Internal wave attractors without sloping boundaries

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Abstract

Internal gravity waves travel at an angle to gravity that is controlled by the strength of the stratification as well as the frequency of the forcing. While both wavelength and frequency are preserved upon reflection from a horizontal or vertical surface, the requirement for the frequency of the incident and reflected waves to be the same means that upon reflection from a sloping boundary, an increase or decrease in wavelength occurs. In previous work in non-rectangular geometries, wave energy has been observed to be focused on to internal wave attractors. This paper compares the wave attractors observed in a two-dimensional trapezoidal tank with those observed when the sloping wall is replaced with a staircase configuration comprising only horizontal and vertical surfaces. The macroscopic shapes of the domains are seen to be similar but the steps, with a length scale comparable to the wave length of the waves, have no sloping boundaries and therefore there is no mechanism for wave focussing. This is explored experimentally and by ray-tracing.

1 Introduction

Internal waves are of interest in the oceans because they are a mechanism for diapycnal mixing. The meridional overturning circulation is a globally observed process where water descends as it cools near to the poles. This needs to be counteracted by an equal upward flux elsewhere or else the potential energy of the ocean would not be conserved. Wunsch and Ferrari (2004) suggested internal waves as a mechanism for this vertical transport of mass. Mixing of stratified layers can raise the centre of mass and balance the potential energy depleted near the poles when the water cools and sinks. Internal waves generated near to ocean boundaries by tidal forcing can propagate through the ocean and cause this required mixing by wavebreaking and other small scale processes. The large spatial and temporal variability over the ocean basin is not well understood so it is an important area of study.

Internal waves are generated in a stratified environment when a disturbance is made that causes isopycnal surfaces to be perturbed away from their equilibrium positions. The resulting restorative buoyancy force causes oscillations whose amplitudes are damped by viscous forces. This restoring force causes a wave to propagate away from the point of disturbance at an angle \( \theta \), measured from the vertical direction, that is a function of the frequency of the disturbance and the strength of the stratification only. The group velocity lies at an angle \( \theta \) to the vertical and is parallel to the wave crests. These waves have been studied previously by Mowbray and Rarity (1967) who found that in a two-dimensional setting there are four possible directions that the waves can propagate. These are shown by the St Andrews cross (Figure 1a) and described by the dispersion relation

\[
\omega = N \cos \theta. \tag{1}
\]

In this paper, the background density of the fluid is denoted \( \rho_0(z) \), which varies linearly with gravity, \( g \), in the vertical \( z \) direction, and the average density of the incompressible
Figure 1: (a)(Left) The St Andrews Cross showing the four possible directions of propagation of internal waves. The direction of the phase of the waves is $c_p$ and $c_g$ is the direction of energy propagation. $c_p$ is perpendicular to $c_g$ at all times. The angle $\theta$ is the angle of the wave relative to the vertical direction and $\alpha$ is the angle of the sloping wall relative to the vertical. (b)(Right) Focussing of an internal wave attractor. This is the predicted attractor from ray tracing using the measured experimental values of density, $\omega$ and height.

fluid is $\rho_*$. The buoyancy frequency can be defined as

$$N = \sqrt{-\frac{g}{\rho_*} \frac{\partial \rho_0}{\partial z}}.$$  
(2)

For a stable linear stratification, $N$ is constant so the slope of the internal waves described by the dispersion relation must also be constant over the height of the fluid.

In the ocean, when an internal wave interacts with non-smooth surfaces and sharp corners this can cause wave breaking and mixing. Longuet-Higgins (1969) and Baines (1971) studied the reflection of internal waves from bumpy surfaces, and Nye (2009) looked at the scattering of waves upon reflection from irregular topography. In a closed domain internal waves are trapped and multiple boundary reflections must occur. The shape of the surrounding surface is extremely important in determining the amount of mixing and the longterm nature of the internal wave field.

For a two-dimensional, non-rotating, incompressible fluid, it is well known that by linearising the governing set of equations appropriately and by introducing a streamfunction $\psi$, we can derive

$$\chi^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = \frac{-i\nu}{\omega} \nabla^4 \psi,$$  
(3)

where

$$u = -\frac{\partial \psi}{\partial z} e^{-i\omega t}, \quad v = \frac{\partial \psi}{\partial x} e^{-i\omega t}$$  
(4)

and

$$\chi^2 = \tan^2 \theta = (N^2 - \omega^2)/\omega^2.$$  
(5)

The direction of the wave us controlled by $\chi^2$ and the right-hand-side of (3) is the viscous contribution. Internal waves differ from most other waves because they are governed by a spatial wave equation which is a hyperbolic instead of an elliptic equation. The consequence of this is that in general the angle of incidence does not equal the angle of
reflection. Furthermore, as the angle $\theta$ is dependant only on the stratification and the frequency of the forcing, it must be the same in the reflected wave as in the incoming wave. It is this fact that causes internal wave attractors to form. Every time a wave reflects from a boundary that is neither horizontal nor vertical, the wavenumber is focused or defocused by a factor

$$\gamma = \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)}.$$  \hspace{1cm} (6)

In a closed domain, multiple reflections occur such that focusing dominates over defocusing and after $n$ focusing reflections the internal wave is focused by $\gamma^n$ onto a limit cycle called an internal wave attractor, (Maas and Lam (1995)).

A ray tracing technique can be employed that tracks the movement of an infinitesimally thin slice of a wave as it reflects around the domain obeying the dispersion relation. These rays converge over time to a limit cycle which shows the location of an internal wave attractor. The possible formation of internal wave attractors are important in predicting the location of internal wave energy and it is here that wave breaking can occur due to the observed decrease in wavenumber and increase in amplitude of the waves. In figure 1b, in two dimensions, we see how an internal wave in a trapezoidal domain can be focused. The ray tracing technique used shows how an internal wave can reflect from the boundaries such that over time it will always approach an attractor. This has been observed experimentally by Maas et al. (1997), Maas (2005) and Hazewinkel et al. (2008).

Figure 2a shows a typical focusing reflection. The wavelength $\lambda_{in}$ is focused to $\lambda_{out} = \lambda_{in}/\gamma$. If instead the reflection occurs from a stepped boundary, we see a forward reflected part (A) and also backwards reflected rays (B) which scatters energy in a different direction. The flux of incoming energy must equal that of the outgoing energy but as there are back reflections then the energy density need not be increased in the forward reflected wave like it was from the sloping wall. The proportion of energy reflected forwards depends on $\gamma$ and the wavelength of the wave in relation to the topographic wavelength. The amount of total energy is constant in the reflections but due to the decreased width of the outgoing wave beams in the sloped case we see an increase in energy density. For an

Figure 2: (a)(Left) Wave focussing from a sloping boundary. $\lambda_{out} = \lambda_{in}/\gamma$ (b)(Right) When the wave reflects from a stepped boundary it is split into a forward reflected part and a beam-like backward reflected part. White arrows show incoming energy and black arrows show the direction of the outgoing energy.
incoming wave with width $W$ we can define

$$A = \frac{W}{\gamma}$$

and

$$B = \frac{(\gamma - 1)W}{\gamma}$$

as the size of the forward and backward reflected rays respectively, see figure 2.

2 Sloping boundaries in the lab

Internal waves attractors were created in the lab by horizontal oscillations of a tank of stratified fluid on an oscillating table. This configuration was initially used by Hazewinkel et al. (2007) and has shown good results, the experimental method used here was similar. The saline stratification was held in a trapezoidal tank that can be considered to be two-dimensional. When the table started oscillating, internal waves were seen to focus on to attractors and were measured by use of the synthetic schlieren method (Dalziel et al. (2000)). This non-intrusive method can accurately measure the density perturbations across the tank by using the refractive properties of the fluid. In this way the internal wave attractor can be measured and then analysed in a similar way to the internal wave beams generated in their paper. The camera is positioned to look through the tank from a distance such that motion is observed in the horizontal and vertical direction. A dot pattern is attached to a lightbox behind the tank and the camera tracks the observed displacements of these dots as the fluid in the tank moves. The closer to the tank the camera is the less accurate the two-dimensional approximation is. However synthetic schlieren also assumes that there is a homogeneous medium between the tank and the camera, so the further away the camera is the more interference from thermal fluctuations in the air in the laboratory there is. This can be picked up by the camera and cause unwanted noise in the results. A careful balance between these two issues was achieved. An optical tunnel was constructed that shields the camera and its field of view towards the tank. This acts to produce more uniform experimental conditions.

Figure 3a shows the distances from the camera to the tank and from the tank to the lightbox behind. The dimensions of the oscillating tank are detailed in Figure 3b.

![Figure 3](image_url)

Figure 3: (a)(Left) The light box, tank, optical tunnel and camera. (b)(Right) The dimensions of the tank when filled 155mm high with water.
2.1 Results

The horizontal motion of the tank has amplitude of around 5 mm and the frequency is 1 rad s\(^{-1}\); the resulting deformation to the free surface is nearly imperceptible. A buoyancy value of \( N = 1.64 \text{ rad s}^{-1} \) is measured, for a fluid depth of 130 mm. This depth is less than the height of the fluid given in figure 3b because a mixed surface layer at the top and a homogeneous salt layer at the bottom of the tank that need to be accounted for. These are due partially to inaccuracies when filling up the tank and partially to the molecular diffusion of the stratification. It is worth noting that the attractor forms despite this. The dense base layer acts as a boundary for the internal wave to reflect from in a similar way to the base of the tank. The observed heights of these layers and the attractors viewed in the results section match up well. Figure 4a in particular shows how the attractor does not reach to the bottom of the tank before its reflection. For these measurements we can use ray tracing to predict the location of an attractor. This is shown in figure 1b and fits the results well. The diamond shaped attractor is the most simple attractor to reproduce experimentally although ray tracing does indicate that many more complicated attractors can also exist. Viscous action of the wave beams will of course influence an attractor if it is too complicated, so this simple attractor is convenient to study in the lab. After \( n \) reflections the amount of focusing can be quantified by \( \gamma^{-n} \). As this tends to zero the attractor must form as all the wave energy have been focused towards the attractor. Our results show \( \gamma = 2.28 \) and so there will be fast convergence.

Synthetic schlieren computes the gradients between the experimental image and a background image of the stationary stratification. This can then be converted in to the perturbations seen in figure 4a. These perturbations were captured during oscillation when the tank was aligned with its stationary position after oscillations are turned off. It shows an internal wave attractor that is fully formed, many of periods after start up. We can see large values at the corners of the attractor where the beams join together at the boundaries. This can be seen more clearly in figure 4b which is the amplitude of the internal wave attractor. The amplitude is greatest as the wave reflects from then sloping wall and it then decreases in an anti-clockwise direction around the attractor due to viscosity acting on the wave beams. The amplitude was calculated by applying harmonic analysis to the first five periods of oscillation of the attractor after the horizontal motion was gradually turned off (over one period). The method used computes the amplitude and phase of the four wave-beams, the phase is shown in figure 5a. The phase diagram agrees with the theory in figure 1b and therefore the internal waves energy propagates in an anti-clockwise direction around the tank.

The wavenumber spectrum has also been calculated for each wave beam using Fourier analysis. This is shown in Figure 5b. We see a decrease in power as the waves travel away from the sloping wall. The dashed line is calculated from a cross-section of the wave reflecting off of the sloping wall. The crosses, lines and circles show the spectra’s of the second, third and fourth wave beams in the anti-clockwise direction. This observed decrease in wave power is due to viscous forces acting on the waves as they propagate away after the focusing reflection from the wall.
3 Discussion and Conclusions

Near to sloping boundaries it is observed that a large amount of focusing takes place and it is here where the largest observed wave amplitudes exist. We expect the steps to partially forward reflect waves on the same trajectory as a sloped wall, and to partially back reflect energy in beams from each step as in figure 2b. The proportions of each are controlled by $\gamma$ as well as the ratio of incoming wavenumber to topographic wavenumber. There is no continuous/microscopic focusing, but there is still a discrete/macroscopic focusing.

The traditional focusing of internal gravity waves is characterised as the transfer of energy from a lower to a higher wavenumber through the process of reflection from a sloping boundary. This description relies on the boundary being continuous and locally flat over lengthscales that are large compared with the wavelength of the incoming wave. The length scales of the steps in relation to the wavelength of the incoming wave will be an important factor in determining the reflection properties of the wave. Ray tracing doesn’t take this in to account.
In the absence of viscosity, the attractor that forms does not have its energy deposited onto a locus of infinitesimal width, but rather the size of the individual steps acts as a cutoff. Looking back to figure 2b we can see that not all the energy is reflected forwards from the steps. If we ignore the ultimate fate of any energy reflected back along the path of incoming rays, then only a fraction $\phi$ of the incident energy is transmitted into the next loop of the attractor. The mean slope of the staircase is known to be focused by $\gamma$ and can be used as a comparison to the steps. Provided $\phi \gamma > 1$, then we can predict that an attractor could form. From figure 2b we see that the proportion of the incoming wave that is forward reflected is $1/\gamma$ and therefore $\phi \gamma = 1$ seems a likely outcome. However this does not necessarily mean that no accumulation of energy will occur. The back reflected energy can still play a role as can the scattering from the sharp corners.

The fate of the back-reflected energy is to propagate away from the steps initially and then to re-approach them. This second reflection will obey the same reflection properties as the initial incoming wave. Some is expected to be forward reflected and will add to the energy that is geometrically accumulated by the steps. In this way we can have an energy build up that is distinct from the traditional focusing.

Scattering from the corners of the staircase will modify the efficiency of the attractor and is most important when the wavelengths are comparable. As the steps decrease in size towards zero we expect to see less back reflections and scattering from the corners as the wave will see the boundary as being more like a continuous sloping wall. It will be considered locally flat in comparison to the wavelength of the incoming wave. This suggests that the amount of forward reflection is not only dependent on the shape of the steps, at small scales viscosity is dominant.

Preliminary experiments have demonstrated that for steps with height and width of 10mm, an attractor is still able to form. Using the same setup as previously mentioned we installed a set of steps on the sloping wall of the tank. This trial run showed the formation of an attractor was possible as well as indicating that the steps can cause large amounts of scattering and back-reflection. Due to the sloping wall of the tank which is at angle $\alpha = 0.47$ rad, the faces of the steps were offset from the horizontal and vertical directions by an angle $0.31$ rad $\approx 18$ degrees. The small slope that this caused is negligible and these results suggest that an attractor may be able to form from a stepped boundary.
Figure 6 shows the amplitude computed from harmonic analysis shortly after the tank stopped oscillating. Comparing this to figure 4b we see some major differences. The largest difference is the back-scatter in the direction of the incoming wave, caused by the steps. This is predicted by ray tracing to carry energy away from the forward reflected beam. In further experiments, Fourier analysis will be applied here to make a spectral comparison between the two boundaries. This will give an idea of the amount of focusing caused by the steps.

Further work will look at the formation of this ‘attractor’, if it does indeed form. This opens questions about the speed at which the energy is captured by the steps and the properties of the back-reflected energy. The area around the staircase is sure to be important and we expect to see scattering and viscous effects influence the behaviour of the waves here. The idealised shape of the steps isn’t a realistic approximation to many ocean features. A more appropriate alteration to the trapezium could be to use a randomly shaped slope such as you might see in the ocean. This needs to be investigated in greater detail, the initial investigations show that the formation of an attractor is a definite possibility.

References


