EQUILIBRIUM SPECTRUM FOR INTERNAL WAVE ATTRACTOR IN A TRAPEZOIDAL BASIN

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Abstract. Reflecting internal gravity waves in a stratified fluid preserve their frequency and thus their angle with the gravitational direction. This leads to a focusing or defocusing of the waves at boundaries that are neither horizontal nor vertical. Previous theoretical and experimental work has demonstrated how this can result in the internal wave energy being focused onto ‘wave attractors’ in relatively simple geometries. We present new experimental and theoretical results on the dynamics of wave attractors in a semi two-dimensional trapezoidal basin. In particular, we demonstrate how a basin-scale mode forced by simple mechanical excitation develops an equilibrium spectrum. We find a balance between focusing of the basin scale waves by reflection from a single sloping boundary and viscous dissipation at the higher wave numbers. Theoretical predictions using a ray-tracing technique are found to agree well with direct experimental observations of the waves.

1. Introduction

The focusing of internal gravity waves through their reflection from sloping boundaries has far-reaching consequences for mixing in density stratified fluids. While the frequency of such reflections is preserved, simple geometric arguments dictate that the wave number cannot be preserved, except in the special cases of horizontal or vertical boundaries. This is a direct consequence of the dispersion relation,

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\[
\frac{\omega}{N} = \cos \theta ,
\]

where \( \omega \) is the wave frequency, \( N = \left( -g/\rho_0 \frac{d\rho_0}{dz} \right)^{1/2} \) is the buoyancy frequency, and \( \theta \) is the angle between the wave crests and the vertical (\( z \) direction). Here, \( g \) is gravity and \( \rho_0 = \rho_0(z) \) is the unperturbed density stratification. The wave vector is then given by \( \mathbf{k} = k(\cos \theta \sin \theta) \).

As shown by Maas & Lam, the focusing of waves dominates and leads to the formation of a ‘wave attractor’ in most closed domains; Maas et al. verified this experimentally with waves excited through the parametric sub-harmonic instability of a vertically oscillated trapezoidal basin containing a linearly stratified fluid. These experiments have been reanalyzed recently by Lam & Maas, with the suggestion that the surface seiche may play an important role.

Hazewinkel et al. extended the earlier experimental work on wave attractors using more sensitive and accurate experimental diagnostics to show the existence of a harmonically generated wave attractor and determined the equilibrium form of its spectrum. Here, we extend this work further, considering both horizontal and vertical oscillations, and examining the velocity spectra.

2. Experimental setup

A rectangular tank divided into two trapezoidal basins (see figure 1), filled to a depth \( H \) with a linear salt water density stratification, was placed on one of two oscillating tables. In the first series of experiments, the large (L) basin was used with \( H = 200 \text{mm} \) and the table was oscillated vertically with a typical amplitude of 120mm. In the second series, the small (S) basin was used with \( H = 230 \text{mm} \) table was oscillated horizontally with a typical amplitude of 2mm. The small basin (S) was also used with vertical oscillations when measuring the velocity field. In all cases a relatively strong initial stratification (typically \( N \sim 3\text{rad/s} \) for basin L and \( N \sim 2.1\text{rad/s} \) for basin S) was used to allow accurate measurement of
the internal wave field by synthetic schlieren for extremely small amplitude internal waves. We studied the initial establishment of the internal wave field after initiating the table oscillation, its steady state characteristics, and the decay of the field after the forcing was ceased.

For the synthetic schlieren measurements, a 1.3MPixel digital video camera was used to capture images of a back-lit pattern of random dots. The camera was located approximately 8m from the tank with its optical path shielded from unwanted thermal fluctuations in the laboratory by a polythene ‘tent’.

Complementary particle image velocimetry (PIV) measurements were obtained during the decay of the attractor. Very fine reflective flakes (Iriodin Glitter Bronze 530) were sprinkled on the surface and settled very slowly through the tank. The smallest particles (smaller than 10µm) proved to be successful tracers, despite the visibility of a given particle depending on its orientation.

The image capture and analysis for both synthetic schlieren and PIV was performed using DigiFlow. Images were captured at four precise points during each period of oscillation of the table. For synthetic schlieren, a separate reference image was used for each of these points to avoid any spurious signals due to optical imperfections in the experimental setup. As the tank was not moving during the decay of the attractor, images were captured at 24 frames per second.

![Figure 2: Snapshot of wave attractor showing the perturbation to the vertical density gradient. (a) Vertically oscillating table (basin L). (b) Horizontally oscillating table (basin S).](image)

3. Results

The internal wave field achieves a steady state approximately 20 periods after the oscillation of the table is started. As can be seen from figure 2a for a vertically oscillating table, the perturbations to the density field lie dominantly on the diamond-shaped attractor first predicted by Maas & Lam. Internal waves propagate around this attractor in a clockwise sense, undergoing amplification during their reflection from the sloping boundary. Here we visualize the vertical gradient of the density perturbation (i.e. $\partial \rho / \partial z$), normalized by the vertical gradient of the background stratification ($d\rho_0/dz$). The waves are clearly linear with the perturbation a factor of $10^5$ smaller than the background stratification.
A time series of the density perturbation reveals the wave frequency is equal to the oscillation frequency. For the present vertically oscillating table the amplitude of the oscillation is insufficient to excite a parametric sub-harmonic forcing; we see instead the result of small imperfections in the vertical oscillation leading to the establishment of a harmonic seiche. The horizontally oscillating table also forces a harmonic seiche. In both cases the coupling of this seiche with the stratification drives the internal waves at the basin scale. Comparison of figures 2a, with the vertically oscillating table, and 2b with the horizontally oscillating table with similar experimental parameters confirm this. The surface modes are visible in the PIV measurements of the vertically oscillated table, as are the smaller length scales of the wave attractor. The velocities due to the attractor are easily recognized after harmonic analysis, as shown in figure 3.

The initial evolution of the wave field is seen in figure 4a, which shows a series of profiles (at a fixed phase) of the horizontal density gradient at the dashed line indicated in figure 2a. The higher wave number component increases along with the overall amplitude as the wave attractor builds up, indicating the transfer of energy from a basin-scale seiche to shorter wave lengths through the action of the attractor. Figure 4b shows how this then decays once the forcing is ceased, with the lower wave numbers disappearing first as the attractor decays.
4. Discussion and conclusions

Hazewinkel et al.\textsuperscript{4} used Fourier techniques to determine the form of the equilibrium spectrum for the density gradients. They showed that energy injected at the basin scale (wave number $k_0$), propagates repeatedly clockwise around the attractor. Each focusing reflection from the sloping boundary transfers energy from wave number $k_n$ to wave number $k_{n+1} = \gamma k_0 = k_0 \gamma^n$, where $\gamma = \sin(\theta + \alpha\sin(\theta - \alpha)$ (here $\gamma = 1.6$ for basin L and 4.2 for basin S). In this paper we extend their analysis to consider the velocity spectrum.

A given packet of energy is introduced by the forcing at wavenumber $k_0 = 2\pi H$, and through repeated reflections is taken to smaller scales. The energy flux associated with the packet is conserved during the reflection, and hence the energy density increases by a factor of $\gamma$ at each reflection, a consequence of the reduction in the group velocity $c_g$. The packet of energy spends a period $t_{\text{loop}} = L_a/c_g$ at each wave number, where $L_a$ is the distance around the attractor (which scales with $H$). However, in a real fluid, viscous dissipation will decrease the energy with both the dissipation rate and $t_{\text{loop}}$ increasing with wave number. Incorporating the dissipation into the recurrence relation gives us the wave attractor’s equilibrium energy spectrum:

$$U_n^2 = U_{n+1}^2 \exp \left[ -\frac{2\nu L_a k_n^3}{N \sin \theta} \right] = U_{n}^2 \gamma^n \exp \left[ -\frac{2 \nu L_a (2\pi)^3 H^2 N \sin \theta}{H^2 N \sin \theta} \gamma^n - 1 \right]. \tag{2}$$

The form of this energy spectrum differs by a factor of $\gamma^n$ from the power spectrum of the density gradient reported by Hazewinkel et al.\textsuperscript{4}.

Figure 5a compares the observed equilibrium velocity spectrum in the vertically oscillated experiments with the recurrence relation (2), demonstrating close agreement in the functional form of the spectrum. (Similar agreement was found with the model for the density gradient spectrum of the attractor in basin L as shown in figure 5b.)

Whilst this model does not tie down the coupling between the forcing and the lowest wave number internal wave, it demonstrates clearly that the linear processes of wave amplification and viscous decay are sufficient to uniquely determine the form of the spectrum. Although in our experiments the wave amplitude and density gradients remain small, and the internal waves are linear, this may not always be the case. For sufficiently large basins, a given package of energy will undergo $p \sim \ln(H^2 N \nu)$ reflections before viscous effects start to reduce the density gradients due to the waves. Thus we might expect to see that the density gradients found in the basin scale mode will be amplified by a factor
approaching $\gamma^{3/2} \sim (H^2 N/\nu)^{3/2} \ln \gamma$. However, molecular viscosity and diffusivity are unlikely to remain the dominant dissipative mechanisms. In these large domains the wave will be amplified to a strongly nonlinear regime and hence exhibit different dynamics, and scattering from rough boundaries will modify the spectra. Lack of sufficient spatial resolution of the internal wave field in natural waters is probably the prime aspect inhibiting the observation of internal wave attractors in the field, although signs of internal wave focusing have been found on several occasions in a canyon$^8$ and in small-scale lakes$^{3,1}$.

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References


Figure 5: Comparison of modeled and observed equilibrium power spectra for (a) kinetic energy ($U_n^2$) and (b) density gradient ($A_n^2$).