COMPLEX METHODS

Summary

0. Introduction

Aims: to explain the application of complex-variable theory to
(i) Harmonic functions (and hence to Laplace's equation in two dimensions).
(ii) Real integration
(iii) Fourier and Laplace -transform theory (and applications to linear ordinary differential equations and linear systems)

N.B. Very much a methods course.

1. Analytic Functions

1.1 Basic Complex-Variable Theory

Definitions and Notation

A complex number z is represented in the complex plane $\mathbb{C}$ by the point with Cartesian coordinates (x, y) and plane polar coordinates (r, θ), where

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \qquad \bar{z} = x - iy = re^{-i\theta}$$

$$r^2 \equiv |z|^2 = z\bar{z} = x^2 + y^2, \qquad \theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$$

$\theta = arg z$ is not well-defined, since $\theta + 2\pi n$, gives the same value of z for any integer n, $n \in \mathbb{Z}$. The *principal value* of arg z satisfies

$$-\pi < \arg z \leq \pi$$

An *open disc* D in $\mathbb{C}$ is a set of the form

$$D = \{z \in \mathbb{C} \text{ such that } |z - z_0| < R\}$$

denoted by $D(z_0, R)$.

A *complex function* is a mapping $f : \mathbb{C} \to \mathbb{C}$, which we write in the form

$$f(z, \bar{z}) \;=\; u(x, y) + iv(x, y)$$

We say that f is *differentiable* if

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. N.B., for the limit to exist, it must be independent of arg h, that is, independent of the direction of approach to z.

Examples

(i) $f(z) = z^3$ is differentiable everywhere, and the derivative is $f'(z) = 3z^2$.

(ii) $f(z) = \bar{z}^3$ is differentiable nowhere.
(iii) $f(z) = 1/z$ is differentiable, except at $z = 0$.

(iv) $f(z) = x$ is not differentiable anywhere.

[Note: f can fail to be differentiable either because it is bad in the real sense (e.g., $1/z$), or because it depends on $\bar{z}$.]

Three more **definitions:**

$f(z)$ is *analytic* (*holomorphic* or *regular*) in D if $f'(z)$ exists $\forall\, z \,\epsilon\, D$.

$f(z)$ is *analytic at* z_0 if $f(z)$ is analytic in $D(z_0, R)$ for some R.

$f(z)$ is *singular* at z_0 if $f(z)$ is not analytic at z_0.

1.2 The Cauchy-Riemann Equations

If f is analytic, then u and v satisfy the CR equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof

From the definition of the derivative, use the fact that the value of the derivative should be independent of the direction in which $h \to 0$. Taking h real gives one expression for the derivative

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Take $h = ik$, with k real:

$$\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Then equate real and imaginary parts.

Remark: We need an extra condition for the converse:
CR and continuous first derivatives $\Rightarrow$ analytic (no proof here).

Harmonic functions

We say that $\phi(x, y)$ is *harmonic* if

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

(*and* ϕ has continuous second derivatives).

If $f = u + iv$ is analytic, then (using CR twice) u (and v) are harmonic. We can use this result to show that a function is harmonic without differentiation.

Conversely, if u is harmonic, then $\exists\, v$ (the **harmonic conjugate** of u), such that $(u + iv)$ is analytic. v is determined, up to an additive constant, by the CR equations. *Level curves:* Curves of $u =$ constant or $v =$ constant. They are orthogonal: $\nabla u \cdot \nabla v = 0$ from CR equations.

Applications in 2-dimensional systems

(i)*Electrostatics.* ϕ is electric potential. The electric field is $\mathbf{E} = -\nabla\phi$. In the absence of charged sources $\nabla \cdot \mathbf{E} = 0$ then $\nabla^2\phi = 0$. Similar for gravitational potential. Level curves of $f(z) = \phi(x, y) + i\psi(x, y)$ are such that $\phi =$ constant are the equipotential lines and $\psi =$ constant are the directions of the field lines.

(ii)*Fluid flow.* $\nabla\phi = \mathbf{v}$ is the velocity of a (non-viscous, incompressible, irrotational) fluid, then $\nabla^2\phi = 0$. Level curves of $f(z) = \phi(x, y) + i\psi(x, y)$: $\phi =$ constant are equipotentials and $\psi =$constant are the streamlines (direction of motion of the fluid particles).

(iii)*Heat flow.* Steady state system. $\phi =$ Temperature, $\nabla^2\phi = 0$. Level curves of $f(z) = \phi(x, y) + i\psi(x, y)$: $\phi =$ constant are isothermals and $\psi =$ constant represents the direction of the heat flow.

1.3 Multivalued Functions: $\log z$ and z^α

Simple functions of real variables such as $x^n, e^x, \cos x, \sin x, \sinh x, \cdots$, have straightforward extensions to the complex plane as functions of the complex variable z, but they have important differences (there exist solutions to $sinz = 10$ in the complex plane, $e^z, \sinh z, \cosh z$ are periodic, etc.). Since the inverse of a periodic function is multivalued then $\log z$ is a multivalued function in the complex plane. First consider log:

The logarithm of a complex number $z = x + iy = re^{i\theta}$ is defined by

$$\log z = \log(re^{i\theta}) = \log r + i\theta$$

However, θ is not uniquely defined by z, since $(\theta + 2n\pi,\ n \in \mathbb{Z})$ define the same value of z. Then we could have also:

$$\log z = \log r + i(\theta + 2n\pi)$$

Thus, $\log z$ is not a well-defined function. This is a problem, because there are closed curves around which $\log z$ varies continuously, but does not return to its original value: This holds if and only if the curve encloses the origin.

The origin is a **branch point** of $\log z$, and similarly any root of $g(z) = 0$ is a branch point of the function $f(z) = \log g(z)$. Also the point at ∞ is also a branch point as can be easily seen by the substitution $z \to t = 1/z$ and analysing the behaviour of the function at $t = 0$.

We can render $\log z$ single-valued by cutting $\mathbb{C}$, so as to prevent curves from encircling $z = 0$.
The cut has to join $z = 0$ and $z = \infty$, but any such cut will do. The resulting function is called a **branch** of log, and is defined by

(i) the position of the cut

(ii) the value of the branch at one point.

The branch is discontinuous across the branch cut, but assumed to be continuous (in fact, analytic) elsewhere on the **cut plane**.

Example

The **principal branch** of log is defined by

(i) Cutting the complex plane at the negative real axis, such that:

$$-\pi < \theta = Arg(z) = \mathrm{Im}(\log z) \leq \pi$$

(ii) For real z $(z = x)$, $\log z = \log x$

Changing the value of θ by integer multiples of 2π define an infinite number of different branches of $\log z$. The domain for each branch will be an identical copy of the cut complex plane above. The set of all these cut planes defines the *Riemann surface* of the log function and each cut plane is referred as a sheet of the Riemann surface. Notice that each sheet of the Riemann surface of the log function is mapped to a different horizontal strip of width 2π in the $w = \log z$ plane.

Next, $f(z) = z^\alpha$

z^α is defined by

$$z^\alpha \;=\; e^{\alpha \log z}$$

Each branch of $\log z$ defines a branch of z^α . If α is real, then

$$z^\alpha \;=\; e^{\alpha(\log r + i\theta)} \;=\; r^\alpha e^{i\theta\alpha}$$

and (say) the principal branch of z^α satisfies

$$-\alpha\pi < \mathrm{Arg} z^\alpha \leq \alpha\pi$$

If $\alpha = 1/n$, $n \in \mathbb{Z}$ then the corresponding Riemann surface will have only n sheets.

See Example: $f(z) = (z^2 - 1)^{1/2}$ with branch points at $z = \pm 1$. Notice that $z = 0$ is *not* a branch point. By making the change $t = 1/z$ we can see that $t = 0$ is not a branch point, so $z = \infty$ is not a branch point. The cut is a line joining the two branch points or from each of them to ∞.

Further Example $f(z) = (z^3 - 1)^{1/2}$ has branch points at $z = 1$, $z = \exp(2\pi i/3)$ and $z = \exp(4\pi i/3)$. For large z, $f(z) \sim z^{3/2}$, which is not single-valued. So ∞ is a branch point. Thus, one needs a cut to ∞.

Aside: Stereographic projection

It is often useful to represent the complex plane $\mathbb{C}$ in terms of a unit sphere with the south pole at the origin of $\mathbb{C}$ and mapping the points in the sphere by straight lines starting at the north pole towards $\mathbb{C}$. The points on the sphere and $\mathbb{C}$ intersecting this line are mapped to each other. This defines the map of the sphere to the complex plane. The north pole would then correspond to the point at infinity in $\mathbb{C}$. This map is called a stereographic projection and the sphere is the *Riemann sphere*. The Riemann sphere helps visualising representations of branch cuts towards ∞.

Summary

To obtain a single-valued function (a branch):

(i) Identify all branch points (usually obvious) in $\mathbb{C}$,

(ii) Investigate large circles (is $z = \infty$ a branch point?).

(iii) Choose a set of cuts joining the branch points (including $z = \infty$).

(iv) Choose a value of f at one point (in each disconnected region of $\mathbb{C}$), and obtain values elsewhere by continuity along paths which do not cross the cuts.

1.4 Conformal mapping

Definition

A mapping which preserves angles (with sense) is called **conformal**.

Theorem

If $\omega(z)$ is analytic in D and $\omega'(z) \neq 0$ in D, then $\omega(z)$ is a conformal mapping of D.

To see this, consider the effect of the mapping on small vectors:

$$\omega(z+h) = \omega(z) + \omega'(z)\, h + \cdots$$

since ω is analytic. So:

$$\mathrm{Arg}\Big(\omega(z+h)-\omega(z)\Big) \sim \mathrm{Arg}\Big(h\omega'(z)\Big) = \mathrm{Arg}\, h + \mathrm{Arg}\, \omega'(z)$$

If $\omega'(z) \neq 0$. This means that all infinitesimal vectors are rotated through the same angle $\mathrm{Arg}\, \omega'(z)$. Therefore their relative angle remains the same.

Note that, if $\omega'(z_0) = 0$, but $\omega''(z_0) \neq 0$, then

$$\mathrm{Arg}\Big(\omega(z_0+h)-\omega(z_0)\Big) \sim \mathrm{Arg}\Big(\frac{h^2\omega''(z_0)}{2}\Big) = 2\mathrm{Arg}\, h + \mathrm{Arg}\, \omega''(z_0)$$

In this case, angles at z_0 are doubled and rotated. This statement generalises if the kth derivative of ω is the first non vanishing derivative at the critical point z_0, then the angle is k times the original angle.

Note: Notice that the argument above shows also that lengths are magnified by $\mid \omega'(z_0) \mid$, and hence that areas are magnified by $\mid \omega'(z_0) \mid^{\,2}$.

Another way to see this is as follows. Compare the $\mathbb{R}^2 \to \mathbb{R}^2$ mapping $(x, y) \to (u, v)$:

$$\omega(z) = u(x, y) + iv(x, y)$$

The Jacobian is

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2$$

(using CR). But further, one has

$$\mid \omega'(z) \mid^2 = \mid \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \mid^2 = J$$

Note also that

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

(using CR). Thus, u and v are orthogonal curvilinear coordinates.

See Example $\omega(z) = z^2$. In this case the grid $x = a =$constant and $y = b =$ constant is mapped to sets of orthogonal parabolae. The point $z = 0$ is critical ($f'(0) = 0, f''(0) \neq 0$) and therefore angles are doubled at the image of $z = 0$.

Further Example

$$\zeta \equiv \omega(z) = e^z.$$

This is conformal, since $\omega'(z) \neq 0$. Note that:

$$\zeta = u + iv = e^{x+iy} = e^x \cos y + ie^x \sin y.$$

The grid maps to:

$$y = b = \text{const} \quad \Rightarrow \quad (v/u) = \ \tan y \quad (\text{radial} \ \ \text{lines}),$$

$$x = a = \text{const} \quad \Rightarrow \quad u^2 + v^2 = \ e^{2a} \quad (\text{circles}).$$

The Möbius map

A Möbius (or fractional linear or bilinear) transformation is

$$\omega(z) = (az + b)/(cz + d) \ ,$$

where $(ad - bc) \neq 0$ is a one-to-one map from the complex plane to itself. The inverse is clearly:

$$z = -\frac{(d\omega - b)}{(c\omega - a)}$$

Particular cases:

rotations: $\omega(z) = e^{i\alpha} \, z \, ,$

dilations: $\omega(z) = kz \, ,$ where k is real,

translations: $\omega(z) = z + a \, ,$

inversions: $\omega(z) = z^{-1} \ ,$

Any Möbius transformation can be written as:

$$w \ = \ \frac{a}{c} + \frac{(bc - ad)}{c} \ \frac{1}{(cz + d)}$$

So it can be seen as a composition of three simple transformations ($cz + d$ followed by $1/z$ followed by $\frac{a}{c} + \frac{(bc-ad)}{c} z$).

The Möbius transformation maps circles and straight lines to circles and straight lines in the ω plane. The less trivial case to prove is the inversion $1/z$. A general circle is $A(x^2 + y^2) + Bx + Cy + D = 0$. In polar coordinates $z = re^{i\theta}$, $Ar^2 + r(B\cos\theta + C\sin\theta) + D = 0$. Under the inversion $1/z \equiv \rho e^{i\phi} = 1/re^{-i\theta}$ this leads to $A + \rho(B\cos\phi - C\sin\phi) + D\rho^2 = 0$ which is also a circle. The straight lines (circles of infinite radii) are obtained in the particular cases $A = 0$ or $D = 0$.

Möbius transformations are useful to map regions bounded by lines to the interior or exterior of circles. For instance $(z - 1)/(z + 1)$ maps the region Re$z \geq 0$ to the unit circle and $1/z$ maps the region Re$z \geq 1/2$ to the circle $(u - 1)^2 + v^2 = 1$.

1.5 Application to Laplace's equation

Suppose we wish to solve $\nabla^2\Psi = 0$ in some region $B \subset \mathbb{C}$, with $\Psi(x,y) = \Psi_0$ on a boundary ∂B with a complicated shape.
The plan is to find a conformal map $\omega(z)$ such that $\omega(B)$ is nice, and then solve

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}$$

on $\omega(B)$, with $\Phi = \Psi_0$ on $\omega(\partial B)$.

Then, $\exists\, f(\zeta)$ analytic such that $\Phi = \mathrm{Re} f(\zeta)$.
Further, $\Psi(x,y)$, defined to be $\mathrm{Re}\, f(\omega(z))$, also satisfies the Laplace equation $\nabla^2\Psi = 0$, [*since* $f(\omega(z))$ *is analytic*], and also obeys $\Psi = \Psi_0$ on ∂B .

An explicit way to see this: $f = \Phi(u,v) + i\Gamma(u,v)$ is analytic in the ω-plane. Since $\omega = u(x,y) + iv(x,y)$ is analytic in the z-plane, we have to check that also $\Psi(x,y) \equiv \Phi(u(x,y), v(x,y))$ and $\Lambda(x,y) \equiv \Gamma(u(x,y), v(x,y)$ are also harmonic conjugates. So:

$$\begin{aligned}
\frac{\partial\Psi}{\partial x} &= \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial x} \\
\frac{\partial\Lambda}{\partial y} &= \frac{\partial\Gamma}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\Gamma}{\partial v}\frac{\partial v}{\partial y} = (-\frac{\partial\Phi}{\partial v})(-\frac{\partial v}{\partial x}) + \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x}
\end{aligned}$$

where in the last step we used the CR relations satisfied by Γ and Φ and also the ones satisfied by u and v. This clearly shows that Ψ and Λ also satisfy the CR equations.

See Example.

N.B. Riemann's mapping theorem: 'any' open region of $\mathbb{C}$ can be mapped conformally onto $D(0,1)$. (Not necessarily smooth on the boundaries.)

Recap:

Solving Laplace's equation by conformal transformation

To solve $\nabla^2\Psi = 0$ on B, with $\Psi = \Psi_0$ on ∂B :

* Find $\omega(z)$ such that $\omega(\partial B)$ is nice;

* Solve

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0$$

with $\Phi = \Psi_0$ on $\omega(\partial B)$;

* Find an analytic function $f(\zeta)$ such that $\Phi(\xi,\eta) = \mathrm{Re} f(\zeta)$;

* Set $\Psi(x,y) = \mathrm{Re} f(\omega(z))$.

* Clearly $\nabla^2\Psi = 0$, and $\Psi = \Psi_0$ on ∂B (since $\Psi(x,y) = \Phi(u(x,y), v(x,y))$).

COMPLEX METHODS

Summary Chapter 2

2. Contour Integration and Cauchy's Theorem

2.1 Contour integrals

Definitions

A curve is **smooth** if $z(t)$ has continuous derivatives $z'(t) \neq 0$ for all points. A curve consisting of a finite number of smooth curves joined end to end is called a **contour**. If only initial and final values of $z(t)$ are the same then it is a **simple closed contour**. A domain D is **simply connected** if every simple contour within D encloses only points on D. Otherwise the domain is **multiply connected**. An easy way to picture this is that simply connected domains have no holes whereas multiply connected domains have holes.

Let γ be a path in $\mathbb{C}$ (assumed to be smooth and non-intersecting; assume further that γ does not cross any branch cuts), parametrised by a real parameter t :

$$z \ = \ h(t), \qquad t_1 \leq t \leq t_2$$

Then we write

$$\int_\gamma f(z)\,dz \equiv \int_{t_1}^{t_2} f[h(t)]\,\frac{dh}{dt}\,dt$$

The real and imaginary parts of the right-hand side are standard integrals in $\mathbb{R}^2$. See Example

Cauchy's Theorem

If $f(z)$ is analytic inside and on a closed contour C, then

$$\oint_C f(z)dz = 0$$

To prove it use $f = u + iv$ and $dz = dx + idy$, separate the product between real and imaginary parts, use Green's theorem in the plane and finally the Cauchy-Riemann equations.

Morera's Theorem. If $f(z)$ is continuous in a domain D and $\int_C f(z)dz = 0$ along every closed curve C, then $f(z)$ is analytic. No proof.

Path Independence Theorem. If $f(z)$ is analytic in a simply connected domain D, then $\int_a^b f(z)\,dz$ is independent of the path joining the points a and b.

To prove it: consider two curves C_1 and C_2 joining points a and b. They make a closed curve C for which the Cauchy theorem applies and the result comes out directly taking care of the sign due to the orientation of the curves.

Anti-Derivative Theorem. Let $f(z)$ be analytic throughout a simply connected domain D, if there exists an analytic function $F(z)$, satisfying $F'(z) = f(z)$ then $\int_a^b f(z)dz = F(b) - F(a)$.

Proof. Use $f(z) = F'(z)$ and the definition of a contour integral in terms of an integral over a real parameter t.

$$\int_C f(z)\, dz = \int_C F'[h(t)]h'(t)dt = \int_C dF[h(t)] = F[[h(t_2)] - F[h(t_1)] = F(b) - F(a)$$

Deformation Theorem. If f is analytic in a region D bounded by two closed curves C_1 and C_2 (and on C_1 and C_2 also), then: $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

To prove it: create a single closed contour by following both curves and a cut between them and use Cauchy's theorem.

2.2 Series expansions and singularities

Taylor series. We state without proof: $f(z)$ is analytic in $D(z_0, R)$

$$\Leftrightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n ,$$

for $| z - z_0 | < R$. This gives the *Taylor series* of f about z_0 with the coefficients given by $n!a_n = f^n(z_0)$.

Thus, $f'(z)$ exists $\Leftrightarrow$ the Taylor series exists, and the radius of convergence of the series is precisely the distance to the nearest singularity of $f(z)$.

Note:
(i)f analytic $\Rightarrow \exists$ Taylor series $\Rightarrow f$ can be written as a function of z only (and in fact the converse holds).

(ii) $f'(z)$ exists in $D \Rightarrow$ Taylor series. Then $f^{(n)}(z)$ exists also.
(iii) Note that, if $f(z) = \sum_0^\infty a_n(z - z_0)^n$, then $f^{(n)}(z_0) = n!\, a_n$, but the a_n can usually be calculated more efficiently by elementary means. See Example.

If f is **not** analytic at $z = z_0$, then we cannot hope for an expansion in the form of a Taylor series. But in complex variables, there is a generalisation of the Taylor expansion.

Laurent Expansions. If $f(z)$ is analytic in the annulus $a \le | z - z_0 | \le b$, then $\exists$ a unique expansion of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

The series converges so well in $a < \mid z - z_0 \mid < b$, that it can be differentiated term-by-term.

Isolated Singularities

z_0 is an **isolated singularity** of $f(z)$ if $f(z)$ is analytic for $0 <\mid z - z_0 \mid< R$ (for some R), but not at z_0.

In this case, for $0 <\mid z - z_0 \mid< R$, one has

$$f(z) = \sum_{n=N}^{\infty} c_n (z - z_0)^n$$

and

(i) z_0 is a **pole of order -N** if $N = -1, -2, ...,$

(ii) z_0 is a **simple pole** if $N = -1$,

(iii) z_0 is an **essential singularity** if $N = -\infty$.

If $N = 0$ we usually refer to a **removable singularity** (as in the case of $\sin z/z$). Singularities can usually be classified by inspection. See Examples.

Some Comments

(i) All points on a branch cut (across which f is discontinuous) are singular, and so a branch point is a **non-isolated singularity**.

(ii) The function $1/[\sin(z^{-1})]$ has simple poles at $z = 1/(n\pi)$, but the singularity at $z = 0$ is non-isolated, being the accumulation point of a sequence of poles.

(iii) The function $1/\big(z^{1/2} - 1\big)$ has a simple pole or no singularity at $z = 1$, depending on the choice of branch (together with a singularity at $z = 0$.)

2.4 Cauchy's integral formula

Laurent Theorem. If $f(z)$ is analytic for $\mid a \mid \leq \mid z - z_0 \mid \leq \mid b \mid$, then the Laurent series:

$$f(z) = \sum_{-\infty}^{\infty} c_m (z - z_0)^m$$

is such that

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{n+1}}$$

where C is a closed curve in D encircling z_0.

To prove it: multiply both sides of the Laurent series above by $(z - z_0)^{-n-1}$ and integrate over C. Then use that $\oint (z - z_0)^{m-n-1} dz = \delta_{mn}$ (as can be easily seen by parameterising $z - z_0 = \rho e^{i\theta}$ with $a \leq \rho \leq b$).

Some Remarks

(i) This holds however badly f behaves in the region $|z| < |a|$.

(ii) Note the special case:

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz\,.$$

The coefficient c_{-1} is called the **residue** of f at z_0.

(iii) If $f(z)$ is analytic for $|z - z_0| \leq b$, then it has the Taylor expansion:$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$ and, setting $z = z_0$, one has:

$$f(z_0) = a_0 = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

This is **Cauchy's integral formula**.

(iv) Cauchy's integral formula can be generalised to obtain an expression for the n-th derivative of f at z_0. Since for $f(z)$ analytic the Taylor expansion coefficient is $f^{(n)}(z_0)$, combining this with Laurent's theorem we get:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n = 1, 2, 3, ...$$

The importance of Cauchy's integral formula and its generalisation is two-fold. First it is a useful tool to compute contour integrals of the type $\oint_C f(z)\, dz/(z - z_0)^n$. Also conceptually these expressions tell us that knowledge of the function $f(z)$ along the one-dimensional contour C is enough to know the value of $f(z)$ (and all its derivatives) in the full two-dimensional region bounded by C.

Some Related Theorems

Cauchy's Inequality. If $f(z)$ is analytic in and on a circle C of radius a with centre at $z = z_0$ and $|f(z)| \leq M$ for positive M, on C then:

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{a^n}, \qquad n = 0, 1, 2, ...$$

To prove it: Start with the expression in (iv) above, take the modulus and use the triangle inequality for integrals, as well as $|z - z_0| = a$ and $|f(z)| \leq M$ on C. There is a factor of $2\pi a$ coming from the integral of $\int d\theta$.

Liouville's Theorem. If $f(z)$ is an entire function (analytic on $\mathbb{C}$) and is bounded, *i.e.* $|f(z)| \leq M$, then f is **constant**.

To prove it: Take $n = 1$ in the previous theorem, then $|f'(z_0)| \leq M/a$ and set $a \to \infty$.

Fundamental Theorem of Algebra. Each polynomial equation $P(z) = 0$ of degree $n \geq 1$ has at least one root (and therefore it has n roots).

To prove it: If the polynomial $P(z)$ has no roots then $f(z) = 1/P(z)$ is analytic and $|f| = 1/|P|$ is bounded when $z \to \infty$, therefore $f(z)$ is a constant (from Liouville's theorem), which is a contradiction.

COMPLEX METHODS
Summary Chapter 3

3. Residue Calculus

3.1 Residue theorem; calculating residues

As we have seen in the previous chapter, the coefficient c_{-1} in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

is called the **residue** of f at z_0 and it satisfies

$$\text{Res}\,[f, z_0] \equiv c_{-1} \;=\; \frac{1}{2\pi i} \oint_C f(z)\, dz\,.$$

for a singularity of the function f at z_0 inside the contour C. Therefore, if we can compute the residue implies we can know the value of the full contour integral.

Computing the residue is usually very easy. If z_0 is a simple pole,

$$f(z) \;=\; \frac{c_{-1}}{z - z_0} \;+\; c_0 \;+\; c_1\,(z - z_0) \;+\; \cdots$$

and so

$$\text{Res}\,[f, z_0] \;\equiv c_{-1} \;= \lim_{z \to z_0} f(z)(z - z_0).$$

Often it is best to calculate this limit using l'Hôpital's rule. Suppose that $f(z) = g(z)/h(z)$ where $g(z)$ is analytic and non-zero at z_0 and $h(z)$ has a simple pole at z_0. Then

$$\text{Res}\,[f, z_0] \;\equiv\; c_{-1} = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{(z - z_0)}{h(z)} \;=\; \frac{g(z_0)}{h'(z_0)}$$

In general, if z_0 is a pole of order k, then a similar argument as above shows that:

$$\text{Res}\,[f, z_0] \;\equiv\; c_{-1} \;=\; \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}\,(z - z_0)^k f(z)$$

N.B. But it is often better not to use this formula, since when k is large enough the calculation of the derivatives is cumbersome and it is better to extract c_{-1} from a direct development of the Laurent expansion. See examples.

The Residue Theorem

Let C be a closed curve and let $f(z)$ be analytic on and inside C, except for a finite number of isolated singularities at $z = z_k$. Then:

$$\oint_C f(z)\,dz = 2\pi i \sum_k \text{Res}\,[f, z_k]$$

For a proof use the deformation theorem to reduce the contour to a series of contours around each of the poles.

3.2 Calculating Definite Integrals

The aim is to evaluate a real definite integral such as $I = \int_a^b f(x)\,dx$ by considering (normally) $J = \oint_C f(z)\,dz$, one part of the contour C including the interval (a, b). We choose C so that the additional contribution is either zero or a multiple of I, and then evaluate J by the residue theorem.

General Classes of Integrals

[**1**] Integrals of the form:

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$$

Method: Consider the contour $|z| = 1$ and substitute $\sin\theta = (z - z^{-1})/2i$, $\cos\theta = (z + z^{-1})/2$ and $d\theta = dz/iz$. See example.

[**2**] Integrals of the form:

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

for which the function $f(x)$ does not have a singularity on the real axis and $zf(z) \to 0$ when $z \to \infty$, then:

Choose a large semi-circle in the upper half-plane or the lower half-plane, provided that $\mid f(z) \mid \;\to\; 0$ fast enough there. See example.

[**3**] Integrals of the form:

$$\int_0^{\infty} f(x)\,dx,$$

which runs from zero to infinity, one should try the possibilities

(i) if $f(x)$ is even, rewrite the integral as $(1/2)\int_{-\infty}^{\infty} f(x)\,dx$, etc.

(ii) If f has a branch point, use a keyhole contour. If the branch point is the origin, the keyhole contour consists of a large circle, a small circle surrounding the branch point and two horizontal lines at $\theta = 0, 2\pi$ for the principal branch, in such a way as to avoid the branch cut on the positive real axis.

(iii) If f has no branch point, introduce one by considering $\int_0^\infty f(x) \log x \, dx$. or work with a sector of a large circle surrounding only one of the poles. Other contours can be chosen, such as large rectangular contours encircling only one of the poles when $f(x)$ includes trigonometric functions in the denominator (this also applies to integrals of type [2]). See examples.

[**4**] Integrals of the form:

$$\int_{-\infty}^{\infty} f(x) \cos mx \, dx, \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x) \sin mx \, dx$$

Consider the integral $\oint_C f(z) e^{imz} dz$ with the contour C consisting of a large semicircle C_R of radius R ($R \to \infty$) closed by the real interval $(-R, R)$. Then use:

Jordan's Lemma
If $zf(z) \to 0$ for $R \to \infty$ and $m > 0$, then $\int_{C_R} f(z) e^{imz} dz \to 0$ when $R \to \infty$.
For the proof, use that $\sin(\pi - \theta) = \sin\theta$ and that in the interval $(0, \pi/2)$, $\sin\theta \geq 2\theta/\pi$. See example.

[**5**] Integrals for which the integrand has poles in the real axis.

For these integrals we need to define the **Cauchy principal value** as follows. We know that if $f(x)$ has a pole at $x = x_0$ with x_0 in the interval (a, b), then,

$$\int_a^b f(x) dx \; = \; \lim_{\epsilon \to 0} \int_a^{x_0 - \epsilon} f(x) \, dx + \lim_{\delta \to 0} \int_{x_0 + \delta}^b f(x) \, dx.$$

In many cases the limit exists only if $\epsilon = \delta$. This defines the principal value (PV) of the integral. See example. In this case a contour similar to the one in [4] can be used but with an extra semicircle of radius ϵ used to avoid the singularity at the point in the real axis. See example.

3.3 Summation of Series

Let us consider the function:

$$f(z) \; = \; \frac{\cot \pi z}{z^2}.$$

It has a pole of order 3 at $z = 0$ and simple poles at $z = n$ for $n \in \mathbb{Z}$. The residue at $z = 0$ can be easily computed by using the ratio of the series for $\cos \pi z$ and $\sin \pi z$ and gives $-\pi/3$. The residues at these poles are easily seen to be $\frac{1}{\pi n^2}$. Let us choose a contour C consisting of a square centred at the origin and of sides N, with N a large integer ($N \to \infty$). In this sense, the upper right-hand corner of the square corresponds to the point $(N+1/2)(1+i)$. Therefore the residues theorem implies that

$$\frac{1}{2\pi i}\oint_C \frac{\cot \pi z}{z^2}\, dz = -\frac{\pi}{3} + \frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n^2}$$

If we can argue that the contour integral vanishes in the limit $N \to \infty$, then we find the expression:

$$\sum_{n=1}^{N}\frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Indeed, it is easy to argue that on C, $|\cot \pi z|$ is bounded whereas $1/z^2 \to 0$ therefore the integral vanishes and we discovered an interesting way to sum an infinite series.

In general it can be shown that:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(n) &= -(\text{sum of residues of } \pi\cot\pi z\, f(z)\,) \\
\sum_{n=-\infty}^{\infty} (-1)^n f(n) &= -(\text{sum of residues of } \pi\csc\pi z\, f(z)\,) \\
\sum_{n=-\infty}^{\infty} f\Big(\frac{2n+1}{2}\Big) &= (\text{sum of residues of } \pi\tan\pi z\, f(z)\,) \\
\sum_{n=-\infty}^{\infty} (-1)^n f\Big(\frac{2n+1}{2}\Big) &= (\text{sum of residues of } \pi\sec\pi z\, f(z)\,)\,.
\end{aligned}$$

COMPLEX METHODS

Summary Chapter 4

4 Fourier and Laplace Transforms

4.1 Fourier Transforms

Most of this is just a review of material covered in Methods course. The only new ingredient is the use of Complex Methods techniques to compute Fourier and inverse Fourier transforms. Something that was not available in the Methods course.

Definition: The Fourier transform of $f(x)$, denoted by $\tilde{f}(k)$, is defined to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\, f(x)\, dx,$$

provided that this integral converges.

Often the variable x represents standard space x or time t. The variable k refers then to wave number or frequency ω. Therefore the Fourier transform is a linear map from a function of standard space x or time t to a function defined on the space of wave numbers k or frequency ω. In quantum mechanics the linear momentum is $p = hk/2\pi$ with h the Planck constant. Therefore the Fourier transform can be seen as a function of momenta.

Properties

Linearity: $\quad (f_1 \tilde{+} f_2) \;=\; \tilde{f}_1 + \tilde{f}_2$

Translation: Let $g(x) = f(x-a)$. Then $\tilde{g}(k) = e^{-ika}\tilde{f}(k)$

Frequency shift: Let $\quad g(x) \;=\; e^{ik'x}\, f(x)$. Then $\tilde{g}(k) = \tilde{f}(k-k')$.

Scaling: Let $g(x) \;=\; f(ax)$. Then $\tilde{g}(k) = \frac{1}{a}\tilde{f}(k/a)$

Derivatives: Let $\quad g(x) = f'(x)$. $\quad$ Then: $\tilde{g}(k) = ik\tilde{f}(k)$

Multiplication by x: Let $\quad g(x) = xf(x)$. $\quad$ Then: $\tilde{g}(k) = i\tilde{f}'(k)$

The Inversion Formula

The *inverse Fourier transform* of $\tilde{f}(k)$ is defined to be

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+ikx}\tilde{f}(k)dk\ .$$

Theorem: If $f(x) \,\epsilon\, L^1 \cap L^2$, that is, if both $\int |\, f \,|$ and $\int |\, f \,|^2$ exist, then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)dk = f(x)$$

If $f(x)$ is continuous at x. Otherwise it gives $(f(x_+) + f(x_-))/2$. See Example.

The **convolution** of $f(x)$ and $g(x)$, denoted by $f * g$, is defined by:

$$f * g \;\; \equiv \;\; \int_{-\infty}^{\infty} f(x-y)g(y)dy \;\; = \;\; g * f$$

Theorem: Let $h(x) = (f * g)(x)$. Then: $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$.

Application to Ordinary Differential Equations (ODE's)

Idea: Use the relation $g(x) = f'(x) \Rightarrow \tilde{g}(k) = ik\tilde{f}(k)$ to reduce a differential equation to an algebraic equation. Then solve the algebraic equation for the Fourier transform and use the inversion theorem or convolution to solve for the original function. This usually requires computation of transforms by contour integration.

N.B. This method assumes that the function and its derivatives are well defined in the sense that the integration by parts used to prove the relation above, holds. This is usually guaranteed if the function and its derivatives are continuous and they vanish at $\pm\infty$.

4.2 Laplace Transforms

Motivation

Often the Fourier transform does not exist (since $f(x)$ does not go to 0 as$x \to \infty$) becuase the integral does not converge. Furthermore in physical applications we are usually interested on a function of time t for $t > 0$ when initial conditions at $t = 0$ are given. Consider then the function $g(t) = e^{-\sigma t}f(t)$ for $t \geq 0$ and $g(t) = 0$ for $t < 0$. Here the parameter σ is taken to be positive. The Fourier transform of this function is

$$\tilde{g}(\omega) \;\; = \;\; \int_0^{\infty} e^{-(\sigma+i\omega)t}f(t)\; dt \;\; = \;\; \int_0^{\infty} e^{-st}f(t)\; dt$$

with $s \equiv \sigma + i\omega$. This leads to the definition of the Laplace transform of the function $f(t)$.

Definition The Laplace transform of a funtion $f(t)$ is defined by:

$$\mathcal{L}\{f(t)\} \equiv F(s) \;\; \equiv \;\; \int_0^{\infty} e^{-st}f(t)\,dt \qquad s = \sigma + i\omega \in \mathbb{C}$$

Conditions for existence: $f(t)$ has to be *piecewise continuous* (on each interval there are at most a finite number of points t_k at which f has finite discontinuities). Also $f(t)$ has to be of *exponential order* which means that there are real constants $K, M > 0$

such that $|f(t)| \leq Me^{Kt}$ for all $t \geq 0$. Many known functions satisfy these conditions and we can see that there are many more functions that have Laplace transform than Fourier transform.

Properties

For $\mathcal{L}\{f(t)\} = F(s)$:

Linearity: $\mathcal{L}\{af_1 + bf_2\} \;=\; a\mathcal{L}\{f_1\} + b\mathcal{L}\{f_2\}$

Translation in s: $\mathcal{L}\{e^{at}f(t)\} \;=\; F(s-a)$

Translation in t: $\mathcal{L}\{H(t-t_0)f(t-t_0)\} \;=\; e^{-st_0}\, F(s)$

Scaling: $\mathcal{L}\{f(at)\} \;=\; \frac{1}{a}\, F\left(\frac{s}{a}\right)$

Derivatives: $\mathcal{L}\{df/dt\} \;=\; s\, F(s) \;-f(0)$

More generally (using $df/dt = f'$, etc.):

$$\mathcal{L}\{d^n f/dt^n\} \;=\; s^n\, F(s) \;\; - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Multiplication by t: $\mathcal{L}\{t\, f(t)\} \;=\; -dF/ds$

More generally: $\mathcal{L}\{t^k f(t)\} \;=\; (-1)^k\, \frac{d^k}{ds^k}\, F(s)$

Examples of Laplace Transforms:

$\mathcal{L}\{t^n\} \;=\; n!/s^{n+1}$, with n a positive integer; $\mathcal{L}\{e^{at}\} \;=\; 1/(s-a)$, for $\mathrm{Re}\,(s) > 0$; $\mathcal{L}\{\delta(t-t_0)\} \;=\; e^{-st_0}$; $\mathcal{L}\{H(t-t_0)\} \;=\; e^{-st_0}/s$. Here and above $H(t-t_0)$ is the step or Heaviside function (1 for $t \geq t_0$ and 0 otherwise). To find the Laplace transform of other functions use the definition and perform the integral or use any of these results combined with the properties above.

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \;=\; \frac{1}{2\pi i} PV \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds \;=\; \mathrm{Lim}_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) ds$$

where $\gamma > s_i$ with s_i the singularities of $F(s)$. This can be determined by a Bromwich contour (usually closing the contour to the left with a section of a circle of large radius R.). Using the residue theorem then we can easily derive the following result: if $F(s)$ has poles at $s_1, s_2, \cdots, s_n$ to the left of $\mathrm{Re}s = \gamma$ and $sF(s)$ is bounded as $R \to \infty$ then:

$$\mathcal{L}^{-1}\{F(s)\} \;=\; \sum_{k=1}^{n} \mathrm{Res}\left(e^{st}F(s), s_k\right).$$

Convolution Theorem

We can define the convolution for functions that vanish at $t < 0$ (causal functions).

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = (g * f)(t)$$

Where we used that the integrand vanishes at $\tau < 0$ and $t - \tau < 0$. Similar to Fourier transforms there is a convolution theorem for Laplace transforms: If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, and $h(t) = (f * g)(t)$ then

$$H(s) = F(s)\,G(s)$$

where $H(s)$ is the Laplace transform of $h(t)$. Convolution can be used for finding inverse Laplace transforms and to solve differential and integral equations.

Applications to Differential Equations

The derivative property of Laplace transforms can be used to solve initial value problems, that is, ordinary and partial differential equations with initial conditions specified. As with Fourier transforms:

Idea: Use the relation $\mathcal{L}\{d^n f/dt^n\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$ to reduce a differential equation to an algebraic equation. Then solve the algebraic equation for the Laplace transform and use the inversion theorem, partial fractions or convolution to solve for the original function. This usually requires computation of transforms by contour integration. Note that this method can be used to sove N differential equations of order N which are then reduced to N algebraic equations with N unknowns. The difficulty again relies in the computation of the inverse transform at the end.

N.B. The advantage over Fourier transforms is that the derivative property of Laplace transforms already includes information on the initial conditions (also there are more functions that have Laplace transforms). Which method to use depends on the particular problem. If initial conditions are given it is better to try with Laplace transforms. Also for Laplace transforms there exist large numbers of tables with explicit expressions for Laplace transforms and their inverses.

Application to Linear Systems

Let $\mathcal{O}$ be a linear operator, which represents the response $y(t)$ of a linear system to an input signal $u(t)$ (for example, in an electrical circuit):

$$\mathcal{O}u(t) \;=\; y(t)$$

The *response* or *transfer* function $h(t)$ is the response to an impulse input, that is to an input $u(t) = \delta(t)$. For an arbitrary input $u(t) = \int_0^\infty u(\tau)\delta(t-\tau)\,d\tau$ the response will be

$$y(t) = \mathcal{O}u(t) = \int_0^\infty u(\tau)\mathcal{O}\delta(t-\tau)\,d\tau \;=\; \int_0^\infty u(\tau)\,h(t-\tau)\,d\tau = (u * h)(t)$$

Therefore knowing the transfer function $h(t)$ is enough to fully determine the response of the system to any other input by convoluting the input signal $u(t)$ with the transfer function $h(t)$. To solve for $y(t)$ we can use the Laplace (or Fourier) transform to find first its Laplace tranform $Y(s)$ determined by

$$Y(s) = H(s)\,U(s)$$

and then find the inverse transform of $Y(s)$.