kz)\cos(kx)$, equivalent to a single pair of progressive deep-water waves travelling in the $\pm x$ directions. The first contour plot on p. 60 gives, to excellent approximation, a side view of the *stream function*, $\propto$ $\exp(kz)\sin(kx)$, of this two-dimensional standing wave. In that plot we had $kh = 5$, hardly distinguishable from $kh = \infty$ for this purpose because $\exp(-5) < 10^{-2}$, making a barely visible difference. (More generally, though, the stream function $\propto \sinh k(z{+}h)$ rather than $\cosh k(z{+}h)$, the difference being noticeable when $\exp(-kh)$ is not negligible.) The particle paths in such a standing wave lie along the streamlines, whose pattern is fixed in space. So the particles oscillate back and forth, synchronously, along small segments of the streamlines. Here is another time-exposure photograph from Van Dyke's book illustrating this, i.e., visualizing the particle paths and hence the streamlines, for a similar but 'shallower' case:



As in the previous photograph, wavelength $\simeq 4.7h$, $\Rightarrow$ $\tanh kh \simeq 0.87$.

Which mode is the gravest, i.e. has the lowest frequency, in this system? If $a > b$, it is the mode for which $(m,\ n) = (1, 0)$; if $b > a$ it is $(m,\ n) = (0, 1)$. So the lowest frequency is $\left\{\dfrac{g\pi}{\max(a,b)}\right\}^{1/2}$.

The general solution, e.g. to a general initial value problem, can be written as a superposition of normal modes, with an infinite sequence of arbitrary constants c_{mn} and d_{mn} each of which can be any complex number:

$$\phi(x, y, z, t) = Re\left\{\sum_{m,n}\left(c_{mn}e^{i\sigma_{mn}t} + d_{mn}e^{-i\sigma_{mn}t}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cosh k_{mn}z\right\} .$$

Rayleigh–Taylor instability: (a simple extension of the above)

Suppose that the fluid is above the free surface rather than below. The above analysis still applies if we take g to be negative! Then

$$\sigma_{mn}^2 = -\,g\,k_{mn}\tanh(k_{mn}h) \ ,$$

and σ_{mn} is a pure imaginary number. Hence $e^{-i\sigma_{mn}t} = e^{\pm|\sigma_{mn}|t}$, corresponding to growing and decaying solutions rather than to oscillations.

Implication: the steady solution with water *above* air and the free surface horizontal is — not surprisingly — unstable, with small displacements growing exponentially. Growth rate increases as m and n increase, implying that smaller-scale structures grow faster.

*If you take a tank of water at rest in a terrestrial laboratory then accelerate it downward with acceleration a, held constant for some finite time — and if ambient air pressures are enough to prevent cavitation in the water — then the same analysis applies with $-g$ in the above relation replaced by $-|a| + g$. If $|a| > g$, the free surface is Rayleigh–Taylor unstable. This is the earliest stage of a sequence of events that, if the acceleration goes on for long enough, ends with most of the water leaving the tank!

In the model just analyzed, arbitrarily small-scale structures (k_{mn} arbitrarily large) grow arbitrarily fast. (This makes the problem for the shape of the free surface ill posed, somewhat like the problem for time-reversed heat diffusion: infinite series like that shown above are catastrophically divergent when σ is imaginary.) But if we refine the model to include either surface tension or viscosity or both, then this pathology is removed. There is then a maximum growth rate $|\sigma|$, occurring at the finite spatial scale k_{mn}^{-1}.

Rayleigh–Taylor instability is lucky for humanity. It makes plutonium fission bombs hard to build, even though plutonium is much easier to obtain than fissile uranium. Plutonium fission is so fast that spherically symmetric implosion of a hollow sphere is used to reach criticality on a short enough timescale. The plutonium becomes a fluid, and its outer surface, driven radially inward by gas at extreme pressure from the surrounding chemical explosives, behaves like the free surface in a tank accelerated downward with $|a| \gg g$. So the design has to maintain spherical symmetry to very high precision.*

§4.3 River flows and weirs

In the final two sections, we look at a class of free-surface potentia flows dominated by horizontal accelerations. These are the simplest models of natural and artificial river flow.

Throughout, we assume that the flow is two-dimensional, irrotational, steady, and approximately horizontal, $\mathbf{u} \simeq (U,\ 0,\ 0)$ say. Throughout §4.3 we also assume that conditions are slowly-varying in the direction of flow, x say. Slowly-varying means that horizontal lengthscales $\gg$ fluid depth $h + \zeta$ (same coordinates and notation as before, p. 56). The free surface can slope gently, producing a slowly-varying horizontal pressure gradient and a corresponding acceleration $D\mathbf{u}/Dt$ that is directed nearly horizontally. The lower boundary $z = -h(x)$ can also slope gently. We no longer make the small-disturbance approximation; thus nonlinear effects may arise.

To a first approximation, any such flow must be independent of depth: $U = U(x)$, a function of x alone. For if the flow depended strongly on depth, then there would be strong y-vorticity $\partial u/\partial z - \partial w/\partial x \simeq \partial u/\partial z \neq 0$, contradicting the assumption of irrotational flow.

Real river flows are not 2-D and do of course depend on depth, though in many cases less strongly than you might think. But more to the point, our simple models share important features with more realistic models, and are a first step, and an important step, toward a more detailed understanding of real rivers.

One such feature is the constraint imposed by Bernoulli's streamline theorem. Both in our simple models and in more realistic models, we can assume that pressure p on the free surface is constant, $p = p_{\text{atm}}$. Then Bernoulli's streamline theorem, applied along the free-surface streamline, says that $\frac{1}{2}U^2 + g\zeta =$ constant. So there are just two 'Bernoulli possibilities', wherever, for one reason or other, the flow properties are changing with x:

Possibility (1): $\zeta \downarrow$ and $U \uparrow$; Possibility (2): $\zeta \uparrow$ and $U \downarrow$.

That's all! Either the flow speeds up and the free surface dips, or the flow slows down and the free surface rises.

We are usually interested in cases where the flow is uniform upstream, with $\zeta = 0$, $U = U_\infty$, and $h = h_\infty$, say. Then Bernoulli gives

$$\tfrac{1}{2}U^2 + g\zeta = \tfrac{1}{2}{U_\infty}^2 ,$$

i.e.,

$$\zeta = \frac{{U_\infty}^2 - U^2}{2g}$$

The other major constraint is mass conservation: for our simple models this implies that $(h + \zeta)U$ is constant. For uniform upstream conditions,

$$(h + \zeta)U = h_\infty U_\infty .$$

We can think of our 2-D flow as being contained in a channel with vertical sides and constant width.

Three examples are investigated, of which the first is one we already know, a recent acquaintance in disguise:

§4.3.1 First example: Long waves brought to rest

Take a water wave in the shallow-water limit $|k|h \ll 1$, and bring it to rest by adopting the frame of reference travelling with the wave. The flow, thus viewed, has all the above properties, apart from being disturbed far upstream. We can equally well describe it as a small-amplitude shallow-water wave brought to rest by a uniform opposing stream with velocity $U_\infty = |c| = (gh)^{1/2}$. (Remember what the shallow-water limit is like: it does mean horizontal lengthscale $|k|^{-1} \gg h$ and disturbance velocities nearly horizontal.) In this case there is no bottom slope: $h = h_\infty =$ constant.

This example illustrates both 'Bernoulli possibilities'. Toward a wave crest, the surface rises and the flow slows down. Toward a wave trough, the surface dips and the flow speeds up. Indeed this, plus mass conservation, is what determines the wave speed.

We can verify the last statement (i.e. rederive the dispersion relation) by linearizing the Bernoulli and mass-conservation relations. Writing $U = U_\infty + u$, we have

$$\tfrac{1}{2}(U_\infty + u)^2 + g\,\zeta = \tfrac{1}{2}{U_\infty}^2 \qquad \text{and} \qquad (h_\infty + \zeta)(U_\infty + u) = h_\infty U_\infty\ .$$

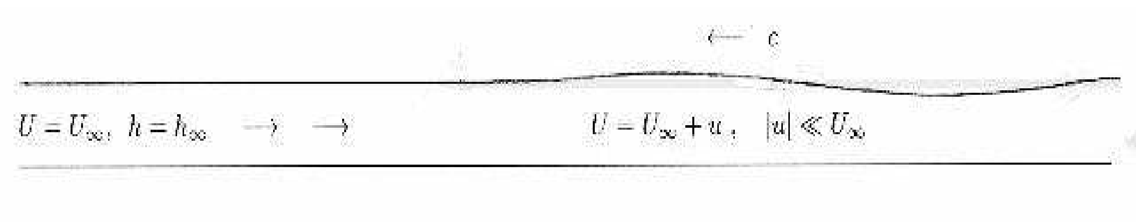
Linearizing (neglecting squares of u and its products with ζ), we have

$$\begin{aligned} U_\infty u + g\,\zeta \quad &= 0\ , \\ h_\infty u + U_\infty \zeta \quad &= 0\ , \end{aligned}$$

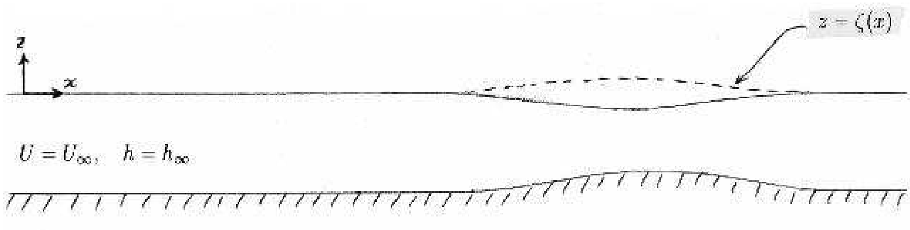
which has nontrivial solutions $\big(u(x),\, \zeta(x)\big)$ if and only if the determinant vanishes:

$$\begin{vmatrix} U_\infty & g \\ h_\infty & U_\infty \end{vmatrix} = 0\ , \qquad \Rightarrow \qquad U_\infty = (gh_\infty)^{1/2}\ .$$

Note that this is valid for arbitrary x-dependence, not necessarily sinusoidal! It is enough for the x dependence to be 'slow' in the above sense. *This is related to the nondispersive property of waves in the shallow-water limit: all Fourier components with $|k|h \ll 1$ propagate with the same speed relative to the undisturbed fluid.* In particular, we can include the case of no disturbance upstream:



§4.3.2 Second example: Free-surface flow over a gentle bump



Does the free surface go up or down? You might think that it would always go down, because the flow must speed up over the bump. This does occur, but only if U_∞ is small enough.

We shall see that the free surface goes down, as it encounters the bump, if $U_\infty < (gh_\infty)^{1/2}$. It goes up if $U_\infty > (gh_\infty)^{1/2}$.

As before,

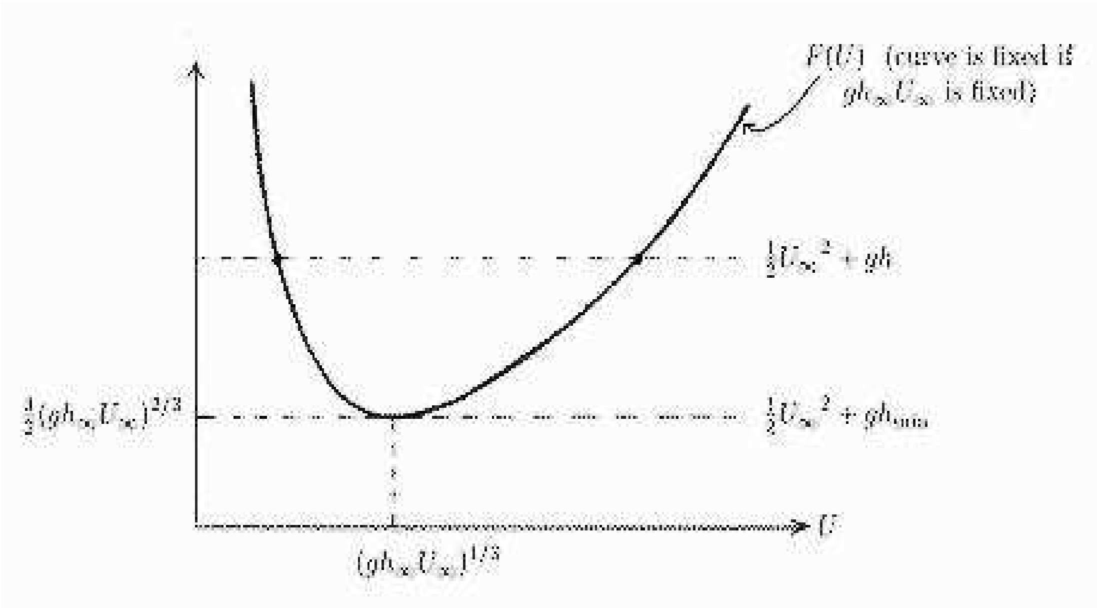
$$\begin{aligned} \tfrac{1}{2}U^2 + g\zeta &= \tfrac{1}{2}{U_\infty}^2 \qquad \text{(Bernoulli on free-surface streamline)}\ , \\ (h+\zeta)U &= h_\infty U_\infty \qquad \text{(mass conservation)}\ , \end{aligned}$$

two equations for two unknowns U and ζ. [We don't use the momentum integral here, because we are not given, nor are we presently interested in, the horizontal force on the bump —*though the force is actually zero in this case: d'Alembert's paradox applies.*]

Eliminate ζ:

$$\tfrac{1}{2}U^2 + \frac{gh_\infty U_\infty}{U} = \tfrac{1}{2}{U_\infty}^2 + gh$$

Use a graphical approach to see what this means in various cases. Denote by $F(U)$ the function of U appearing on left-hand side. Its graph is the curve sketched below. If we fix the value of $gh_\infty U_\infty$ (gravity times volume flux) then we have just one such curve. Solutions to the above equation, for given values of U_∞, gh_∞ and gh, are given by intersections of the graph of $F(U)$ with the horizontal straight line $\tfrac{1}{2}{U_\infty}^2 + gh$:



There is also a negative root (U times the above equation being a cubic): we ignore this as unphysical, representing negative U and negative thickness of the fluid layer.

Minimum of curve $F(U)$ occurs at $U = (gh_\infty U_\infty)^{1/3}$; U_∞ falls to the right or left of this according as $U_\infty \gtrless (gh_\infty)^{1/2}$.

These cases are called 'supercritical' and 'subcritical' respectively. The case $U_\infty = (gh_\infty)^{1/2}$ will be called 'critical'. In a subcritical flow, long waves can penetrate arbitrarily far upstream. In a supercritical flow, no waves can penetrate upstream: all waves, even the longest, fastest-propagating waves, are swept downstream.

There are two real positive roots if and only if $h > h_{\min}$ $\forall x$, where

$$gh_{\min} = \tfrac{3}{2}(gh_\infty U_\infty)^{2/3} - \tfrac{1}{2}U_\infty{}^2 \ .$$

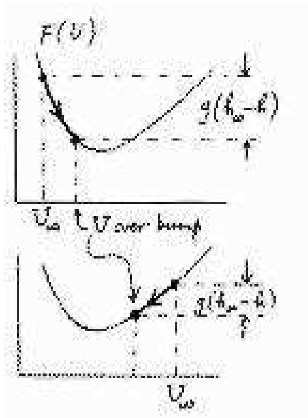
If $h < h_{\min}$ (bump too tall), a flow of the kind we have assumed is not possible!

(What happens in reality, when $h < h_{\min}$, is that the oncoming fluid will pile up on the upstream side of the bump, until h does exceed $h_{\min}$ and locally steady flow does become possible. In the critical case, we have $h_{\min} = h_\infty$, and the fluid will begin to pile up even for an arbitrarily thin bump! This can be viewed as a case of resonant forcing of long waves.)

When $h > h_{\min} > 0$, then, there are two positive roots:

Smallest root: The oncoming flow is subcritical, and U is a decreasing and ζ an increasing function of h. As the flow rides up the left-hand slope of the bump, $h\downarrow$, $U\uparrow$ and $\zeta\downarrow$ (free surface lowered); see arrow in sketch below (top):

Largest root: The oncoming flow is supercritical, and U is an increasing and ζ a decreasing function of h. As the flow rides up the left-hand slope of the bump, $h\downarrow$, $U\downarrow$ and $\zeta\uparrow$ (free surface raised); see arrow in sketch below (bottom):

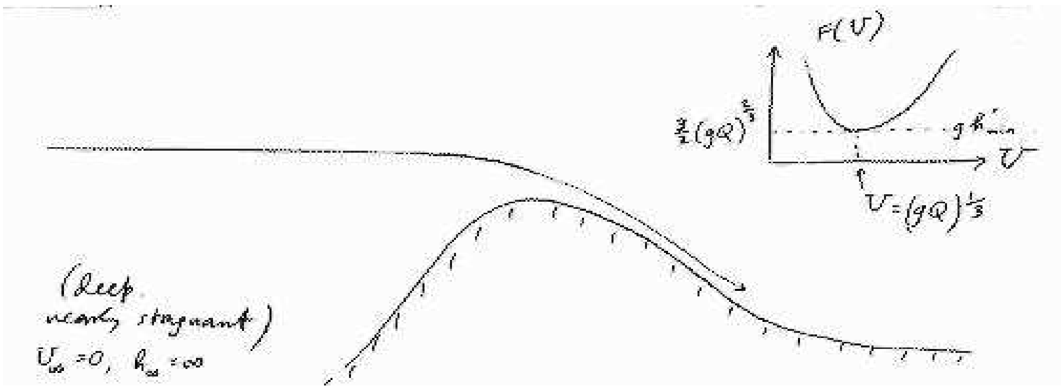


§4.3.3 Third example: Flow out of a reservoir over a broad weir

Reservoir: Special case of the above in which h_∞ is very large, U_∞ negligibly small. Assume *volume flux or flow rate* $Q = h_\infty U_\infty$ is finite.

Previous analysis implies

$$\tfrac{1}{2}U^2 + \frac{gQ}{U} \qquad [\ = F(U)\ ] \qquad = \quad gh.$$



In the reservoir we evidently start with the 'smallest-root' situation, the left-hand branch in the graphical solution above. (Infinitely slow flow is subcritical flow.) If we were to stay on this same branch, as we cross the weir, then surface would dip as the weir is crossed and then rise again. This is not what we see happening at real weirs on real rivers; and it won't happen unless someone were to force the downstream water level to rise by building a dam somewhere downstream.

So flow over a real weir — when it is acting as a weir, i.e. when downstream levels are low — must involve changing the branch. The flow conditions must cross over from the left-hand (subcritical) to the right-hand (supercritical) branch as x varies across the weir.

We can change branch smoothly only if minimum h (at the crest of the weir), h'_{min} say, corresponds to minimum $F(U) = \frac{1}{2}U^2 + gQU^{-1}$ $(Q = h_\infty U_\infty)$. Hence

$$h'_{\text{min}} \;=\; \frac{3}{2}\left(\frac{Q^2}{g}\right)^{1/3} ;$$

h'_{min} is a limiting case of the earlier h_{min}:

$$h'_{\text{min}} \;=\; g^{-1} \lim_{\substack{U_\infty \to 0 \\ Q \text{ fixed}}} \left(\tfrac{3}{2}(gh_\infty U_\infty)^{2/3} - \tfrac{1}{2}{U_\infty}^2\right) .$$

We know h'_{min} (the difference in elevation between the top of the weir and the level of the free surface in the reservoir), and can therefore deduce Q, as

$$Q \;=\; \left(\frac{8}{27}\, g\, {h'_{\text{min}}}^3\right)^{1/2} .$$

In other words Q can be controlled by choice of h'_{min}, i.e. by choosing the height of the weir. Q is therefore called the 'hydraulically controlled' flow rate.

At the crest of the weir:

$$\begin{array}{lrcl} \text{flow speed} & U & = & (\tfrac{2}{3}\, g\, h'_{\text{min}})^{1/2} , \\ \text{depth} & h+\zeta & = & Q/U = \tfrac{2}{3}h'_{\text{min}} , \\ \text{free surface height} & \zeta & = & -\tfrac{1}{3}h'_{\text{min}} . \end{array}$$

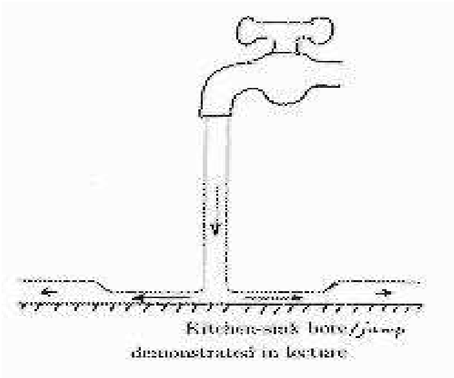
Speed of long waves at crest of weir $= U = \left(g\,(\tfrac{2}{3}h'_{\text{min}})\right)^{1/2}$, so flow speed = wave speed. I.e., the flow at the crest is critical.

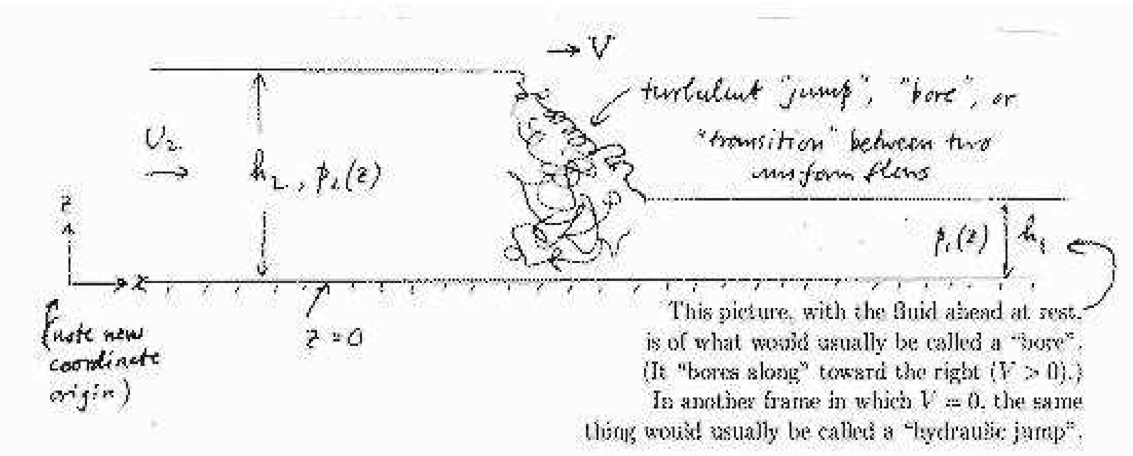
The flow downstream is supercritical: no waves generated at positions downstream can penetrate upstream. Weirs and waterfalls send no disturbances back into the reservoir: unwary canoe enthusiasts watch out!

§4.4 Bores and hydraulic jumps

[‘bore’ as in ‘drill’, or ‘penetrate’. E.g. the famous ‘Severn bore’.]

Bores and hydraulic jumps are essentially the same thing viewed in different frames of reference: an abrupt change in velocity U and surface elevation ζ, propagating at constant speed, V say, relative to the fluid ahead. Significant turbulent energy loss is involved in the transition region, as one might guess even from casual observation of these phenomena:





Turbulent energy loss in the transition region can be so strong, in fact, that Bernoulli cannot be used, even as a rough approximation.

Take the simplest case of a flat bottom (i.e. depth variations on lengthscale much larger than width of bore itself). There are no longer any horizontal forces, so we can now use the *momentum integral* to supplement mass conservation.

Apply mass conservation to *fixed* box containing the bore:

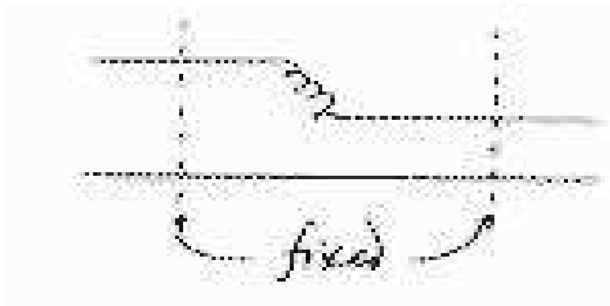
$$\text{mass flux out} \;=\; -\frac{d}{dt}(\text{ mass in box}) \;;$$

mass in (fixed) box is increasing. Therefore

$$-\rho h_2 U_2 \;=\; -V\rho(h_2 - h_1)\ ,$$

i.e.

$$h_2 U_2 \;=\; V(h_2 - h_1)\ .(1)$$



(Note that a segment of the box of length $V\delta t$ changes from being 'ahead' of the bore to being 'behind' the bore in time δt.)

Similarly apply momentum conservation to the box:

(force exerted by fluid in box on fluid outside)

$$+\ \text{(advective momentum flux out)} = -\frac{d}{dt}\ \text{(momentum in box)}:$$

$$\int_0^{h_1} p_1(z)dz + \int_{h_1}^{h_2} p_{\rm atm}\,dz - \int_0^{h_2} p_2(z)dz - \rho h_2 U_2^2 \;=\; -\rho V h_2 U_2\ .$$

The pressure distributions $p_1(z)$ and $p_2(z)$ are hydrostatic and given by

$$p_1(z) = p_{\rm atm} + \rho g(h_1 - z) \qquad (0 < z < h_1)$$

$$p_2(z) = p_{\rm atm} + \rho g(h_2 - z) \qquad (0 < z < h_2)$$

Hence the momentum balance equation simplifies to

$$\tfrac{1}{2}gh_1^2 - \tfrac{1}{2}gh_2^2 - h_2U_2^2 = -Vh_2U_2\ .(2)$$

We now have two equations, with unknowns V and U_2 (if regard h_2 as given).

Eliminate U_2:

$$\tfrac{1}{2}g(h_1^2 - h_2^2) \;=\; -\,h_2 \frac{V(h_2 - h_1)}{h_2}\,\frac{Vh_1}{h_2}\ .$$

Hence

$$\text{speed of bore} \qquad V \;=\; \left(g\frac{(h_1+h_2)h_2}{2h_1}\right)^{1/2}\ ,$$

$$\text{flow speed behind bore} \qquad U_2 \;=\; \frac{h_2 - h_1}{h_2}\,V\ .$$

Note that $V \to \bigl(gh_1\bigr)^{1/2}$ if $h_2 \to h_1$, again checking consistency with small-amplitude wave theory, whereas, at finite amplitude, $h_2 > h_1$, we have

$$V \;>\; \bigl(gh_1\bigr)^{1/2}\ .$$

So the bore travels faster than, and will catch up with, any gravity waves ahead of it. But $V - U_2 = \bigl(h_1/h_2\bigr)V < \bigl(gh_2\bigr)^{1/2}$; i.e.

$$V \;<\; U_2 + \bigl(gh_2\bigr)^{1/2}\ ,$$

which says that waves upstream can catch up with the bore.

Exercise (last question on Ex. Sheet 3): Show that the rate of energy dissipation in the bore is positive if $h_2 > h_1$; this must be so if the model is to be physically reasonable.

~~~~~~~~~~~~~~~~~~~~~~~~ End of notes ~~~~~~~~~~~~~~~~~~~~~~~~
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