Supersymmetry and Extra Dimensions

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Chapter 1

Physical Motivation for Supersymmetry and Extra Dimensions

Let us start with a simple question in high energy physics: What do we know so far about the universe we live in?

1.1 Basic Theory: QFT

Microscopically we have Quantum Mechanics and Special Relativity as our two basic theories.

The consistent framework to make these two theories consistent with each other is Quantum Field Theory (QFT). In this theory the fundamental entities are quantum fields. Their excitations correspond to the physically observable elementary particles which are the basic constituents of matter as well as the mediators of all the known interactions. Therefore, fields have particle - like character. Particles can be classified in two general classes: bosons (spin $s = n \ \in \ \mathbb{Z}$) and fermions ($s = n + \frac{1}{2} \ \ n \in \mathbb{Z}$). Bosons and fermions have very different physical behaviour. The main difference is that fermions can be shown to satisfy the Pauli "exclusion principle" , which states that two identical fermions cannot occupy the same quantum state, and therefore explaining the vast diversity of atoms. All elementary matter particles: the leptons (including electrons and neutrinos) and quarks (that make protons, neutrons and all other hadrons) are fermions. Bosons on the other hand are not constrained by the Pauli principle. They include the photon (particle of light and mediator of electromagnetic interaction), and the mediators of all the other interactions. As we will see, supersymmetry is a symmetry that unifies bosons and fermions despite all their differences.

1.2 Basic Principle: Symmetry

If Quantum Field Theory is the basic framework to study elementary process, the basic tool to learn about these processes is the concept of symmetry. A symmetry is a transformation that can be made to a physical system leaving the physical observables unchanged. Throughout the history of science symmetry has played a very important role to better understand nature.

1.2.1 Classes of Symmetries

For elementary particles, we can define two general classes of symmetries:

- *Spacetime symmetries.* These symmetries correspond to transformations on a field theory acting explicitly on the spacetime coordinates.

$$x^{\mu} \rightarrow x'^{\mu}\left(x^{\nu}\right), \qquad \mu, \nu = 0, 1, 2, 3 \tag{1.1}$$

  Examples are rotations, translations and, more generally, Lorentz and Poincaré transformations defining Special Relativity as well as General Coordinate Transformations that define General Relativity.

- *Internal symmetries.* These are symmetries that correspond to transformations to the different fields on a field theory.

$$\Phi^{a}(x) \rightarrow M^{a}_{b} \Phi^{b}(x) \tag{1.2}$$

  Where the indices a, b label the corresponding field. If M^{a}_{b} is constant then the symmetry is a *global symmetry.* If they depend on the spacetime coordinates: $M^{a}_{b}(x)$ then the symmetry is called a *global symmetry.*

1.2.2 Importance of Symmetries

Symmetry is important for various reasons:

- *Labelling and classifying particles.* Symmetries label and classify particles according to the different conserved quantum numbers identified by the spacetime and internal symmetries (mass, spin, charge, colour, etc.). In this regard symmetries actually "define" an elementary particle according to the behaviour of the corresponding field with respect to the different symmetries.

- Symmetries determine the interactions among particles by means of the gauge principle, e.g. consider the Lagrangian

$$\mathcal{L} \;=\; \partial_{\mu}\phi\partial^{\mu}\phi^{*} - V(\phi, \phi^{*}) \;,$$

which is invariant under rotation in the complex plane

$$\phi \quad \longmapsto \quad \exp(i\alpha)\phi \ ,$$

as long as α is a constant (global symmetry). If $\alpha = \alpha(x)$, the kinetic term is no longer invariant:

$$\partial_\mu \phi \quad \longmapsto \quad \exp(i\alpha)\big(\partial_\mu \phi + i(\partial_\mu \alpha)\phi\big)$$

However, the covariant derivative D_μ, defined as

$$D_\mu \phi \quad := \quad \partial_\mu \phi + i A_\mu \phi \ ,$$

transforms like ϕ itself, if the gauge - potential A_μ transforms to $A_\mu - \partial_\mu \alpha$:

$$D_\mu \quad \longmapsto \quad \exp(i\alpha)\big(\partial_\mu \phi + i(\partial_\mu \alpha)\phi + i(A_\mu - \partial_\mu \alpha)\phi\big) \quad = \quad \exp(i\alpha) D_\mu \phi \ ,$$

so rewrite the Lagrangian to ensure gauge - invariance:

$$\mathcal{L} \quad = \quad D_\mu \phi D^\mu \phi^* - V(\phi, \phi^*)$$

The scalar field ϕ couples to the gauge - field A_μ via $A_\mu \phi A^\mu \phi$, similarly, the Dirac - Lagrangian

$$\mathcal{L} \quad = \quad \overline{\Psi} \gamma^\mu D_\mu \Psi$$

has an interaction - term $\overline{\Psi} A_\mu \Psi$. This interaction provides the three point vertex that describes interactions of electrons and photons and illustrate how photons mediate the electromagnetic interactions.

- *Symmetries can hide or be "spontaneously broken"*. Consider the potential $V(\phi, \phi^*)$ in the scalar field Lagrangian above. If $V(\phi, \phi^*) = V(|\phi|^2)$, then it is symmetric for $\phi \mapsto \exp(i\alpha)\phi$. If the potential is of the type

$$V = a|\phi|^2 + b|\phi|^4, \qquad a, b \geq 0 \tag{1.3}$$

the minimum is at $< \phi >= 0$ (here $\langle \phi \rangle \equiv \langle 0|\phi|0 \rangle$ denotes the ‘vacuum expectation value (vev) of the field ϕ). The vacuum state is then also symmetric under the symmetry since the origin is invariant. However if the potential is of the form

$$V = \big(a - b|\phi|^2\big)^2 \qquad a, b \geq 0 \tag{1.4}$$

the symmetry of V is lost in the ground state $\langle \phi \rangle \neq 0$. The existence of hidden symmetries is important for at least two reasons. First, this is a natural way to introduce an energy scale in the system. In particular, we will see that for the standard model $M_{EW} \sim 10^3$ GeV, defines the basic scale of mass for the particles of the standard model, the electroweak gauge bosons and the matter fields obtain their mass from this effect. Second, the existence of hidden symmetries implies that the fundamental symmetries of nature may be huge despite the fact that we observe a limited amount of symmetry. This is because the only manifest symmetries we can observe are the symmetries of the vacuum we live in and not those of the full underlying theory.

1.3 Basic Example: The Standard Model

The concrete example is the particular QFT known as the Standard Model which describes all known particles and interactions in four-dimensional spacetime.

- *Matter particles.* Quarks and leptons. They come in three identical families differing only by their mass. Only the first family participate in making the atoms and all composite matter we observe. Quarks and leptons are fermions of spin $1/2\hbar$ and therefore satisfy Pauli's exclusion principle. Leptons include the electron e^-, muon μ and τ as well as the three neutrinos. Quarks come in three colours and are the building blcoks of strongly interacting particles such as the proton and neutron in the atoms.

- *Interaction particles.* The three non-gravitational interactions (strong, weak and electromagnetic) are described by a gauge theory based on an internal symmetry:

$$G_{SM} \quad = \quad \underbrace{SU(3)_c}_{\text{strong}} \otimes \underbrace{SU(2)_L \otimes U(1)}_{\text{electroweak}}$$

  Here $SU(3)_c$ refers to quantum chromodynamics part of the standard model describing the strong interactions, the subindex c refers to colour. Also $SU(2)_L \otimes U(1)$ refers to the electroweak part of the standard model, describing the electromagnetic and weak interactions. The subindex L in $SU(2)_L$ refers to the fact that the standard model does not preserve parity and differentiates between left-handed and right-handed particles. In the standard model only left-handed particles transform non-trivially under $SU(2)_L$. The gauge particles have all spin $s = 1\hbar$ and mediate each of the three forces: photons (γ) for $U(1)$ electromagnetism, gluons for $SU(3)_c$ of strong interactions, and the massive $W^\pm$ and Z for the weak interactions.

- *The Higgs particle.* This is the spin $s = 0$ particle that has a potential of the Mexican hat shape and is responsible for the breaking of the Standard Model gauge symmetry This is the way in which symmetry is spontaneously broken, in the Standard Model:

$$SU(2)_L \otimes U(1) \quad \stackrel{\langle\phi\rangle\approx 10^3\text{GeV}}{\longrightarrow} \quad U_{EM}(1)$$

  For the gauge particles this is the Higgs effect, that explains how the $W^\pm$ and Z particles get a mass and therefore the weak interactions are short range. This is also the source of masse for all quarks and leptons.

- *Gravity particle?.* The standard model only describe gravity at the classical level since, contrary to gauge theories which are consistent quantum mechanical theories, there is not known QFT that describes gravity in a consistent manner. The behaviour of gravity at the classical level would correspond toa particle, the graviton of spin $s = 2\hbar$.

1.4 Problems of the Standard Model

The Standard Model is one of the cornerstones of all science and one of the great triumphs of the XX century. It has been carefully experimentally verified in many ways, especially during the past 20 years, but there are many questions it cannot answer:

- Quantum Gravity. The standard model describes three of the four fundamental interactions at the quantum level and therefore microscopically. However, gravity is only treated classically and any quantum discussion of gravity has to be considered as an effective field theory valid at scales smaller than the Planck scale ($M_{pl} = \sqrt{\frac{Gh}{c^3}} \approx 10^{19}$GeV). At this scale quantum effects of gravity have to be included and then Einstein theory has the problem of being non-renormalizable and therefore it cannot provide proper answers to observables beyond this scale.

- Why $G_{SM} = SU(3) \otimes SU(2) \otimes U(1)$? Why there are four interactions and three families of fermions? Why 3 + 1 spacetime - dimensions? Why there are some 20 parameters (masses and couplings between particles) in the standard model for which their values are only determined to fit experiment without any theoretical understanding of these values?

- Confinement. Why quarks can only exist confined in hadrons such as protons and neutrons? The fact that the strong interactions are asymptotically free (meaning that the value of the coupling increases with decreasing energy) indicates that this is due to the fact that at the relatively low energies we can explore the strong interactions are so strong that do not allow quarks to separate. This is an issue about our ignorance to treat strong coupling field theories which are not well understood because standard (Feynman diagrams) perturbation theory cannot be used.

- The "hierarchy problem". Why there are totally different energy scales

$$M_{EW} \approx 10^2\text{GeV}\ , \qquad M_{pl} = \sqrt{\frac{Gh}{c^3}} \approx 10^{19}\text{GeV} \quad \Longrightarrow \quad \frac{M_{EW}}{M_{pl}} \approx 10^{-15}$$

  This problem has two parts. First why these fundamental scales are so different which may not look that serious. The second part refers to a naturalness issue. A fine tuning of many orders of magnitude has to be performed order by order in perturbation theory in order to avoid the electroweak scale M_{EW} to take the value of the "cut-off" scale which can be taken to be M_{pl}.

- The strong CP problem. There is a coupling in the standard model of the form $\theta F^{\mu\nu}\tilde{F}_{\mu\nu}$ where θ is a parameter, $F^{\mu\nu}$ refers to the field strength of quantum chromodynamics (QCD) and $\tilde{F_{\mu\nu}} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. This term breaks the symmetry CP (charge conjugation followed by parity). The problem refers to the fact that the parameter θ is unnaturally small $\theta < 10^{-8}$. A parameter can be made naturally small by the t'Hooft naturalness criterion in which a parameter is naturally small if setting it to zero implies there is a symmetry protecting its value. For this problem, there is a concrete proposal due to Peccei and Quinn in which, adding a new particle, the axion, with coupling $aF^{\mu\nu}\tilde{F}_{\mu\nu}$, then the corresponding Lagrangian will be symmetric under $a \to a + c$ which is the PQ symmetry. This solves the strong CP problem because non-perturbative QCD effects introduce a potential for a with minimum at $a = 0$ which would correspond to $\theta = 0$.

- The "cosmological constant problem". Observations about the accelerated expansion of the universe indicate that the cosmological constant interpreted as the energy of the vacuum is near zero, $\Lambda \approx 10^{-120} M_{pl}^4$

$$\frac{M_\Lambda}{M_{EW}} \approx 10^{-15}$$

This is probably the biggest puzzle in theoretical physics. The problem, similar to the hierarchy problem, is the issue of naturalness. There are many contributions within the standard model to the value of the vacuum energy and they all have to cancel to 60-120 orders of magnitude in order to keep the cosmological constant small after quantum corrections for vacuum fluctuations are taken into account.

All of this indicates that the standard model is not the fundamental theory of the universe and we need to find extension that could solve some or all of the problems mentioned above in order to generalize the standard model. The standard model is expected to be only an effective theory describing the fundamental theory at low energies.

In order to go beyond the standard model we can follow several avenues.

- Experiments. This is the traditional way of making progress in science. We need experiments to explore energies above the currently attainable scales and discover new particles and underlying principles that generalize the standard model. This avenue is presently important due to the imminent starting of the LHC collider experiemt at CERN, Geneva in the summer of 2007. This experiment will explore physics at the 10^3 GeV scale and may discover the last remaining particle of the standard model, known as the Higgs particle, as well as new physics beyond the standard model. Notice that to explore energies closer to the Planck scale $M_{pl} \sim 10^{18}$ GeV is out of the reach for many years to come.

- Add new particles/interactions. This is an *ad hoc* technique is not well guided but it is possible to follow if by doing this we are addressing some of the questions mentioned before.

- More general symmetries. We understand by now the power of symmetries in the foundation of the standard model, it is then natural to use this as a guide and try to generalize it by adding more symmetries. These can be of the two types mentioned before: more general internal symmetries leads to consider Grand Unified Theories (GUTs) in which the symmetries of the standard model are themselves the result of the breaking of yet a larger symmetry group.

$$G_{GUT} \quad \stackrel{M\approx 10^{17}\mathrm{GeV}}{\longrightarrow} \quad G_{SM} \quad \stackrel{M\approx 10^{2}\mathrm{GeV}}{\longrightarrow} \quad SU(3)\otimes U(1)\ ,$$

This proposal is very elegant because it unifies, in one single symmetry, the three gauge interactions of the standard model. It leaves unanswered most of the open questions above, except for the fact that it reduces the number of independent parameters due to the fact that there is only one gauge coupling at large energies. This is expected to "run" at low-energies and give rise to the three different couplings of the standard model (one corresponding to each group factor). Unfortunately, with our present precision understanding of the gauge couplings and spectrum of the standard model, the running of the three gauge couplings *does not* unify at a single coupling at higher energies but they cross each other at different energies.

More general spacetime symmetries open-up many more interesting avenues. These can be of two types. First we can add more dimensions to spacetime, therefore the Poincaré - symmetries of the standard model and more generally the general coordinate transformations of general relativity, get substantially enhanced. This is the well known Kaluza - Klein theory in which our observation of a four-dimensional universe is only due to the fact that we have limitations about "seeing" other dimensions of spacetime that may be hidden to our experiments.

In recet years this has been extended to the "brane - world" scenario in which our four-dimensional universe is only a brane or surface inside a larger dimensional universe. These ideas approach very few of the problems of the standard model. They may lead to a different perspective of the hierarchy problem and also about the possibility to unify internal and spacetime symmetries.

The second option is supersymmetry. Supersymmetry is a spacetime symmetry, despite the fact that it is seen as a transformation that exchanges bosons and fermions. Supersymmetry solves the naturalness issue (the most important part) of the hierarchy problem due to cancellations between the contributions of bosons and fermions to the electroweak scale, defined by the Higgs mass. Combined with the GUT idea, it solves the unification of the three gauge couplings at one single point at larger energies. Supersymmetry also provides the best example for dark matter candidates. It also provides well defined QFTs in which issues of strong coupling can be better studied than in the non-supersymmetric models.

- Beyond QFT. Supersymmetry and extra dimensions do not address the most fundamental problem mentioned above, that is the problem of quantising gravity. For this the best hope is string theory which goes beyond our basic framework of QFT. It so happens that for its consistency string theory requires supersymmetry and extra dimensions also. This gives a further motivation to study these two areas which are the subject of this course.

Chapter 2

Supersymmetry Algebra and Representations

2.1 Poincaré Symmetry and Spinors

The Poincaré group corresponds to the basic symmetries of special relativity, it acts on spacetime coordinates x^μ as follows:

$$x^\mu \quad \longmapsto \quad x'^\mu \quad = \quad \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} \, x^\nu + \underbrace{a^\mu}_{\text{translation}}$$

Lorentz transformations leave the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1 \ , \ -1 \ , \ -1 \ , \ -1)$ invariant:

$$\Lambda^T \eta \Lambda \quad = \quad \eta$$

They can be separated between those that are connected to the identity and this that are not (like parity for which $\Lambda = \mathrm{diag}(1 \ , \ -1 \ , \ -1 \ , \ -1)$. We will mostly discuss those Λ connected to identity, i.e. the proper orthochronous group $SO(3,1)^\uparrow$. Generators for the Poincaré group are the $M^{\mu\nu}$, P^σ with algebra

$$\begin{aligned} \Big[P^\mu \ , \ P^\nu\Big] \quad &= \quad 0 \\ \Big[M^{\mu\nu} \ , \ P^\sigma\Big] \quad &= \quad i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \\ \Big[M^{\mu\nu} \ , \ M^{\rho\sigma}\Big] \quad &= \quad i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}) \end{aligned}$$

A four-dimensional matrix representation for the $M^{\mu\nu}$ is

$$(M^{\rho\sigma})^\mu{}_\nu \quad = \quad i(\eta^{\mu\nu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\sigma{}_\nu)$$

2.1.1 Properties of Lorentz - Group

- Locally, we have a correspondence

$$SO(3,1) \quad \cong \quad SU(2) \oplus SU(2) \ ,$$

the generators J_i of rotations and K_i of Lorentz - boosts can be expressed as

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} , \qquad K_i = M_{0i} ,$$

and their linear combinations (which are not hermitian)

$$A_i = \frac{1}{2}(J_i + iK_i) , \qquad B_i = \frac{1}{2}(J_i - iK_i)$$

satisfy $SU(2)$ commutation - relations

$$\Big[A_i , A_j\Big] = i\epsilon_{ijk}A_k , \qquad \Big[B_i , B_j\Big] = i\epsilon_{ijk}B_k , \qquad \Big[A_i , B_j\Big] = 0 .$$

Under parity P ($x^0 \mapsto x^0$ and $\vec{x} \mapsto -\vec{x}$) we have

$$J_i \longmapsto J_i , \qquad K_i \longmapsto -K_i \quad \Longrightarrow \quad A_i \longleftrightarrow B_i .$$

We can interpret $\vec{J} = \vec{A} + \vec{B}$ as the physical spin.

- On the other hand, there is a homeomorphism (not an isomorphism)

$$SO(3,1) \cong SL(2,\mathbb{C}) :$$

Take a 4 - vector X and a corresponding 2×2 - matrix $\tilde{x}$,

$$X = x_\mu e^\mu = (x_0 , x_1 , x_2 , x_3) , \qquad \tilde{x} = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} ,$$

where σ^μ is the 4 - vector of Pauli - matrices

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} .$$

Transformations $X \mapsto \Lambda X$ under $SO(3,1)$ leaves the square

$$|X|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

invariant, whereas the action of $SL(2,\mathbb{C})$ mapping $\tilde{x} \mapsto N\tilde{x}N^\dagger$ with $N \in SL(2,\mathbb{C})$ preserves the determinant

$$\det \tilde{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2 .$$

The map between $SL(2,\mathbb{C})$ is 2 - 1, since $N = \pm\mathbb{1}$ both correspond to $\Lambda = \mathbb{1}$, but $SL(2,\mathbb{C})$ has the advantage to be simply connected, so $SL(2,\mathbb{C})$ is the universal covering group.

2.1.2 Representations and Invariant Tensors of SL(2,C)

The basic representations of $SL(2,\mathbb{C})$ are:

- The fundamental representation

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta , \qquad \alpha, \beta = 1, 2$$

The elements of this representation ψ_α are called left-handed Weyl spinors.

- The conjugate representation

$$\bar{\chi}'_{\dot{\alpha}} \;=\; N^{*\;\dot{\beta}}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}}\;, \qquad \dot{\alpha},\dot{\beta} = 1,2$$

  Here $\bar{\chi}_{\dot{\beta}}$ are called right-handed Weyl spinors.

- The contravariant representations

$$\psi'^{\alpha} \;=\; \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha}\;, \qquad \bar{\chi}'^{\dot{\alpha}} \;=\; \bar{\chi}^{\dot{\beta}}(N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}$$

The fundamental and conjugate representations are the basic representations of $SL(2,\mathbb{C})$ and the Lorentz group, giving then the importance to spinors as the basic objects of special relativity, a fact that could be missed by not realising the connection of the Lorentz group and $SL(2,\mathbb{C})$. We will see next that the contravariant representations are however not independent.

To see this we will consider now the different ways to raise and lower indices.

- The metric tensor $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$ is invariant under $SO(3,1)$.

- The analogy within $SL(2,\mathbb{C})$ is

$$\epsilon^{\alpha\beta} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; -\epsilon_{\alpha\beta}\;,$$

  since

$$\epsilon'^{\alpha\beta} \;=\; \epsilon^{\rho\sigma}N_{\rho}{}^{\alpha}N_{\sigma}{}^{\beta} \;=\; \epsilon^{\alpha\beta}\cdot\det N\;.$$

  That is why ϵ is used to raise and lower indices

$$\psi^{\alpha} \;=\; \epsilon^{\alpha\beta}\psi_{\beta}\;, \qquad \bar{\chi}^{\dot{\alpha}} \;=\; \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}\;,$$

  so contravariant representations are not independent.

- To handle mixed $SO(3,1)$- and $SL(2,\mathbb{C})$ - indices, recall that the transformed components x_{μ} should look the same, whether we transform the vector X via $SO(3,1)$ or the matrix $\tilde{x} \;=\; x_{\mu}\sigma^{\mu}$

$$(x_{\mu}\sigma^{\mu})_{\alpha\dot{\alpha}} \;\longmapsto\; N_{\alpha}{}^{\beta}(x_{\nu}\sigma^{\nu})_{\beta\dot{\gamma}}N^{*\;\dot{\gamma}}_{\dot{\alpha}} \;=\; \Lambda_{\mu}{}^{\nu}x_{\nu}\sigma^{\mu}\;,$$

  so the right transformation rule is

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} \;=\; N_{\alpha}{}^{\beta}(\sigma^{\nu})_{\beta\dot{\gamma}}(\Lambda^{-1})^{\mu}{}_{\nu}N^{*\;\dot{\gamma}}_{\dot{\alpha}}\;.$$

  Similar relations hold for

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} \;:=\; \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^{\mu})_{\beta\dot{\beta}} \;=\; (\mathbb{1}\;,\;-\vec{\sigma})\;.$$

2.1.3 Generators of SL(2,C)

Define tensors $\sigma^{\mu\nu}$, $\bar{\sigma}^{\mu\nu}$

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} \;=\; \frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}{}^{\beta}$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \;=\; \frac{i}{4}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}$$

which satisfy the Lorentz - algebra. Spinors transform like

$$\psi_\alpha \quad \longmapsto \quad \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\psi_\beta \qquad \text{(left - handed)}$$
$$\bar{\chi}^{\dot{\alpha}} \quad \longmapsto \quad \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\chi}^{\dot{\beta}} \qquad \text{(right - handed)}$$

Now consider the spins with respect to the $SU(2)$s spanned by the A_i and B_i:

$$\psi_\alpha : \qquad (A\ ,\ B) = \left(\frac{1}{2}\ ,\ 0\right) \quad \Longrightarrow \quad J_i \ = \ \frac{1}{2}\sigma_i\ , \qquad K_i \ = \ -\frac{i}{2}\sigma_i$$
$$\bar{\chi}^{\dot{\alpha}} : \qquad (A\ ,\ B) = \left(0\ ,\ \frac{1}{2}\right) \quad \Longrightarrow \quad J_i \ = \ \frac{1}{2}\sigma_i\ , \qquad K_i \ = \ +\frac{i}{2}\sigma_i$$

Here are some useful identities concerning the σ^μ and $\sigma^{\mu\nu}$,

$$\begin{aligned}
\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu &= 2\eta^{\mu\nu}\mathbb{1} \\
\mathrm{Tr}\Big\{\sigma^\mu\bar{\sigma}^\nu\Big\} &= 2\eta^{\mu\nu} \\
(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} &= 2\delta_\alpha^{\ \beta}\delta_{\dot{\alpha}}^{\ \dot{\beta}} \\
\sigma^{\mu\nu} &= \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \\
\bar{\sigma}^{\mu\nu} &= -\frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}\ ,
\end{aligned}$$

the last of which are known as self - duality and anti - self - duality. These are important because naively $\sigma^{\mu\nu}$ being antisymmetric seems to have $4 \times 3/2$ components, but the self-duality conditions reduces this by half. A reference - book illustrating many of the calculations for two - component spinors is "Supersymmetry" (Müller, Kristen, Wiedermann).

2.1.4 Products of Weyl - Spinors

Define the product of two Weyl - spinors as

$$\begin{aligned}
\chi\psi &:= \chi^\alpha\psi_\alpha &&= -\chi_\alpha\psi^\alpha \\
\bar{\chi}\bar{\psi} &:= \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} &&= -\bar{\chi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}\ ,
\end{aligned}$$

particularly,

$$\psi^2 \ = \ \psi\psi \ = \ \psi^\alpha\psi_\alpha \ = \ \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha \ = \ \psi_2\psi_1 - \psi_1\psi_2\ .$$

Choose the ψ_α to be anticommuting Grassmann - numbers: $\psi_1\psi_2 = -\psi_2\psi_1$, so

$$\psi_\alpha\psi_\beta \ = \ \frac{1}{2}\epsilon_{\alpha\beta}(\psi\psi)\ , \qquad \chi\psi \ = \ \psi\chi\ , \qquad (\chi\psi)(\chi\psi) \ = \ -\frac{1}{2}(\psi\psi)(\chi\chi)\ .$$

From the definitions

$$\psi_\alpha^\dagger \ := \ \bar{\psi}_{\dot{\alpha}}\ , \qquad \bar{\psi}^{\dot{\alpha}} \ := \ \psi_\beta^*(\sigma^0)^{\beta\dot{\alpha}}$$

it follows that

$$\begin{aligned}
\psi\sigma^{\mu\nu}\chi &= -(\chi\sigma^{\mu\nu}\psi) \\
(\chi\psi)^\dagger &= \bar{\chi}\bar{\psi} \\
(\psi\sigma^\mu\bar{\chi})^\dagger &= \chi\sigma^\mu\bar{\psi}\ .
\end{aligned}$$

In general we can generate all higher dimensional representations of the Lorentz group by products of the fundamental representation $(1/2,0)$ and its conjugate $(0,1/2)$. For instance:

$$\psi_\alpha \bar{\chi}_{\dot{\alpha}} \;=\; \frac{1}{2}\left(\psi\sigma_\mu\bar{\chi}\right)\,\sigma^\mu_{\alpha\dot{\alpha}}.$$

In terms of the spins (A,B) this corresponds to the decomposition $(1/2,0)\otimes(0,1/2) \;=\; (1/2,1/2)$. Similarly:

$$\psi_\alpha\chi_\beta \;=\; \frac{1}{2}\,\epsilon_{\alpha\beta}\left(\psi\chi\right)+\frac{1}{2}\left(\sigma^{\mu\nu}\epsilon^T\right)_{\alpha\beta}\left(\psi\sigma_{\mu\nu}\chi\right)$$

Which corresponds to $(1/2,0)\otimes(1/2,0) \;=\; (0,0)\oplus(1,0)$. Notice that the counting of independent components of $\sigma^{\mu\nu}$ from its self-duality property, precisely provides the right number of components for the $(1,0)$ representation.

2.1.5 Dirac - Spinors

To connect the ideas of Weyl spinors with the more standard Dirac theory, define

$$\gamma^\mu \;:=\; \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

then these γ^μ satisfy the Clifford - algebra

$$\left\{\gamma^\mu\,,\,\gamma^\nu\right\} \;=\; 2\eta^{\mu\nu}\mathbb{1}\ .$$

The matrix γ^5, defined as

$$\gamma^5 \;:=\; i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

can have eigenvalues ± 1 (chirality). The generators of the Lorentz - group are

$$\Sigma^{\mu\nu} \;=\; \frac{i}{4}\gamma^{\mu\nu} \;=\; \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}.$$

Define Dirac - spinors to be

$$\Psi_D \;:=\; \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

such that the action of γ^5 is

$$\gamma^5\Psi_D \;=\; \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \;=\; \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

We can define the following projection operators P_L, P_R,

$$P_L \;:=\; \frac{1}{2}(\mathbb{1}-\gamma^5)\ , \qquad P_R \;:=\; \frac{1}{2}(\mathbb{1}+\gamma^5)\ ,$$

eliminate the part of one chirality, i.e.

$$P_L\Psi_D \;=\; \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad P_R\Psi_D \;=\; \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

Finally, define the Dirac - conjugate $\overline{\Psi}_D$ and charge - conjugate spinor ${\Psi_D}^C$ by

$$\begin{aligned} \overline{\Psi}_D \;&:=\; (\chi^\alpha \; , \; \bar{\psi}_{\dot{\alpha}}) \;=\; \Psi_D^\dagger \gamma^0 \\ {\Psi_D}^C \;&:=\; C\overline{\Psi}_D^T \;=\; \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \; , \end{aligned}$$

where C denotes the charge - conjugation - matrix

$$C \;:=\; \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \; .$$

Majorana - spinors Ψ_M have property $\psi_\alpha = \chi_\alpha$, so

$$\Psi_M \;=\; \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \;=\; {\Psi_M}^C \; .$$

Decompose general Dirac - spinors (and their charge - conjugates) as

$$\Psi_D \;=\; \Psi_{M1} + i\Psi_{M2} \; , \qquad {\Psi_D}^C \;=\; \Psi_{M1} - i\Psi_{M2} \; .$$

Note from this discussion that there can be no spinors in 4 dimensions which are both Majorana and Weyl.

2.2 Supersymmetry - Algebra

2.2.1 History of Supersymmetry

- In the 1960's, from the study of strong interactions, many hadrons have been discovered and were successfully organized in multiplets of $SU(3)_{flavour}$ In what was known as the "eightfold way" of Gell-Mann and Neeman. Questions arouse about bigger multiplets including particles of different spins.
- No - go - theorem (Coleman - Mandula 1967): most general symmetry of the S - matrix is Poincaré $\otimes$ internal, that cannot mix different spins
- Golfand + Licktman (1971): extended the Poincaré algebra to include spinor generators Q_α, where $\alpha = 1, 2$.
- Ramond + Neveu - Schwarz + Gervais + Sakita (1971): supersymmetry in 2 dimensions (from string theory).
- Volkov + Akulov (1973): neutrinos as Goldstone - particles ($m = 0$)

- Wess + Zumino (1974): supersymmetric field - theories in 4 dimensions. They opened the way to many other contributions to the field. This is generally seen as the actual starting point on systematic study of supersymmetry.

- Haag + Lopuszanski + Sohnius (1975): Generalized Coleman - Mandula - theorem including spinor - generators Q_α^A ($\alpha = 1, 2$ and $A = 1, ..., N$) corresponding to spins $(A\ ,\ B) = \left(\frac{1}{2}\ ,\ 0\right)$ and $\bar{Q}_{\dot{\alpha}}^A$ with $(A\ ,\ B) = \left(0\ ,\ \frac{1}{2}\right)$ in addition to P^μ and $M^{\mu\nu}$; but no further generators transforming in higher dimensional representations of the Lorentz group such as $\left(1\ ,\ \frac{1}{2}\right)$, etc.

2.2.2 Graded Algebra

In order to have a supersymmetric extension of the Poincaré algebra, we need to introduce the concept of "graded algebras". Let O_a be a operators of a Lie - algebra, then

$$O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a \quad = \quad i C^e{}_{ab} O_e \ ,$$

where gradings η_a take values

$$\eta_a \quad = \quad \begin{cases} 0 & : O_a \text{ bosonic generator} \\ 1 & : O_a \text{ fermionic generator} \end{cases} \quad .$$

For supersymmetry, generators are the Poincaré - generators P^μ, $M^{\mu\nu}$ and the spinor - generators Q_α^A, $\bar{Q}_{\dot{\alpha}}^A$, where $A = 1, ..., N$. In case $N = 1$ we speak of a simple SUSY, in case $N > 1$ of an extended SUSY. In this chapter, we will only discuss $N = 1$.

We know the commutation - relations $[P^\mu, P^\nu]$, $[P^\mu, M^{\rho\sigma}]$ and $[M^{\mu\nu}, M^{\rho\sigma}]$ from Poincaré - algebra, so we need to find

$$\text{(a)}\ \Big[Q_\alpha\ ,\ M^{\mu\nu}\Big]\ , \quad \text{(b)}\ \Big[Q_\alpha\ ,\ P^\mu\Big]\ ,$$
$$\text{(c)}\ \Big\{Q_\alpha\ ,\ Q_\beta\Big\}\ , \quad \text{(d)}\ \Big\{Q_\alpha\ ,\ \bar{Q}_{\dot{\beta}}\Big\}\ ,$$

also (for internal symmetry - generators T_i)

$$\text{(e)}\ \Big[Q_\alpha\ ,\ T_i\Big]\ .$$

- (a) $\Big[Q_\alpha\ ,\ M^{\mu\nu}\Big]$

  Since Q_α is a spinor, it transforms under the exponential of the $SL(2, \mathbb{C})$ - generators $\sigma^{\mu\nu}$:

$$Q'_\alpha \quad = \quad \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta \quad \approx \quad \left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta \ ,$$

  but Q_α is also an operator transforming under Lorentz - transformations $U = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$ to

$$Q'_\alpha \quad = \quad U^\dagger Q_\alpha U \quad \approx \quad \left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q_\alpha \left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \ .$$

Compare these two expressions for Q'_α up to first order in $\omega_{\mu\nu}$,

$$Q_\alpha - \frac{i}{2}\omega_{\mu\nu}\,(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \;=\; Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(Q_\alpha M^{\mu\nu} - M^{\mu\nu}Q_\alpha) + \mathcal{O}(\omega^2)$$

$$\Longrightarrow \quad \Big[Q_\alpha \,,\, M^{\mu\nu}\Big] \;=\; (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

- (b) $\Big[Q_\alpha \,,\, P^\mu\Big]$

  $c\cdot(\sigma^\mu)_{\alpha\dot\alpha}\bar Q^{\dot\alpha}$ is the only way of writing a sensible term with free indices μ, α which is linear in Q. To fix the constant c, consider $[\bar Q^{\dot\alpha}, P^\mu] = c^*\cdot(\bar\sigma)^{\dot\alpha\beta}Q_\beta$ (take adjoints using $(Q_\alpha)^\dagger = \bar Q_{\dot\alpha}$ and $(\sigma^\mu\bar Q)^\dagger_\alpha = (Q\sigma^\mu)_{\dot\alpha}$). The Jacobi - identity for P^μ, P^ν and Q_α

$$\begin{aligned}
0 \;&=\; \Big[P^\mu \,,\, \Big[P^\nu \,,\, Q_\alpha\Big]\Big] + \Big[P^\nu \,,\, \Big[Q_\alpha \,,\, P^\mu\Big]\Big] + \Big[Q_\alpha \,,\, \underbrace{\Big[P^\mu \,,\, P^\nu\Big]}_{0}\Big] \\
&=\; -c(\sigma^\nu)_{\alpha\dot\alpha}\Big[P^\mu \,,\, \bar Q^{\dot\alpha}\Big] + c(\sigma^\mu)_{\alpha\dot\alpha}\Big[P^\nu \,,\, \bar Q^{\dot\alpha}\Big] \\
&=\; |c|^2(\sigma^\nu)^{\alpha\dot\alpha}(\bar{si}^\mu)^{\dot\alpha\beta}Q_\beta - |c|^2(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta}Q_\beta \\
&=\; |c|^2\,\underbrace{(\sigma^\nu\bar\sigma^\mu - \sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta}_{\neq 0}\,Q_\beta
\end{aligned}$$

  can only hold for general Q_β, if $c=0$, so

$$\Big[Q_\alpha \,,\, P^\mu\Big] \;=\; \Big[\bar Q^{\dot\alpha} \,,\, P^\mu\Big] \;=\; 0$$

- (c) $\Big\{Q_\alpha \,,\, Q_\beta\Big\}$

  Due to index - structure, that commutator should look like

$$\Big\{Q_\alpha \,,\, Q^\beta\Big\} \;=\; k\cdot(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}\ .$$

  Since the left hand side commutes with P^μ and the right hand side doesn't, the only consistent choice is $k=0$, i.e.

$$\Big\{Q_\alpha \,,\, Q_\beta\Big\} \;=\; 0$$

- (d) $\Big\{Q_\alpha \,,\, \bar Q_{\dot\beta}\Big\}$

  This time, index - structure implies an ansatz

$$\Big\{Q_\alpha \,,\, \bar Q_{\dot\beta}\Big\} \;=\; t(\sigma^\mu)_{\alpha\dot\beta}P_\mu\ .$$

  There is no way of fixing t, so, by convention, set $t=2$:

$$\Big\{Q_\alpha \,,\, \bar Q_{\dot\beta}\Big\} \;=\; 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu$$

Notice that two symmetry - transformations $Q_\alpha\bar Q_{\dot\beta}$ have the effect of a translation. Let $|B\rangle$ be a bosonic state and $|F\rangle$ a fermionic one, then

$$Q_\alpha|F\rangle \;=\; |B\rangle\ , \qquad \bar Q_{\dot\beta}|B\rangle \;=\; |F\rangle \quad\Longrightarrow\quad Q\bar Q:\ |B\rangle \;\longmapsto\; |B\rangle\ \text{(translated)}\ .$$

- (e) $\Big[Q_\alpha\ ,\ T_i\Big]$

  Usually, this commutator vanishes, exceptions are $U(1)$ - automorphisms of the supersymmetry algebra known as R-symmetry.

$$Q_\alpha \quad \longmapsto \quad \exp(i\lambda) Q_\alpha \ , \qquad \bar{Q}_{\dot\alpha} \quad \longmapsto \quad \exp(-i\lambda) \bar{Q}_{\dot\alpha} \ .$$

  Let R be a $U(1)$ - generator, then

$$\Big[Q_\alpha\ ,\ R\Big] \ = \ Q_\alpha \ , \qquad \Big[\bar{Q}_{\dot\alpha}\ ,\ R\Big] \ = \ -\bar{Q}_{\dot\alpha} \ .$$

2.2.3 Representations of the Poincaré - Group

Recall the rotation - group $\{J_i\}$ satisfying

$$\Big[J_i\ ,\ J_j\Big] \ = \ i\epsilon_{ijk} J_k \ .$$

The Casimir operator

$$J^2 \ = \ \sum_{i=1}^{3} J_i^2$$

commutes with all the J_i labels irreducible representations by eigenvalues $j(j+1)$ of J^2. Within these representations, diagonalize J_3 to eigenvalues $j_3 = -j, -j+1, ..., j-1, j$. States are labelled like $|j, j_3\rangle$.

Also recall the two Casimirs in Poincaré - group, one of which involves the Pauli - Ljubanski - vector W_μ,

$$W_\mu \ = \ \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$$

given by

$$C_1 \ = \ P^\mu P_\mu \ , \qquad C_2 \ = \ W^\mu W_\mu \ .$$

The C_i commute with all generators. Multiplets are labelled $|m, \omega\rangle$, eigenvalues m^2 of C_1 and eigenvalues of C_2. States within those irreducible representations carry the eigenvalue p^μ of the generator P^μ as a label. Notice that at this level the Pauli-Ljubanski vector only provides a short way to express the second Casimir. Even though W_μ has standard commutation relations with the generators of the Poincaré group $M_{\mu\nu}, P_\mu$ statitng that it transfrom as a vector under Lorentz transformations and commutes with $P\mu$ (invariant under translations), the commutator $[W_\mu, W_\nu] \sim \epsilon_{\mu\nu\rho\sigma} W_\rho \P_\sigma$ states that te W_μ's by themselves are not generators of any algebra.

To find more labels, take P^μ as given and look for all elements of the Lorentz - group that commute with P^μ. This defines little groups:

- Massive particles, $p^\mu = (m\ ,\ \underbrace{0\ ,\ 0\ ,\ 0}_{\text{invariant under rot.}})$, have rotations as their little group. Due to the antisymmetric $\epsilon_{\mu\nu\rho\sigma}$ in the W_μ, it follows

$$W_0 \;=\; 0\ , \qquad W_i \;=\; -mJ_i\ .$$

Every particle with nonzero mass is an irreducible representation of Poincaré - group with labels $|m, j; p^\mu, j_3\rangle$.

- Massless particles' momentum has the form $p^\mu = (E\ ,\ 0\ ,\ 0\ ,\ E)$ which implies

$$W_0 \;=\; EJ_3\ , \qquad W_1 \;=\; E(-J_1 + K_2)\ , \qquad W_2 \;=\; E(J_2 - K_1)\ , \qquad W_3 \;=\; EJ_3$$

$$\Longrightarrow \quad \Big[W_1\ ,\ W_2\Big] \;=\; 0\ , \qquad \Big[W_3\ ,\ W_1\Big] \;=\; iW_2\ , \qquad \Big[W_3\ ,\ W_2\Big] \;=\; -iW_1\ .$$

Commutation - relations are those for Euclidean group in two dimensions. For finite - dimensional representations, $SO(2)$ is a subgroup and W_1, W_2 have to be zero. In that case, $W^\mu = \lambda P^\mu$ and states are labelled $|0, 0; p^\mu, \lambda\rangle =: |p^\mu, \lambda\rangle$, where λ is called helicity. Under CPT, those states transform to $|p^\mu, -\lambda\rangle$. The relation

$$\exp(2\pi i\lambda)|p^\mu, \lambda\rangle \;=\; \pm|p^\mu, \lambda\rangle$$

requires λ to be integer or half - integer $\lambda = 0, \frac{1}{2}, 1, ...$, e.g. $\lambda = 0$ (Higgs), $\lambda = \frac{1}{2}$ (quarks, leptons), $\lambda = 1$ (γ, $W^\pm$, Z^0, g) and $\lambda = 2$ (graviton).

2.2.4 N = 1 Supersymmetry Representations

For Supersymmetry with $N = 1$, $C_1 = P^\mu P_\mu$ is still a good casimir, $C_2 = W^\mu W_\mu$, however, is not. So one can have particles of different spin within one multiplet. To get a new casimir $\tilde{C}_2$ (corresponding to superspin), define

$$B_\mu \;:=\; W_\mu - \frac{1}{4}\bar{Q}_{\dot\alpha}(\sigma_\mu)^{\dot\alpha\beta}Q_\beta\ , \qquad C_{\mu\nu} \;:=\; B_\mu P_\nu - B_\nu P_\mu$$

$$\tilde{C}_2 \;:=\; C_{\mu\nu}C^{\mu\nu}\ .$$

Proposition 1

In any supersymmetric multiplet, the number n_B of bosons equals the number n_F of fermions,

$$n_B \;=\; n_F\ .$$

Proof 1

Consider the fermion - number - operator $(-1)^F = (-)^F$, defined via

$$(-)^F|B\rangle = |B\rangle \ , \qquad (-)^F|F\rangle = -|F\rangle \ .$$

The new operator $(-)^F$ anticommutes with Q_α since

$$(-)^F Q_\alpha|F\rangle = (-)^F|B\rangle = |B\rangle = Q_\alpha|F\rangle = -Q_\alpha(-)^F|F\rangle \implies \Big\{(-)^F \ , \ Q_\alpha\Big\} = 0 \ .$$

Next, consider the trace

$$\begin{aligned}\mathrm{Tr}\Big\{(-)^F\Big\{Q_\alpha \ , \ \bar{Q}_{\dot{\beta}}\Big\}\Big\} &= \mathrm{Tr}\Big\{\underbrace{(-)^F Q_\alpha}_{\text{anticommute}} \ \bar{Q}_{\dot{\beta}} + \underbrace{(-)^F\bar{Q}_{\dot{\beta}}Q_\alpha}_{\text{cyclic perm.}}\Big\} \\ &= \mathrm{Tr}\Big\{-Q_\alpha(-)^F\bar{Q}_{\dot{\beta}} + Q_\alpha(-)^F\bar{Q}_{\dot{\beta}}\Big\} = 0 \ .\end{aligned}$$

On the other hand, it can be evaluated using $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$,

$$\mathrm{Tr}\Big\{(-)^F\Big\{Q_\alpha \ , \ \bar{Q}_{\dot{\beta}}\Big\}\Big\} = \mathrm{Tr}\Big\{(-)^F 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\Big\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}p_\mu \mathrm{Tr}\Big\{(-)^F\Big\} \ ,$$

where P^μ is replaced by its eigenvalues p^μ for the specific state. The conclusion is

$$0 = \mathrm{Tr}\Big\{(-)^F\Big\} = \sum_{\text{bosons}}\langle B|(-)^F|B\rangle + \sum_{\text{fermions}}\langle F|(-)^F|F\rangle = \sum_{\text{bosons}}\langle B|B\rangle - \sum_{\text{fermions}}\langle F|F\rangle = n_B - n_F \ .$$

2.2.5 Massless Supermultiplet

States of massless particles have P^μ - eigenvalues $p^\mu = (E \ , \ 0 \ , \ 0 \ , \ E)$. The casimirs $C_1 = P^\mu P_\mu$ and $\tilde{C}_2 = C_{\mu\nu}C^{\mu\nu}$ are zero. Consider the algebra

$$\Big\{Q_\alpha \ , \ \bar{Q}_{\dot{\beta}}\Big\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P^\mu = 2E(\sigma^0+\sigma^3)_{\alpha\dot{\beta}} = 4E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \ ,$$

which implies that Q_2 is zero in the representation:

$$\Big\{Q_2 \ , \ \bar{Q}_{\dot{2}}\Big\} = 0 \implies \langle p^\mu, \lambda|\bar{Q}_{\dot{2}}Q_2|\tilde{p}^\mu, \tilde{\lambda}\rangle = 0 \implies Q_2 = 0$$

The Q_1 satisfy $\{Q_1, \bar{Q}_{\dot{1}}\} = 4E$, so defining creation- and annihilation - operators a and $a^\dagger$ via

$$a := \frac{Q_1}{2\sqrt{E}} \ , \qquad a^\dagger := \frac{\bar{Q}_{\dot{1}}}{2\sqrt{E}} \ ,$$

get the anticommutation - relations

$$\Big\{a \ , \ a^\dagger\Big\} = 1 \ , \qquad \Big\{a \ , \ a\Big\} = \Big\{a^\dagger \ , \ a^\dagger\Big\} = 0 \ .$$

Also, since $[a, J^3] = \frac{1}{2}(\sigma^3)_{11}a = \frac{1}{2}a$,

$$J^3\big(a|p^\mu,\lambda\rangle\big) = \left(aJ^3 - \left[a\ ,\ J^3\right]\right)|p^\mu,\lambda\rangle = \left(aJ^3 - \frac{1}{2}a\right)|p^\mu,\lambda\rangle = \left(\lambda - \frac{1}{2}\right)a|p^\mu,\lambda\rangle\ .$$

$a|p^\mu,\lambda\rangle$ has helicity $\lambda - \frac{1}{2}$, and by similar reasoning, find that the helicity of $a^\dagger|p^\mu,\lambda\rangle$ is $\lambda + \frac{1}{2}$. To build the representation, start with a vacuum - state of mimimum helicity λ, let's call it $|\Omega\rangle$. Obviously $a|\Omega\rangle = 0$ (otherwise $|\Omega\rangle$ would not have lowest helicity) and $a^\dagger a^\dagger|\Omega\rangle = 0|\Omega\rangle = 0$, so the whole multiplet consists of

$$|\Omega\rangle = |p^\mu,\lambda\rangle\ , \qquad a^\dagger|\Omega\rangle = |p^\mu,\lambda + \frac{1}{2}\rangle\ .$$

Add the CPT - conjugate to get

$$|p^\mu,\pm\lambda\rangle\ , \qquad |p^\mu,\pm\left(\lambda + \frac{1}{2}\right)\rangle\ .$$

There are, for example, chiral multiplets with $\lambda = 0, \frac{1}{2}$, vector- or gauge - multiplets ($\lambda = \frac{1}{2}, 1$ - gauge and gaugino)

$\lambda = 0$ scalar	$\lambda = \frac{1}{2}$ fermion
squark	quark
slepton	lepton
Higgs	Higgsino

,

$\lambda = \frac{1}{2}$ fermion	$\lambda = 1$ boson
photino	photon
gluino	gluon
Wino , Zino	W , Z

,

as well as the graviton with its partner

$\lambda = \frac{3}{2}$ fermion	$\lambda = 2$ boson
gravitino	graviton

2.2.6 Massive Supermultiplet

In case of $m \neq 0$, there are P^μ - eigenvalues $p^\mu = (m\ ,\ 0\ ,\ 0\ ,\ 0)$ and Casimirs

$$C_1 = P^\mu P_\mu = m^2\ , \qquad \tilde{C}_2 = C_{\mu\nu}C^{\mu\nu} = 2m^4 Y^i Y_i\ ,$$

where Y_i denotes superspin

$$Y_i = J_i - \frac{1}{4m}\bar{Q}\bar{\sigma}_i Q = \frac{B_i}{m}\ , \qquad \left[Y_i\ ,\ Y_j\right] = i\epsilon_{ijk}Y_k\ .$$

Eigenvalues to $Y^2 = Y^i Y_i$ are $y(y+1)$, so label irreducible representations by $|m, y\rangle$. Again, the anti-commutation - relation for Q and $\bar{Q}$ is the key to get the states:

$$\left\{Q_\alpha\ ,\ \bar{Q}_{\dot{\beta}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu = 2m(\sigma^0)_{\alpha\dot{\beta}} = 2m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}}$$

Since both Q's have nonzero anticommutators with their $\bar{Q}$ - partner, define two sets of ladder - operators

$$a_{1,2} := \frac{Q_{1,2}}{\sqrt{2m}}\ , \qquad a^\dagger_{1,2} := \frac{\bar{Q}_{\dot{1},\dot{2}}}{\sqrt{2m}}\ ,$$

with anticommutation - relations

$$\left\{a_p\ ,\ a_q^\dagger\right\} \;=\; \delta_{pq}\ , \qquad \left\{a_p\ ,\ a_q\right\} \;=\; \left\{a_p^\dagger\ ,\ a_q^\dagger\right\} \;=\; 0\ .$$

Let $|\Omega\rangle$ be the vacuum state, annihilated by $a_{1,2}$. Conseqently,

$$Y_i|\Omega\rangle \;=\; J_i|\Omega\rangle - \frac{1}{4m}\bar{Q}\sigma_i\sqrt{2m}\,\underbrace{a|\Omega\rangle}_{0} \;=\; J_i|\Omega\rangle\ ,$$

i.e. for $|\Omega\rangle$ the spin number j and superspin - number y are the same. So for given m, y:

$$|\Omega\rangle \;=\; |m, j = y; p^\mu, j_3\rangle$$

Obtain the rest of the multiplet using

$$\begin{aligned} a_1|j_3\rangle &= |j_3 - \frac{1}{2}\rangle\ , & a_1^\dagger|j_3\rangle &= |j_3 + \frac{1}{2}\rangle \\ a_2|j_3\rangle &= |j_3 + \frac{1}{2}\rangle\ , & a_2^\dagger|j_3\rangle &= |j_3 - \frac{1}{2}\rangle\ , \end{aligned}$$

where $a_p^\dagger$ acting on $|\Omega\rangle$ behave like coupling of two spins j and $\frac{1}{2}$. This will yield a linear combination of two possible total spins $j+\frac{1}{2}$ and $j-\frac{1}{2}$ with Clebsch - Gordan - coefficients k_i (recall $j\otimes 1/2 = |j-1/2|\oplus$J$+1/2|$):

$$\begin{aligned} a_1^\dagger|\Omega\rangle &= k_1|m, j = y + \frac{1}{2}; p^\mu, j_3 + \frac{1}{2}\rangle + k_2|m, j = y - \frac{1}{2}; p^\mu, j_3 + \frac{1}{2}\rangle \\ a_2^\dagger|\Omega\rangle &= k_3|m, j = y + \frac{1}{2}; p^\mu, j_3 - \frac{1}{2}\rangle + k_4|m, j = y - \frac{1}{2}; p^\mu, j_3 - \frac{1}{2}\rangle\ . \end{aligned}$$

The remaining states

$$a_2^\dagger a_1^\dagger|\Omega\rangle \;=\; -a_1^\dagger a_2^\dagger|\Omega\rangle \;\propto\; |\Omega\rangle$$

represent spin j - objects. In total, we have

$$\underbrace{2\cdot|m, j = y; p^\mu, j_3\rangle}_{(4y+2)\ \text{states}}\ , \qquad \underbrace{1\cdot|m, j = y + \frac{1}{2}; p^\mu, j_3\rangle}_{(2y+3)\ \text{states}}\ , \qquad \underbrace{1\cdot|m, j = y - \frac{1}{2}; p^\mu, j_3\rangle}_{(2y+1)\ \text{states}}\ ,$$

in a $|m, y\rangle$ - multiplet, which is of course an equal number of bosonic and fermionic states. Notice that in labelling the states we have the value of m and y *fixed* throughout the multplet and the values of j change state by state, as it should since in a supersymmetric multplet there are states of different spin.

The case $y = 0$ needs to be treated separately:

$$\begin{aligned} |\Omega\rangle &= |m, j = 0; p^\mu, j_3 = 0\rangle \\ a_{1,2}^\dagger|\Omega\rangle &= |m, j = \frac{1}{2}; p^\mu, j_3 = \pm\frac{1}{2}\rangle \\ a_1^\dagger a_2^\dagger|\Omega\rangle &= |m, j = 0; p^\mu, j_3 = 0\rangle \;=:\; |\Omega'\rangle \end{aligned}$$

Parity interchanges $(A\ ,\ B) \leftrightarrow (B\ ,\ A)$, i.e. $(\frac{1}{2}\ ,\ 0) \leftrightarrow (0\ ,\ \frac{1}{2})$. Since $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu$, need the following transformation - rules for Q_α and $\bar{Q}_{\dot\alpha}$ under parity P (with phase factor η_P such that $|\eta_P| = 1$):

$$\begin{aligned} PQ_\alpha P^{-1} &= \eta_P(\sigma^0)_{\alpha\dot\beta}\bar{Q}^{\dot\beta} &= \eta_P(\sigma^0)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar{Q}_{\dot\gamma} \\ P\bar{Q}_{\dot\alpha}QP^{-1} &= \eta_P^*(\bar\sigma^0)_{\dot\alpha\beta}Q^\beta &= \eta_P^*(\bar\sigma^0)_{\dot\alpha\beta}\epsilon^{\beta\gamma}Q_\gamma \end{aligned}$$

That ensures $P^\mu \mapsto (P^0\ ,\ -\vec{P})$ and has the interesting effect $P^2QP^{-2} = -Q$. Moreover, consider the two $j = 0$ - states $|\Omega\rangle$ and $|\Omega'\rangle$: The first is annihilated by a_i, the second one by $a_i^\dagger$. Due to $Q \leftrightarrow \bar{Q}$, partiy interchanges a_i and $a_i^\dagger$ and therefore $|\Omega\rangle \leftrightarrow |\Omega'\rangle$. To get vacuum - states with a definied parity, we need linear combinations

$$|\pm\rangle \ := \ |\Omega\rangle \pm |\Omega'\rangle \ , \qquad P|\pm\rangle \ = \ \pm 1 \cdot |\pm\rangle \ .$$

Those states are called scalar ($|+\rangle$) and pseudoscalar ($|-\rangle$).

2.3 Extended Supersymmetry

Having discussed the algebra and representations of simple ($N = 1$) supersymmetry, we will turn now to the more general case of extended supersymmetry $N > 1$.

2.3.1 Algebra of Extended Supersymmetry

Now, the spinor - generators get an additional label $A, B = 1, 2, ..., N$. The algebra is the same as for $N = 1$ except for

$$\begin{aligned}
\left\{Q_\alpha^A\ ,\ \bar{Q}_{\dot{\beta}B}\right\} \ &= \ 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^A{}_B \\
\left\{Q_\alpha^A\ ,\ Q_\beta^B\right\} \ &= \ \epsilon_{\alpha\beta}Z^{AB}
\end{aligned}$$

with antisymmetric central - charges $Z^{AB} = -Z^{BA}$ commuting with all the generators

$$\left[Z^{AB}\ ,\ P^\mu\right] \ = \ \left[Z^{AB}\ ,\ M^{\mu\nu}\right] \ = \ \left[Z^{AB}\ ,\ Q_\alpha^A\right] \ = \ \left[Z^{AB}\ ,\ Z^{CD}\right] \ = \ \left[Z^{AB}\ ,\ T_a\right] \ = \ 0 \ .$$

They form an abelian invariant subalgebra of internal symmetries. Recall that $[T_a, T_b] = iC_{abc}T_c$. Let G be an internal symmetry group, then define the R - symmetry $H \subset G$ to be the set of G - elements that do not commute with the Supersymmetry - generators, e.g. $T_a \in G$ satisfying

$$\left[Q_\alpha^A\ ,\ T_a\right] \ = \ S_a{}^A{}_B Q_\alpha^B \ \neq \ 0$$

is an element of H. If $Z^{AB} = 0$, then the R - symmetry is $H = U(N)$, but with $Z^{AB} \neq 0$, H will be a subgroup. The existence of central charges is the main new ingredient of extended supersymmetries. The derivation of the previous algebra is a straightforward generalisation of the one for $N = 1$ supersymmetry.

2.3.2 Massless Representations of $N > 1$ - Supersymmetry

As we did for $N = 1$, we will procede now to discuss massless and massive representations. We will start with the massless case which is simpler and has very important implications.

Let $p_\mu = (E\ ,\ 0\ ,\ 0\ ,\ E)$, then (similar to $N=1$).

$$\left\{Q^A_\alpha\ ,\ \bar{Q}_{\dot\beta B}\right\} \quad = \quad 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^A{}_B \quad \Longrightarrow \quad Q^A_2 \quad = \quad 0$$

We can immmediately see from this that the central charges Z^{AB} vanish since $Q^A_2 = 0$ implies $Z^{AB} = 0$ from the anticommutators $\left\{Q^A_\alpha\ ,\ Q^B_\beta\right\} \quad = \quad \epsilon_{\alpha\beta} Z^{AB}$.

In order to obtain the full representation, define N creation- and annihilation - operators

$$a^A \quad := \quad \frac{Q^A_1}{2\sqrt{E}}\ , \qquad a^{A\dagger} \quad := \quad \frac{\bar{Q}^A_{\dot 1}}{2\sqrt{E}} \quad \Longrightarrow \quad \left\{a^A\ ,\ a^\dagger_B\right\} \quad = \quad \delta^A{}_B\ ,$$

to get the following states (starting from vacuum $|\Omega\rangle$, which is annihilated by all the a^A):

states	helicity	number of states
$\|\Omega\rangle$	λ_0	$1 = \begin{pmatrix} N \\ 0 \end{pmatrix}$
$a^{A\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{1}{2}$	$N = \begin{pmatrix} N \\ 1 \end{pmatrix}$
$a^{A\dagger}a^{B\dagger}\|\Omega\rangle$	$\lambda_0 + 1$	$\frac{1}{2!}N(N-1) = \begin{pmatrix} N \\ 2 \end{pmatrix}$
$a^{A\dagger}a^{B\dagger}a^{C\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{3}{2}$	$\frac{1}{3!}N(N-1)(N-2) = \begin{pmatrix} N \\ 3 \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$
$a^{N\dagger}a^{(N-1)\dagger}...a^{1\dagger}\|\Omega\rangle$	$\lambda_0 + \frac{N}{2}$	$1 = \begin{pmatrix} N \\ N \end{pmatrix}$

Note that the total number of states is given by

$$\sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} \quad = \quad \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} 1^k 1^{N-k} \quad = \quad 2^N\ .$$

Consider the following examples:

- $N=2$ vector - multiplet ($\lambda_0 = 0$)

$$\begin{array}{ccc} & \lambda = 0 & \\ \lambda = \frac{1}{2} & & \lambda = \frac{1}{2} \\ & \lambda = 1 & \end{array}$$

We can see that this $N=2$ multiplet can be decomposed in terms of $N=1$ multplets: one $N=1$ vector and one $N=1$ chiral multiplet.

- $N=2$ hyper - multiplet ($\lambda_0 = -\frac{1}{2}$)

$$\begin{array}{ccc} & \lambda = -\frac{1}{2} & \\ \lambda = 0 & & \lambda = 0 \\ & \lambda = \frac{1}{2} & \end{array}$$

Again this can be decomposed in terms of two $N=1$ chiral multiplets.

- $N = 4$ vector - multiplet ($\lambda_0 = -1$)

$$\begin{array}{ll} 1\times & \lambda = -1 \\ 4\times & \lambda = -\frac{1}{2} \\ 6\times & \lambda = \pm 0 \\ 4\times & \lambda = +\frac{1}{2} \\ 1\times & \lambda = +1 \end{array}$$

  This is the sinlge $N = 4$ multiplet with states of helicity $\lambda < 2$. It consists of one $N = 2$ vector multiplet and two $N = 2$ hypermultiplets plus their CPT conjugates (with opposite helicities). Or one $N = 1$ vector and three $N = 1$ chiral multiplets plus their CPT conjugates.

- $N = 8$ maximum - multiplet ($\lambda_0 = -2$)

$$\begin{array}{ll} 1\times & \lambda = \pm 2 \\ 8\times & \lambda = \pm\frac{3}{2} \\ 28\times & \lambda = \pm 1 \\ 56\times & \lambda = \pm\frac{1}{2} \\ 70\times & \lambda = \pm 0 \end{array}$$

From these results we can extract very important general conclusions:

- In every multiplet: $\lambda_{max} - \lambda_{\min} \quad = \quad \frac{N}{2}$

- Renormalizable theories have $|\lambda| \leq 1$ implying $N \leq 4$. Therefore $N = 4$ supersymmetry is the largest supersymmetry for renormalizable field theories. Gravity is not renormalizable!

- *The maximum number of supersymmetries is* $N = 8$. There is a strong belief that no massless particles of helicity $|\lambda| > 2$ exist (so only have $N \leq 8$). One argument is the fact that massless particle of $|\lambda| > \frac{1}{2}$ and low momentum couple to some conserved currents ($\partial_\mu j^\mu = 0$ in $\lambda = \pm 1$ - electromagnetism, $\partial_\mu T^{\mu\nu}$ in $\lambda = \pm 2$ - gravity). But there are no further conserved currents for $|\lambda| > 2$ (something that can also be seen from the Coleman-Mandula theorem). Also, $N > 8$ would imply that there is more than one graviton. See chapter 13 in Weinberg I on soft photons for a detailed discussion of this and the extension of his argument to supersymmetry in an article by Grisaru and Pendleton (1977). Notice this is not a full no-go theorem, in particular the constraint of low momentum has to be used.

- $N > 1$ - *supersymmetries are non - chiral.* We know that the Standard Model - particles live on complex fundamental representations. They are chiral since right handed quarks and leptons do not feel the weak interactions whereas left-handed ones do feel it (they are doublets under $SU(2)_L$). All $N > 1$ - multiplets, except for the $N = 2$ - hypermultiplet, have $\lambda = \pm 1$ - particles transforming in the adjoint representation which is real (recall that in $SU(N)$ theories the adjoint representation is obtained from theproduct of fundamental and complex conjugate representations and so is real) and therefore non - chiral. Then the $\lambda = \pm\frac{1}{2}$ - particle within the multiplet would transform in the same representation and therefore be non - chiral. The only exception is the $N = 2$ - hypermultiplets - for this the previous argument doesn't work because they do not include $\lambda = \pm 1$ states, but since $\lambda = \frac{1}{2}$- and $\lambda = -\frac{1}{2}$ - states are in the same multiplet, there can't be chirality either in this multiplet. Therefore only $N = 1, 0$ can be chiral, for instance $N = 1$ with $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ predicting at least one extra particle for each Standard Model - particle. But they have not been observed. Therefore the only hope for a realistic supersymmetric theory is: broken $N = 1$ - supersymmetry at low energies $E \approx 10^2$ GeV.

2.3.3 Massive Representations of $N > 1$ Supersymmetry and BPS States

Now consider $p_\mu = (m\ ,\ 0\ ,\ 0\ ,\ 0)$, so

$$\left\{ Q^A_\alpha\ ,\ \bar{Q}_{\dot{\beta}B} \right\} \ = \ 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^A{}_B\ .$$

Contrary to the massless case, here the central charges can be non-vanishing. Therefore we have to distinguish two cases:

- $Z^{AB} = 0$

  There are $2N$ creation- and annihilation - operators

$$a^A_\alpha \ := \ \frac{Q^A_\alpha}{\sqrt{2m}}\ , \qquad a^{A\dagger}_{\dot{\alpha}} \ := \ \frac{\bar{Q}^A_{\dot{\alpha}}}{\sqrt{2m}}$$

  leading to 2^{2N} states, each of them with dimension $(2y + 1)$. In the $N = 2$ case, we find:

$$\begin{array}{rc} |\Omega\rangle & 1 \times \text{spin } 0 \\ a^{A\dagger}_{\dot{\alpha}}|\Omega\rangle & 4 \times \text{spin } \frac{1}{2} \\ a^{A\dagger}_{\dot{\alpha}} a^{B\dagger}_{\dot{\beta}}|\Omega\rangle & 3 \times \text{spin } 0\ ,\ 3 \times \text{spin } 1 \\ a^{A\dagger}_{\dot{\alpha}} a^{B\dagger}_{\dot{\beta}} a^{C\dagger}_{\dot{\gamma}}|\Omega\rangle & 4 \times \text{spin } \frac{1}{2} \\ a^{A\dagger}_{\dot{\alpha}} a^{B\dagger}_{\dot{\beta}} a^{C\dagger}_{\dot{\gamma}} a^{D\dagger}_{\dot{\delta}}|\Omega\rangle & 1 \times \text{spin } 0 \end{array}\ ,$$

  i.e. as predicted $16 = 2^4$ states in total. Notice that these multplets are much larger than the massless ones with only 2^N states, due to the fact that in that case, half of the supersymmetry generators vanish ($Q^A_2 = 0$).

- $Z^{AB} \neq 0$

  Define the scalar quantity $\mathcal{H}$ to be

$$\mathcal{H} \ := \ (\bar{\sigma}^0)^{\dot{\beta}\alpha} \left\{ Q^A_\alpha - \Gamma^A_\alpha\ ,\ \bar{Q}_{\dot{\beta}A} - \bar{\Gamma}_{\dot{\beta}A} \right\} \ \geq \ 0\ .$$

  As a sum of products $AA^\dagger$, $\mathcal{H}$ is semi-positive, and the Γ^A_α are defined as

$$\Gamma^A_\alpha \ := \ \epsilon_{\alpha\beta} U^{AB} \bar{Q}_{\dot{\gamma}} (\bar{\sigma}^0)^{\dot{\gamma}\beta}$$

  for some unitary matrix U (satisfying $UU^\dagger = \mathbb{1}$). Anticommutation - relations from the supersymmetry - algebra imply

$$\mathcal{H} \ = \ 8mN - 2\ \mathrm{Tr}\left\{ ZU^\dagger + UZ^\dagger \right\} \ \geq \ 0\ .$$

Due to the polar - decomposition - theorem, each matrix Z can be written as a product $Z = HV$ of a positive hermitian $H = H^\dagger$ and a unitary phase - matrix $V = (V^\dagger)^{-1}$. Choose $U = V$, then

$$\mathcal{H} \;=\; 8mN - 4\,\mathrm{Tr}\Big\{H\Big\} \;=\; 8mN - 4\,\mathrm{Tr}\Big\{\sqrt{Z^\dagger Z}\Big\} \;\geq\; 0\,.$$

This is the BPS - bound for the mass m:

$$m \;\geq \frac{1}{2N}\,\mathrm{Tr}\Big\{\sqrt{Z^\dagger Z}\Big\}$$

States of minimal $m = \frac{1}{2N}\mathrm{Tr}\Big\{\sqrt{Z^\dagger Z}\Big\}$ are called BPS (Bogomolnyi-Prasad-Sommerfeld) - states. For these states the combination , $Q^A_\alpha - \Gamma^A_\alpha = 0$ so the multiplet is shorter (similar to the massless case in which $Q^a_2 = 0$) having only 2^N instead of 2^{2N} states.

In $N = 2$, define the components of the antisymmetric Z^{AB} to be

$$Z^{AB} \;=\; \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \;\Longrightarrow\; m \;\geq\; \frac{1}{2}q_1\,.$$

More generally, if $N > 2$ (but N even)

$$Z^{AB} \;=\; \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & \cdots & & \\ -q_1 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & q_2 & 0 & \cdots & & \\ 0 & 0 & -q_2 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ & & & & & & 0 & q_{\frac{N}{2}} \\ & & & & & & -q_{\frac{N}{2}} & 0 \end{pmatrix},$$

the BPS - conditions holds block by block: $2m \geq q_i$. To see that, define an $\mathcal{H}$ for each block. If k of the q_i are equal to $2m$, there are $2N - 2k$ creation - operators and $2^{2(N-k)}$ states.

$$\begin{aligned} k \;&=\; 0 &&\Longrightarrow\; 2^{2N} \text{ states, long multiplet} \\ 0 \;<\; k \;&<\; \frac{N}{2} &&\Longrightarrow\; 2^{2(N-k)} \text{ states, short multiplets} \\ k \;&=\; \frac{N}{2} &&\Longrightarrow\; 2^{N} \text{ states, ultra - short multiplet} \end{aligned}$$

Remarks:

– BPS - states and -bounds started in soliton - (monopole-) solutions of Yang - Mills - systems, which are localised finite-energy solutions of the classical equations of motion. The bound refers to an energy bound.

– The BPS - states are stable since they are the lightest charged particles.

– The equivalence of mass and charge reminds that of charged black holes. Actually, extremal black holes (which are the end points of the Hawking evaporation and therefore stable) happen to be BPS states for extended supergravity theories.

– BPS - states are important in understanding strong- / weak - coupling - dualities in field- and string - theory. In particular the fact that they correspond to short multiplets allows to extend them from weak to strong coupling since the size of a multplet is not expected to change by changing continuously the coupling from weak to strong.

– In string theory the extended objects known as D - branes are BPS.

Chapter 3

Superfields and Superspace

So far, we just considered supermultiplets, 1 particle - states. The goal is a supersymmetric field theory describing interactions. Recall that particles are described by fields $\varphi(x^\mu)$ with properties:

- function of coordinates x^μ in Minkowski - spacetime
- transformation of φ under Lorentz - group

We want objects $\Phi(X)$,

- function of coordinates X in superspace
- transformation of Φ under Super - Poincaré

But what is that superspace?

3.1 Basics

3.1.1 Groups and Cosets

We know that every continuous group G defines a manifold M_G via

$$\Lambda : \; G \; \longrightarrow \; M_G \; , \qquad \Big\{ g = \exp(i\alpha_a T^a) \Big\} \; \longrightarrow \; \Big\{ \alpha_a \Big\} \; ,$$

where $\dim G = \dim M_G$. Consider for example:

- $G = U(1)$ with elements $g = \exp(i\alpha Q)$, then $\alpha \in [0, 2\pi]$, so the corresponding manifold is the 1 - sphere (a circle) $M_{U(1)} = S^1$.
- $G = SU(2)$ with elements $g = \begin{pmatrix} p & q \\ -q^* & p^* \end{pmatrix}$, where complex parameters p and q satisfy $|p|^2 + |q|^2 = 1$. Write $p = x_1 + ix_2$ and $q = x_3 + ix_4$ for $x_k \in \mathbb{R}$, then the constraint for p, q implies $\sum_{k=1}^4 x_k^2 = 1$, so $M_{SU(2)} = S^3$

- $G = SL(2,\mathbb{C})$ with elements $g = HV$, $V \in SU(2)$ and $H = H^\dagger$ positive, $\det H = 1$. Writing the generic element $h \in H$ as $h = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$, the determinant - constraint is $(x_0)^2 - \sum_{k=1}^3 (x_k)^2 = 1$, so $M_{SL(2,\mathbb{C})} = \mathbb{R}^3 \times S^3$.

To be more general, let's define a coset G/H where $g \in G$ is identified with $gh \; \forall \; h \in H$, e.g.

- $G = U_1(1) \times U_2(1) \;\ni\; g = \exp\big(i(\alpha_1 Q_1 + \alpha_2 Q_2)\big)$, $H = U_1(1) \;\ni\; h = \exp(i\beta Q_1)$. In $G/H = \big(U_1(1) \times U_2(1)\big)/U_1(1)$, the identification is

$$gh \;=\; \exp\Big\{i\big((\alpha_1 + \beta)Q_1 + \alpha_2 Q_2\big)\Big\} \;=\; \exp\big(i(\alpha_1 Q_1 + \alpha_2 Q_2)\big) \;=\; g \; ,$$

  so only α_2 contains an effective information, $G/H = U_2(1)$.

- $SU(2)/U(1) \sim SO(3)/SO(2) = S^2$ This is the 2-sphere since $g \in SU(2)$ can be written as $g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$, identifying this by a $U(1)$ element $(e^{i\gamma}, e^{-i\gamma})$ makes α effectively real and therefore the parameter space is the 2-sphere $(\beta_1^2 + \beta_2^2 + \alpha^2 = 1)$.

- More generally $SO(n+1)/SO(n) = S^n$.

heightwidthdepthSUSY02.png

- Minkowski = Poincaré / Lorentz = $\{\omega^{\mu\nu}, a^\mu\}/\{\omega^{\mu\nu}\}$ simplifies to the translations $\{a^\mu = x^\mu\}$ which can be identified with Minkowski - space.

We define $N = 1$ - superspace to be the coset

$$\text{Super - Poincaré / Lorentz} \;=\; \Big\{\omega^{\mu\nu}, a^\mu, \theta^\alpha, \bar\theta_{\dot\alpha}\Big\}/\Big\{\omega^{\mu\nu}\Big\} \; .$$

Recall that the general element g of Super - Poincaré - group is given by

$$g \;=\; \exp\big(i(\omega^{\mu\nu} M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha} \bar Q^{\dot\alpha})\big) \; ,$$

where Grassmann - parameters θ^α, $\bar\theta_{\dot\beta}$ reduce anticommutation - relations for Q_α, $\bar Q^{\dot\beta}$ to commutation - relations:

$$\Big\{Q_\alpha \; , \; \bar Q_{\dot\alpha}\Big\} \;=\; 2(\sigma^\mu)_{\alpha\dot\alpha} P_\mu \quad\Longrightarrow\quad \Big[\theta^\alpha Q_\alpha \; , \; \bar\theta^{\dot\beta} \bar Q_{\dot\beta}\Big] \;=\; 2\theta^\alpha (\sigma^\mu)_{\alpha\dot\beta} \bar\theta^{\dot\beta} P_\mu$$

3.1.2 Properties of Grassmann - Variables

Recommendable books about Superspace are (Berezin), Supermanifolds (Bryce de Witt). Superspace was first introduced in (Salam + Strathdee 1974).

Let's first consider one single variable θ. When trying to expand a generic (analytic) function in θ as a power - series, the fact $\theta^2 = 0$ cancels all the terms except for two,

$$f(\theta) \;=\; \sum_{k=0}^\infty f_k \theta^k \;=\; f_0 + f_1\theta + f_2 \underbrace{\theta^2}_{0} + \underbrace{\ldots}_{0} \;=\; f_0 + f_1\theta \; ,$$

so the most general function $f(\theta)$ is linear. Of course, its derivative is given by $\frac{df}{d\theta} = f_1$. For integrals, define

$$\int d\theta \, \frac{df}{d\theta} \;:=\; 0 \quad\Longrightarrow\quad \int d\theta \;=\; 0 \;,$$

as if there were no boundary - terms. Integrals over θ are left to talk about: To get a non - trivial result, define

$$\int d\theta \; \theta \;:=\; 1 \quad\Longrightarrow\quad \delta(\theta) \;=\; \theta \;.$$

The integral over a function $f(\theta)$ is equal to its derivative,

$$\int d\theta \; f(\theta) \;=\; \int d\theta \; (f_0 + f_1\theta) \;=\; f_1 \;=\; \frac{df}{d\theta} \;.$$

Next, let θ^α, $\bar{\theta}_{\dot{\alpha}}$ be spinors of Grassmann - numbers. Their squares are defined by

$$\theta\theta \;:=\; \theta^\alpha\theta_\alpha \;, \qquad \bar{\theta}\bar{\theta} \;:=\; \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$$

$$\Longrightarrow \quad \theta^\alpha\theta^\beta \;=\; -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \;, \qquad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} \;=\; \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \;.$$

Derivatives work in analogy to Minkowski - coordinates:

$$\frac{\partial\theta^\beta}{\partial\theta^\alpha} \;=\; \delta_\alpha{}^\beta \quad\Longrightarrow\quad \frac{\partial\bar{\theta}^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}} \;=\; \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$

As to multi - integrals,

$$\int d\theta^1 \int d\theta^2 \; \theta^2\theta^1 \;=\; \frac{1}{2}\int d\theta^1 \int d\theta^2 \; \theta\theta \;=\; 1 \;,$$

which justifies the definition

$$\frac{1}{2}\int d\theta^1 \int d\theta^2 \;=:\; \int d^2\theta \;, \qquad \int d^2\theta \; \theta\theta \;=\; 1 \;, \qquad \int d^2\theta \int d^2\bar{\theta} \; (\theta\theta)(\bar{\theta}\bar{\theta}) \;=\; 1 \;,$$

or written in terms of ϵ:

$$d^2\theta \;=\; -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta} \;, \qquad d^2\bar{\theta} \;=\; \frac{1}{4}d\bar{\theta}^{\dot{\alpha}} d\bar{\theta}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} \;.$$

Identifying integration and differentiation,

$$\int d^2\theta \;=\; \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} \;, \qquad \int d^2\bar{\theta} \;=\; -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}$$

3.1.3 Definition and Transformation of the General Scalar Superfield

To define a superfield, recall properties of scalar fields $\varphi(x^\mu)$:

- function of spacetime - coordinates x^μ

- transformation under Poincaré, e.g. under translations:

  Treating φ as an operator, a translation with parameter a_μ will change it to

$$\varphi \quad \longmapsto \quad \exp(-ia_\mu P^\mu)\varphi \exp(ia_\mu P^\mu) \ .$$

  But $\varphi(x^\mu)$ is also a Hilbert - vector in some function - space $\mathcal{F}$, so

$$\varphi(x^\mu) \quad \longmapsto \quad \exp(-ia_\mu \mathcal{P}^\mu)\varphi(x^\mu) \quad =: \quad \varphi(x^\mu - a^\mu) \quad \Longrightarrow \quad \mathcal{P}_\mu \quad = \quad -i\partial_\mu \ .$$

  $\mathcal{P}$ is a representation of the abstract operator P^μ acting on $\mathcal{F}$. Comparing the two transformation - rules to first order in a_μ, get the following relationship:

$$(1 - ia_\mu P^\mu)\varphi(1 + ia_\mu P^\mu) \quad = \quad (1 - ia_\mu \mathcal{P}^\mu)\varphi \quad \Longrightarrow \quad i\Big[\varphi \ , \ a_\mu P^\mu\Big] \quad = \quad -ia^\mu \mathcal{P}_\mu \varphi \quad = \quad -a^\mu \partial_\mu \varphi$$

For a general scalar superfield $S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$, do an expansion in powers of θ_α, $\bar\theta_{\dot\alpha}$ which has a finite number of nonzero terms:

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \quad = \quad & \varphi(x) \ + \ \theta\psi(x) \ + \ \bar\theta\bar\chi(x) \ + \ \theta\theta M(x) \ + \ \bar\theta\bar\theta N(x) \ + \ (\theta\sigma^\mu\bar\theta)V_\mu(x) \\ & + (\theta\theta)\bar\theta\bar\lambda(x) \ + \ (\bar\theta\bar\theta)\theta\rho(x) \ + \ (\theta\theta)(\bar\theta\bar\theta)D(x) \end{aligned}$$

Transformation of $S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$ under Super - Poincaré, firstly as a field - operator

$$S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \quad \longmapsto \quad \exp\big(-i(\epsilon Q + \bar\epsilon\bar Q)\big) S \exp\big(i(\epsilon Q + \bar\epsilon\bar Q)\big) \ ,$$

secondly as a Hilbert - vector

$$S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \quad \longmapsto \quad \exp\big(i(\epsilon \mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\big) S(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) \quad = \quad S\big(x^\mu - ic(\epsilon\sigma^\mu\bar\theta) + ic^*(\theta\sigma^\mu\bar\epsilon), \theta + \epsilon, \bar\theta + \bar\epsilon\big) \ .$$

Here, ϵ denotes a parameter, $\mathcal{Q}$ a representation of the spinor - generators Q_α acting on functions of θ, $\bar\theta$, and c is a constant to be fixed later, which is involved in the translation

$$x^\mu \quad \longmapsto \quad x^\mu - ic(\epsilon\sigma^\mu\bar\theta) + ic^*(\theta\sigma^\mu\bar\epsilon) \ .$$

The translation of arguments x^μ, θ_α, $\bar\theta_{\dot\alpha}$ imply,

$$\mathcal{Q}_\alpha \quad = \quad -i\frac{\partial}{\partial\theta^\alpha} - c(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial x^\mu} \ , \qquad \bar{\mathcal{Q}}_{\dot\alpha} \quad = \quad i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + c^*\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\frac{\partial}{\partial x^\mu} \ , \qquad \mathcal{P}_\mu \quad = \quad -i\partial_\mu \ ,$$

where c can be determined from the commutation - relation

$$\Big\{\mathcal{Q}_\alpha \ , \ \bar{\mathcal{Q}}_{\dot\alpha}\Big\} \quad = \quad 2(\sigma^\mu)_{\alpha\dot\alpha}\mathcal{P}_\mu \quad \Longrightarrow \quad \mathrm{Re}\{c\} \quad = \quad 1$$

which, of course, holds in any representation. It is convenient to set $c = 1$. Again, a comparison of the two expressions (to first order in ϵ) for the transformed superfield S is the key to get its commutation - relations with Q_α:

$$i\Big[S \ , \ \epsilon Q + \bar\epsilon\bar Q\Big] \quad = \quad i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})S \quad = \quad \delta S$$

Knowing the $\mathcal{Q}$, $\bar{\mathcal{Q}}$ and S, get explicit terms for the change in the different parts of S:

$$\begin{aligned}
\delta\varphi &= \epsilon\psi + \bar{\epsilon}\bar{\chi} \\
\delta\psi &= 2\epsilon M + \sigma^\mu\bar{\epsilon}(i\partial_\mu\varphi + V_\mu) \\
\delta\bar{\chi} &= 2\bar{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\varphi - V_\mu) \\
\delta M &= \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\
\delta V_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}) \\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon})\partial_\mu V_\nu + i(\bar{\sigma}^\mu\epsilon)\partial_\mu M \\
\delta\rho &= 2\epsilon D - \frac{i}{2}(\sigma^\nu\bar{\sigma}^\mu\epsilon)\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N \\
\delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon})
\end{aligned}$$

Note that δD is a total derivative.

3.1.4 Remarks on Superfields

- S_1, S_2 superfields $\Rightarrow$ S_1S_2 superfields:

$$\begin{aligned}
\delta(S_1S_2) &= i\Big[S_1S_2\ ,\ \epsilon Q + \bar{\epsilon}\bar{Q}\Big] \;=\; iS_1\Big[S_2\ ,\ \epsilon Q + \bar{\epsilon}\bar{Q}\Big] + i\Big[S_1\ ,\ \epsilon Q + \bar{\epsilon}\bar{Q}\Big]S_2 \\
&= S_1\big(i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})S_2\big) + i\big(i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})S_1\big)S_2 \;=\; i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})(S_1S_2)
\end{aligned}$$

  In the last step, we used the Leibnitz - property of the $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ as differential - operators.

- Linear combinations of superfields are superfields again (straigtforward proof).

- $\partial_\mu S$ is a superfield but $\partial_\alpha S$ is not:

$$\delta(\partial_\alpha S) = i\Big[\partial_\alpha S\ ,\ \epsilon Q + \bar{\epsilon}\bar{Q}\Big] = i\partial_\alpha\Big[S\ ,\ \epsilon Q + \bar{\epsilon}\bar{Q}\Big] = i\partial_\alpha(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})S \neq i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})(\partial_\alpha S)$$

  The problem is $[\partial_\alpha, \epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}] \neq 0$. We need to define a covariant derivative,

$$\mathcal{D}_\alpha := \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}} := -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu$$

  which satisfies

$$\Big\{\mathcal{D}_\alpha\ ,\ \mathcal{Q}_\beta\Big\} = \Big\{\mathcal{D}_\alpha\ ,\ \bar{\mathcal{Q}}_{\dot{\beta}}\Big\} = \Big\{\bar{\mathcal{D}}_{\dot{\alpha}}\ ,\ \mathcal{Q}_\beta\Big\} = \Big\{\bar{\mathcal{D}}_{\dot{\alpha}}\ ,\ \bar{\mathcal{Q}}_{\dot{\beta}}\Big\} = 0$$

  and therefore

$$\Big[\mathcal{D}_\alpha\ ,\ \epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}\Big] = 0 \quad\Longrightarrow\quad \mathcal{D}_\alpha S \text{ superfield}\ .$$

  Also note that $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu$.

- $S = f(x)$ is a superfield only if $f = const$, otherwise, there would be some $\delta\psi \propto \epsilon\partial^\mu f$. For constant spinor c, $S = c\theta$ is not a superfield due to $\delta\phi = \epsilon c$.

S is *not* an irreducible representation of supersymmetry, so we can eliminate some of its components keeping it still as a superfield. In general we can impose consistent constraints on S, leading to smaller superfields that can be irreducible representations of the supersymmetry algebra. The relevant superfields are:

- Chiral superfield Φ such that $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$
- Antichiral superfield $\bar{\Phi}$ such that $\mathcal{D}_\alpha \bar{\Phi} = 0$
- Vector (or real) superfield $V = V^\dagger$
- Linear superfield L such that $\mathcal{D}\mathcal{D}L = 0$ and $L = L^\dagger$.

3.2 Chiral Superfields

We want to find the components of a superfields Φ satisfying $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. Define

$$y^\mu \ := \ x^\mu + i\theta\sigma^\mu\bar{\theta} \ .$$

If $\Phi = \Phi(y, \theta, \bar{\theta})$, then

$$\begin{aligned} \bar{\mathcal{D}}_{\dot{\alpha}}\Phi &= -\bar{\partial}_{\dot{\alpha}}\Phi - \frac{\partial \Phi}{\partial y^\mu}\frac{\partial y^\mu}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu\Phi \\ &= -\bar{\partial}_{\dot{\alpha}}\Phi - \partial_\mu\Phi(-i\theta\sigma^\mu)_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu\Phi \\ &= -\bar{\partial}_{\dot{\alpha}}\Phi \ = \ 0 \ , \end{aligned}$$

so there is no $\bar{\theta}^{\dot{\alpha}}$ - dependence and Φ depends only on y and θ. In components,

$$\Phi(y^\mu, \theta^\alpha) \ = \ \varphi(y^\mu) + \sqrt{2}\theta\psi(y^\mu) + \theta\theta F(y^\mu) \ ,$$

The physical components of a chiral superfield are: φ represents a scalar part (squarks, sleptons, Higgs), ψ some $s = \frac{1}{2}$ - particles (quarks, leptons, Higgsino) and F is an auxiliary - field in a way to be defined later. There are 4 bosonic (complex φ, F) and 4 fermionic (complex ψ_α) components. Reexpress Φ in terms of x^μ:

$$\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\alpha'}) \ = \ \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi(x)$$

Under supersymmetry - transformation

$$\delta\Phi \ = \ i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\Phi \ ,$$

find for the change in components

$$\begin{aligned} \delta\varphi &= \sqrt{2}\epsilon\psi \\ \delta\psi &= i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\varphi + \sqrt{2}\epsilon F \\ \delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ . \end{aligned}$$

So δF is another total derivative - term, just like δD in a general superfield. Note that:

- The product of chiral superfields is a chiral superfield. In general, any *holomorphic* function $f(\Phi)$ of chiral Φ is chiral.

- If Φ is chiral, then $\bar{\Phi} = \Phi^\dagger$ is antichiral.

- $\Phi^\dagger\Phi$ and $\Phi^\dagger + \Phi$ are real superfields but neither chiral nor antichiral.

3.3 Vector Superfields

3.3.1 Definition and Transformation of the Vector Superfield

The most general vector superfield $V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$ has the form

$$\begin{aligned} V(x,\theta,\bar{\theta}) \;\; = \;\; & C(x) \; + \; i\theta\chi(x) \; - \; i\bar{\theta}\bar{\chi}(x) \; + \; \frac{i}{2}\theta\theta\big(M(x)+iN(x)\big) \; - \; \frac{i}{2}\bar{\theta}\bar{\theta}\big(M(x)-iN(x)\big) \\ & + \; \theta\sigma^\mu\bar{\theta}V_\mu(x) \; + \; i(\theta\theta)\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) \\ & - \; i(\bar{\theta}\bar{\theta})\theta\left(\lambda(x) \pm \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \; + \; \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\left(D - \frac{1}{2}\partial_\mu\partial^\mu C\right) \; . \end{aligned}$$

These are 8 bosonic components C, M, N, D, V_μ and 4 + 4 fermionic ones (χ_α , λ_α).

If Λ is a chiral superfield, then $i(\Lambda - \Lambda^\dagger)$ is a vector - superfield. It has components:

$$\begin{aligned} C \;\; &= \;\; i(\varphi - \varphi^\dagger) \\ \chi \;\; &= \;\; \sqrt{2}\psi \\ \frac{1}{2}(M+iN) \;\; &= \;\; F \\ V_\mu \;\; &= \;\; -\partial_\mu(\varphi + \varphi^\dagger) \\ \lambda \;\; = \;\; D \;\; &= \;\; 0 \end{aligned}$$

We can define a generalized gauge - transformations to vector fields via

$$V \;\; \longmapsto \;\; V + i(\Lambda - \Lambda^\dagger) \; ,$$

which induces a standard gauge - transformation for the vector - component of V

$$V_\mu \;\; \longmapsto \;\; V_\mu - \partial_\mu(\varphi + \varphi^\dagger) \;\; =: \;\; V_\mu - \partial_\mu\alpha \; .$$

Then we can choose φ, ψ, F within Λ to gauge away some of the components of V.

3.3.2 Wess - Zumino - Gauge

We can choose the components of Λ above: φ, ψ, F in such a way to set $C = \chi = M = N = 0$. This defines the Wess-Zumino (WZ) gauge. A vector superfield in Wess - Zumino - gauge reduces to the form

$$V_{WZ}(x,\theta,\bar{\theta}) \;=\; (\theta\sigma^\mu\bar{\theta})V_\mu(x) \;+\; i(\theta\theta)(\bar{\theta}\bar{\lambda}) \;-\; i(\bar{\theta}\bar{\theta})(\theta\lambda) \;+\; \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x) \;,$$

The physical components of a vector superfield are: V_μ corresponding to gauge - particles (γ, $W^\pm$, Z, gluon), the λ and $\bar{\lambda}$ to gauginos and D is an auxiliary - field in a way to be defined later. Powers of V_{WZ} are given by

$$V_{WZ}^2 \;=\; \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})V^\mu V_\mu \;, \qquad V_{WZ}^{2+n} \;=\; 0 \;\forall\; n \in \mathbb{N} \;.$$

Note that the Wess - Zumino - gauge is not supersymmetric, since $V_{WZ} \mapsto V'_{\not{W}\not{Z}}$ under supersymmetry. However, under a combination of supersymmetry and generalized gauge - transformation $V'_{\not{W}\not{Z}} \mapsto V''_{WZ}$ we can end up with a vector - field in Wess - Zumino - gauge.

3.3.3 Field - Strength - Superfield

A non - supersymmetric complex scalar - field φ transforms like

$$\varphi(x) \;\longmapsto\; \exp\big(i\alpha(x)q\big)\varphi(x) \;, \qquad V_\mu(x) \;\longmapsto\; V_\mu(x) + \partial_\mu\alpha(x)$$

under local $U(1)$ with charge q and parameter $\alpha(x)$. Now, under supersymmetry

$$\Phi \;\longmapsto\; \exp(i\Lambda q)\Phi \;, \qquad V \;\longmapsto\; V + i(\Lambda - \Lambda^\dagger) \;,$$

where Λ is the chiral superfield defining the generalised gauge transformations, then $\exp(i\Lambda q)\Phi$ is also chiral if Φ is.

Before supersymmetry, we defined

$$F_{\mu\nu} \;=\; \partial_\mu V_\nu - \partial_\nu V_\mu$$

as a field - strength. The supersymmetric analogy is

$$W_\alpha \;=\; -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}})\mathcal{D}_\alpha V$$

which is both chiral and invariant under generalized gauge - transformations. In components,

$$W_\alpha(y,\theta) \;=\; -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + (\theta\theta)(\sigma^\mu)_{\alpha\beta'}\partial_\mu\bar{\lambda}^{\beta'} \;.$$

Chapter 4

4 D Supersymmetric Lagrangians

4.1 N = 1 Global Supersymmetry

We want to determine couplings among superfields Φ's, V's and W_α which include the particles of the Standard Model. For this we need a prescription to build Lagrangians which are invariant (up to a total derivative) under a supersymmetry transformation. We will start with the simplest case of only chiral superfields.

4.1.1 Chiral Superfield - Lagrangian

Look for an object $\mathcal{L}(\Phi)$ such that $\delta\mathcal{L}$ is a total derivative under supersymmetry - transformation. We know that

- For a general scalar superfield $S = ... + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)$, the D-term transforms as:

$$\delta D = \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon})$$

- For a chiral superfield $\Phi = ... + (\theta\theta)F(x)$, the F-term transforms as:

$$\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ ,$$

Therefore, the most general Lagrangian for a chiral superfield Φ's can be written as:

$$\mathcal{L} = \underbrace{K(\Phi, \Phi^\dagger)}_{\text{Kähler - potential}}\Big|_D + \left(\underbrace{W(\Phi)}_{\text{super - potential}}\Big|_F + h.c.\right) .$$

Where $|_D$ refers to the D-term of the corresponding superfield and similar for F-terms. The function K is known as the *Kähler potential*, it is a real function of Φ and $\Phi^\dagger$. $W(\Phi)$ is known as the *superpotential*, it is a holomorphic function of the chiral superfield Φ (and therefore is a chiral superfield itself).

In order to construct a renormalisable theory, we need to construct a Lagrangian in terms of operators of dimensionality such that the Lagrangian has dimensionality 4. We know $[\varphi] = 1$ (where the square brackets stand for dimensionality of te field) and want $[\mathcal{L}] = 4$. Terms of dimension 4, such as $\partial^\mu\varphi\partial_\mu\varphi^*$, $m^2\varphi\varphi^*$ and $g|\varphi|^4$, are renormalizable, but $\frac{1}{M^2}|\varphi|^6$ is not. The dimensionality of the superfield Φ is the same as that of its scalar component and that of ψ is as any standard fermion, that is

$$[\Phi] \;=\; [\varphi] \;=\; 1 \; , \qquad [\psi] \;=\; \frac{3}{2}$$

From the expansion $\Phi = \varphi + \sqrt{2}\theta\psi + \theta\theta F + ...$ it follows that

$$[\theta] \;=\; -\frac{1}{2} \; , \qquad [F] \;=\; 2 \; .$$

This already hints that F is not a standard scalr field. In order to have $[\mathcal{L}] = 4$ we need:

$$\begin{aligned} [K_D] \;&\le\; 4 \;\text{ in }\; K \;=\; ... + (\theta\theta)(\bar\theta\bar\theta)K_D \\ [W_F] \;&\le\; 4 \;\text{ in }\; W \;=\; ... + (\theta\theta)W_F \\ \Longrightarrow \;\; [K] \;&\le\; 2 \; , \qquad [W] \;\le\; 3 \; . \end{aligned}$$

A possible term for K is $\Phi^\dagger\Phi$, but no $\Phi + \Phi^\dagger$ nor $\Phi\Phi$ since those are linear combinations of chiral superfields. Therefore we are lead to the following general expressions for K and W:

$$K \;=\; \Phi^\dagger\Phi \; , \qquad W \;=\; \alpha + \lambda\Phi + m\Phi^2 + g\Phi^3 \; ,$$

whose Lagrangian is known as Wess - Zumino - model:

$$\begin{aligned} \mathcal{L} \;&=\; \Phi^\dagger\Phi\Big|_D + \left(\left(\alpha + \lambda\Phi + m\Phi^2 + g\Phi^3\right)\Big|_F + h.c. \right) \\ &=\; \partial^\mu\varphi^*\partial_\mu\varphi + i\bar\psi\bar\sigma^\mu\partial_\mu\psi + FF^* + \left(\frac{\partial W}{\partial\varphi}F + h.c.\right) - \frac{1}{2}\frac{\partial^2 W}{\partial\varphi^2}\psi\psi \end{aligned}$$

Note that

- The expression for $\Phi^\dagger\Phi\Big|_D$ is justified by

$$\Phi \;=\; \varphi(x) \;+\; \sqrt{2}\theta\psi \;+\; \theta\theta F \;+\; i\theta\sigma^\mu\bar\theta\partial_\mu\varphi \;-\; \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar\theta \;-\; \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu\varphi$$

- In general, the procedure to obtain the expansion of the Lagrangian in terms of the components of the superfield is to perform a Taylor - expansion around $\Phi = \varphi$, for instance (where $\frac{\partial W}{\partial\varphi} = \frac{\partial W}{\partial\Phi}\Big|_{\Phi=\varphi}$):

$$W(\Phi) \;=\; W(\varphi) + \underbrace{(\Phi - \varphi)}_{...+\theta\theta F+...}\frac{\partial W}{\partial\varphi} + \underbrace{\frac{1}{2}(\Phi-\varphi)^2}_{...+(\theta\psi)(\theta\psi)+...}\frac{\partial^2 W}{\partial\varphi^2}$$

The part of the Lagrangian depending on the auxiliary field F takes the simple form:

$$\mathcal{L}_{(F)} \;=\; FF^* + \frac{\partial W}{\partial\varphi}F + \frac{\partial W^*}{\partial\varphi^*}F^*$$

Notice that this is quadratic and without any derivatives. This means that the field F does not propagate. Also, we can easily eliminate F using the field - equations

$$\begin{aligned} \frac{\delta\mathcal{S}_{(F)}}{\delta F} \;&=\; 0 \;\;\Longrightarrow\;\; F^* + \frac{\partial W}{\partial\varphi} \;=\; 0 \\ \frac{\delta\mathcal{S}_{(F)}}{\delta F^*} \;&=\; 0 \;\;\Longrightarrow\;\; F + \frac{\partial W^*}{\partial\varphi^*} \;=\; 0 \end{aligned}$$

and substitute the result back into the Lagrangian,

$$\mathcal{L}_{(F)} \quad \longmapsto \quad -\left|\frac{\partial W}{\partial \varphi}\right|^2 \quad =: \quad -V_{(F)}(\varphi) \ ,$$

This defines the scalar potential. From its expression we can easily see thet it is a positive definite scalar potential $V_{(F)}(\varphi)$.

We finish the section about chiral superfield - Lagrangian with two remarks,

- The $N = 1$ - Lagrangian is a particular case of standard $N = 0$ - Lagrangians: the scalar potential is semipositive ($V \geq 0$). Also the mass for scalar field φ (as it can be read from the quadratic term in the scalar potential) equals the one for the spinor ψ (as can be read from the term $\frac{1}{2}\frac{\partial^2 W}{\partial \varphi^2}\psi\psi$) . Moreover, the coefficient g of Yukawa - coupling $g(\varphi\psi\psi)$ also determines the scalar self - coupling, $g^2|\varphi|^4$. This is the source of "miraculous" cancellations in SUSY perturbation - theory. Divergences are removed from diagrams:

  heightwidthdepthSUSY03.png

- In general, expand $K(\Phi^i, \Phi^{j^\dagger})$ and $W(\Phi^i)$ around $\Phi^i = \varphi^i$, in components

$$\left(\frac{\partial^2 K}{\partial \varphi^i \partial \varphi^{j*}}\right) \partial_\mu \varphi^i \partial^\mu \varphi^{j*} \quad = \quad K_{i\bar{j}} \partial_\mu \varphi^i \partial^\mu \varphi^{j*} \ .$$

  $K_{i\bar{j}}$ is a metric in a space with coordinates φ^i which is a complex Kähler - manifold:

$$g_{i\bar{j}} \quad = \quad K_{i\bar{j}} \quad = \quad \frac{\partial^2 K}{\partial \varphi^i \partial \varphi^{j*}}$$

4.1.2 Vector Superfield - Lagrangian

Let's first discuss how we ensured gauge - invariance of $\partial^\mu \varphi \partial_\mu \varphi^*$ under local transformations $\varphi \mapsto \exp\big(i\alpha(x)q\big)$ for non - supersymmetric Lagrangians.

- Introduce covariant derivative D_μ depending on gauge - potential A_μ

$$D_\mu \varphi \quad := \quad \partial_\mu \varphi - iqA_\mu \varphi \ , \qquad A_\mu \quad \longmapsto \quad A_\mu + \partial_\mu \alpha$$

  and rewrite kinetic term as

$$\mathcal{L} \quad = \quad D^\mu \varphi (D_\mu \varphi)^* + ...$$

- Add kinetic term for A_μ to $\mathcal{L}$

$$\mathcal{L} \quad = \quad ... + \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \ , \qquad F_{\mu\nu} \quad = \quad \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

With SUSY, the Kähler - potential $K = \Phi^\dagger \Phi$ is not invariant under

$$\Phi \longmapsto \exp(iq\Lambda)\Phi \ , \qquad \Phi^\dagger\Phi \longmapsto \Phi^\dagger \exp\big(iq(\Lambda - \Lambda^\dagger)\big)\Phi$$

for chiral Λ. Our procedure to construct a suitable Lagrangian is analogous to the non-supersymmetric case (although the expressions look slightly different):

- Introduce V such that

  $$K = \Phi^\dagger \exp(qV)\Phi \ , \qquad V \longmapsto V - i(\Lambda - \Lambda^\dagger) \ ,$$

  i.e. K is invariant under general gauge - transformation.

- Add kinetic term for V with coupling τ

  $$\mathcal{L}_{kin} = \tau(W^\alpha W_\alpha)\Big|_F + h.c.$$

  which is renormalizable if τ is a constant. In general it is non - renormalizable for $\tau = f(\Phi)$ (need $\tau = const$). We will call f *the gauge - kinetic function.*

- A new ingredient of supersymmetric theories is that an extra term can be added to $\mathcal{L}$ which is also invariant (for $U(1)$ gauge theories) and is known as the *Fayet - Iliopoulos - term*:

  $$\mathcal{L}_{FI} = \xi V\Big|_D = \xi D$$

  Where ξ a constant. Notice that the FI term is gauge invariant for a $U(1)$ theory because the corresponding gauge field is not charged under $U(1)$ (the photon is chargeless), whereas for a non-abelian gauge theory the gauge fields are charged and therefore their corresponding D terms is also and then a FI term would not be gauge invariant and therefore would be forbidden. This is the reason it exists only for abelian gauge theories.

So the renormalizable Lagrangian of super - QED is given by

$$\mathcal{L} = \big(\Phi^\dagger \exp(qV)\Phi\big)\Big|_D + \left(W(\Phi)\Big|_F + h.c.\right) + \left(\frac{1}{4}W^\alpha W_\alpha\Big|_F + h.c.\right) + \xi V\Big|_D \ .$$

If there were only one superfield Φ charged under $U(1)$ then $W = 0$. For several superfields the superpotential W is constructed out of holomorphic combinations of the superfields which are gaguge invariant. In components (using Wess - Zumino - gauge):

$$\begin{aligned}\big(\Phi^\dagger \exp(qV)\Phi\big)\Big|_D &= F^*F \ + \ \partial_\mu\varphi\partial^\mu\varphi^* \ + \ i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \ + \ qV^\mu\left(\frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi + \frac{i}{2}\varphi^*\partial_\mu\varphi - \frac{i}{2}\varphi\partial_\mu\varphi^*\right) \\ &\quad + \ \frac{i}{\sqrt{2}}q(\varphi\bar{\lambda}\bar{\psi} - \varphi^*\lambda\psi) \ + \ \frac{q}{2}\left(D + \frac{1}{2}V_\mu V^\mu\right)|\varphi|^2\end{aligned}$$

Note that

- $V^{n\geq 3} = 0$ due to Wess - Zumino - gauge

- can complete ∂_μ to D_μ using the term $qV^\mu(...)$

$W(\Phi) = 0$ if there is only one Φ. In case of several Φ_i, only chargeless combinations of products of Φ_i contribute, since $W(\Phi)$ has to be invariant under $\Phi \mapsto \exp(i\Lambda)\Phi$.

Let's move on to the $W^\alpha W_\alpha$ - term:

$$W^\alpha W_\alpha\Big|_F \;=\; 2\left(\frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{i}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}\right) ,$$

where the last term involving $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is a total derivative i.e. contains no local physics.

With the last term,

$$\xi V\Big|_D \;=\; \xi D ,$$

the collection of the D - dependent terms in $\mathcal{L}$

$$\mathcal{L}_{(D)} \;=\; \frac{q}{2}D|\varphi|^2 + \frac{1}{2}D^2 + \xi D$$

yields field - equations

$$\frac{\partial \mathcal{L}}{\partial D} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu D)} \;=\; 0 \quad\Longrightarrow\quad D \;=\; -\xi - \frac{q}{2}|\varphi|^2 .$$

Substituting those back into $\mathcal{L}_{(D)}$,

$$\mathcal{L}_{(D)} \;=\; -\frac{1}{2}\left(\xi + \frac{q}{2}|\varphi|^2\right)^2 \;=\; -V_{(D)}(\varphi) ,$$

get a scalar potential $V_{(D)}(\varphi)$. Together with $V_{(F)}(\varphi)$ from the previous section, the total potential is given by

$$V(\varphi) \;=\; V_{(F)}(\varphi) + V_{(D)}(\varphi) \;=\; \left|\frac{\partial W}{\partial\varphi}\right|^2 + \frac{1}{2}\left(\xi + \frac{1}{2}q|\varphi|^2\right)^2 .$$

4.1.3 Action as a Superspace - Integral

Without SUSY, the relationship between the action $\mathcal{S}$ and $\mathcal{L}$ is

$$\mathcal{S} \;=\; \int d^4x\ \mathcal{L} .$$

To write down a similar expression for SUSY - actions, recall

$$\int d^2\theta\ (\theta\theta) \;=\; 1 , \qquad \int d^4\theta\ (\theta\theta)(\bar{\theta}\bar{\theta}) \;=\; 1 .$$

This provides elegant ways of expressing $K\Big|_D$ and so on:

$$\mathcal{L} \;=\; K\Big|_D + \left(W\Big|_F + h.c.\right) + \left(W^\alpha W_\alpha\Big|_F + h.c.\right) \;=\; \int d^4\theta\ K + \left(\int d^2\theta\ W + h.c.\right) + \left(\int d^2\theta\ W^\alpha W_\alpha + h.c.\right)$$

With non - abelian generalizations

$$\begin{aligned}
\Phi' \;&=\; \exp(i\Lambda)\Phi \\
\exp(V') \;&=\; \exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda) \\
W'_\alpha \;&=\; \exp(-2i\Lambda)W_\alpha\exp(2i\Lambda) \\
W^\alpha W_\alpha \;&\longmapsto\; \mathrm{Tr}\Big\{W^\alpha W_\alpha\Big\}
\end{aligned}$$

end up with the most general action

$$\mathcal{S}\Big[K\big(\Phi_i^\dagger,\exp(qV),\Phi_i\big),W\big(\Phi_i\big),f\big(\Phi_i\big),\xi\Big] \;=\; \int d^4x\int d^4\theta\;(K+\underline{\xi V_{U(1)}})+\int d^4x\int d^2\theta(W+fW^\alpha W_\alpha+h.c.)\,.$$

Recall that the underlined Fayet - Ikopoulos - term ξV only appears for $U(1)$ - gauge theories.

4.2 Non - Renormalization - Theorems

We have seen that in general the functions K, W, f and the FI constant ξ determine the full structure of $N = 1$ supersymmetric theories (up to two derivatives of the fields as usual). if we know their expressions we know all the interactions among the fields.

In order to understand the important properties of supersymmetric theories under quantization, we most address the following question: How do K, W, f and ξ behave under quantum - corrections? We will show now that:

- K gets corrections order by order in perturbation - theory
- only one loop - corrections for $f(\Phi)$
- $W(\Phi)$ and ξ *not* renormalized in perturbation - theory.

The non-renormalization of the superpotential is one of the most important results of supersymmetric field theories. The simple behaviour of f and the non-renormalization of ξ have also interesting consequences. We will procede now to address these issues.

4.2.1 History

- In 1977 Grisaru, Siegel, Rocek showed using "supergraphs" that except for one loop - corrections for f, quantum corrections only come in the form

$$\int d^4x\int d^4\theta\Big\{\ldots\Big\}\,.$$

- 1993: Seiberg (based on string theory - arguments by Witten 1985) used symmetru and holomorphy arguments to establish these results in a simple an elegant way. We will follow here this approach following closely the discusion of Weinberg's section 27.6.

4.2.2 Proof of the Non - Renormalization - Theorem

Let's follow Seiberg's path of proving the non - renormalization - theorem. Introduce "spurious" superfields X, Y,

$$X = (x, \psi_x, F_x) \ , \qquad Y = (y, \psi_y, F_y)$$

involved in the action

$$\mathcal{S} = \int d^4x \int d^4\theta \left[K + \xi V_{U(1)} \right] + \int d^4x \int d^2\theta \left[Y W(\Phi_i) + X W^\alpha W_\alpha + h.c. \right] .$$

We will use:

- symmetries
- holomorphicity
- limits $X \to \infty$ and $Y \to 0$

Symmetries

- SUSY and gauge - symmetries
- R - symmetry $U(1)_R$: Fields have different $U(1)_R$ - charges determining how they transform under that group

fields	Φ_i	V	X	Y	θ	$\bar{\theta}$	W^α
$U(1)_R$ - charge	0	0	0	2	-1	1	1

$$\text{e.g.} \qquad Y \longmapsto \exp(2i\alpha) Y \ , \qquad \theta \longmapsto \exp(-i\alpha)\theta \ , \ \text{etc.}$$

- Peccei - Quinn - symmetry

$$X \longmapsto X + ir \ , \qquad r \in \mathbb{R}$$

Since $XW^\alpha W_\alpha$ involves terms like

$$\text{Re}\{X\} F_{\mu\nu} F^{\mu\nu} + \text{Im}\{X\} F_{\mu\nu} \tilde{F}^{\mu\nu} \ ,$$

a change in the imaginary - part of X would only add total derivatives to $\mathcal{L}$,

$$\mathcal{L} \longmapsto \mathcal{L} + r F_{\mu\nu} \tilde{F}^{\mu\nu}$$

without any local physics. Call X an axion - field.

Holomorphicity

Consider the quantum - corrected Wilsonian - action

$$\mathcal{S}_\lambda \equiv \int \mathcal{D}\varphi e^{iS}$$

where the path integral is understood to go for all the fields in the system and the integration is only over all momenta greater than λ in the standard Wilsonian formalism (different to the 1PI action in which the integral is over all momenta). If supersymmetry is preserved by the quantisation process, we can write the effective action as:

$$\mathcal{S}_\lambda \;=\; \int d^4x \int d^4\theta \left[J\big(\Phi, \Phi^\dagger, e^V, X, Y, \mathcal{D}...\big) + \xi(X, X^\dagger, Y, Y^\dagger)V_{U(1)}\right] + \int d^4x \int d^2\theta \left[\underbrace{H(\Phi, X, Y, W^\alpha)}_{\text{holomorphic}} + h.c.\right] .$$

Due to $U(1)_R$ - transformation - invariance, H must have the form

$$H \;=\; Yh(X, \Phi) + g(X, \Phi)W^\alpha W_\alpha \ .$$

Invariance under shifts in X imply that $h = h(\Phi)$ (independent of X). But a linear X - dependence is allowed in front of $W^\alpha W_\alpha$ (due to $F_{\mu\nu}\tilde{F}^{\mu\nu}$ as a total derivative). So the X - dependence in h and g is restricted to

$$H \;=\; Yh(\Phi) + \big(\alpha X + g(\Phi)\big)W^\alpha W_\alpha \ .$$

Limits

In the limit $Y \to 0$, there is an equality $h(\Phi) = W(\Phi)$ at tree - level, so $W(\Phi)$ is not renormalized! The gauge - kinetic function $f(\Phi)$, however, gets a one - loop correction

$$f(\Phi) \;=\; \underbrace{\alpha X}_{\text{tree - level}} + \underbrace{g(\Phi)}_{\text{1 loop}} \ .$$

Note that gauge - field - propagators are proportional to $\frac{1}{x}$ (gauge - couplings $\sim xF^{\mu\nu}F_{\mu\nu} \propto X\partial^{[\mu}A^{\nu]}\partial_{[\mu}A_{\nu]}$, gauge self - couplings to X^3 corresponding to a vertex of 3 X - lines).

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Count the number N_x of x - powers in any diagram; it is given by

$$N_x \;=\; V_W - I_W$$

and is therefore related to the numbers of loops L:

$$L = I_W - V_W + 1 = -N_x + 1 \quad \Longrightarrow \quad N_x = 1 - L$$

$$\begin{aligned} L &= 0 \text{ (tree - level)}: \ N_x = 1 \,, \qquad \alpha = 1 \\ L &= 1 \text{ (one loop)}: \ N_x = 0 \end{aligned}$$

Therefore the gauge kinetic term $X + g(\Phi)$ is corrected only at one-loop! (all other (infinite) loop corrections just cancel).

On the other hand, the Kähler - potential, being non-holomorphic, is corrected to all orders $J(Y, Y^\dagger, X + X^\dagger, ...)$. For the Fayet - Iliopoulos - term $\xi(X, X\dagger, Y, Y^\dagger) V_{U(1)}\Big|_D$, gauge - invariance under $V \mapsto V + i(\Lambda - \Lambda^\dagger)$ implies that ξ is a constant. Only contributions are

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$$\propto \sum q_i = \mathrm{Tr}\Big\{ Q_{U(1)} \Big\} \,.$$

But if $\mathrm{Tr}\{Q\} \neq 0$, the theory is "inconsistent" due to gravitational anomalies:

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Therefore, if there are no gravitational anomalies, there are no corrections to the Fayet - Iliopoulos - term.

4.3 N = 2,4 Global Supersymmetry

For $N = 1$ - SUSY, we had an action $\mathcal{S}$ depending on K, W, f and ξ. What will the $N \geq 2$ - actions depend on? We know that in global supersymmetry, the $N = 1$ actions are particular cases of non-supersymmetric actions (in which some of the couplings are related, potential is positive, etc.). In the same way, actions for extended supersymmetries are particular cases of $N = 1$ supersymmetric actions, and therefore will be determined by K, W, f and ξ. The extra supersymmetry will put constraints to these functions and therefore the corresponding actions will be more rigid. The larger the number of supersymmetries the more constrained actions.

4.3.1 N = 2

Consider the $N = 2$ vector - multiplet

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \varphi & \end{array}$$

where the A_μ and λ are described by a vector superfield V and the φ, ψ by a chiral superfield Φ.

$W = 0$ in the $N = 2$ - action. K, f can be written in terms of a single holomorphic function $\mathcal{F}(\Phi)$ called prepotential:

$$f(\Phi) = \frac{\partial^2 \mathcal{F}}{\partial \Phi^2} \, , \qquad K(\Phi, \Phi^\dagger) = \frac{1}{2i} \left(\Phi^\dagger \exp(2V) \frac{\partial \mathcal{F}}{\partial \Phi} - h.c. \right)$$

Full perturbative action doen't contain any corrections for more than one loop,

$$\begin{aligned} \mathcal{F} &= \Phi^2 && \text{(tree - level)} \\ &= \Phi^2 \ln \left(\frac{\Phi^2}{\lambda^2} \right) && \text{(one loop)} \end{aligned}$$

λ denotes some cut - off. These statements apply to the "Wilson" effective - action distinct from 1 particle - irreducible $\Gamma[\Phi]$. Note that

- Perturbative processes usually involve series $\sum_n a_n g^n$ with coupling $g < 1$.
- $\exp\left(-\frac{c}{g^2}\right)$ is a non - perturbative example (no expansion in powers of g).

There are obviously more things in QFT than Feynman - diagrams can tell, e.g. instantons, monopoles.

Decompose the $N = 2$ - prepotential $\mathcal{F}$ as

$$\mathcal{F}(\Phi) = \mathcal{F}_{\text{1loop}} + \mathcal{F}_{\text{non - pert}}$$

where $\mathcal{F}_{\text{non - pert}}$ for instance could be the "instanton" - expansion $\sum_k a_k \exp\left(-\frac{c}{g^2} k\right)$. In 1994, Seiberg - Witten achieved such an expansion in $N = 2$ SUSY.

Of course, there are still vector- and hypermultiplets in $N = 2$, but those are much more complicated. We will now consider a particularly simple combination of these multplets.

4.3.2 N = 4

As an $N = 4$ - example, consider the vector multiplet,

$$\underbrace{\begin{pmatrix} & A_\mu & \\ \lambda & & \psi_1 \\ & \varphi_1 & \end{pmatrix}}_{N=2 \text{ vector}} + \underbrace{\begin{pmatrix} & \varphi_2 & \\ \psi_3 & & \psi_2 \\ & \varphi_3 & \end{pmatrix}}_{N=2 \text{ hyper}} .$$

We are more constrained than in above theories, there are no free functions at all, only 1 free parameter:

$$f \;=\; \tau \;=\; \underbrace{\frac{\Theta}{2\pi}}_{F_{\mu\nu}\tilde{F}^{\mu\nu}} + \underbrace{i\frac{4\pi}{g^2}}_{F_{\mu\nu}F^{\mu\nu}}$$

$N = 4$ is a finite theory, with vanishing β - function. Couplings remain constant at any scale, we have conformal invariance. There are nice - transformation - properties under S - duality,

$$\tau \;\longmapsto\; \frac{a\tau + b}{c\tau + d} ,$$

where a, b, c, d form a $SL(2, \mathbb{Z})$ - matrix.

Finally, as an aside, major developments in string and field theories have led to the realization that certain theories of gravity in anti de sitter space are 'dual' to field theories (without gravity) in one less dimension, that happen to be invariant under conformal transformations. This is the AdS/CFT correspondence. This has allowed to extend gravity (and string) theories to domains where they are not well understood and field theories also. The prime example of this correspondence is AdS in five dimensions dual to a conformal field theory in four dimensions that happens to be $N = 4$ supersymmetry.

4.3.3 Aside on Couplings

For all kinds of renormalizations, couplings g depend on a scale μ. The coupling changes under RG - transformations scale - by - scale. Define the β - function to be

$$\mu\frac{dg}{d\mu} \;=\; \beta(g) \;=\; \underbrace{-bg^3}_{1-\text{loop}} + \ldots .$$

The theory's - cutoff depends on the particle - content.

Solve for $g(\mu)$ up to one loop - order:

$$\int_m^M \frac{dg}{g^3} \;=\; -b\int_{-\infty}^{+\infty} \frac{d\mu}{\mu} \;\Longrightarrow\; -\frac{1}{2}\left(\frac{1}{g_M^2} - \frac{1}{g_m^2}\right) \;=\; -b\ln\left(\frac{M}{m}\right)$$

$$\Longrightarrow \quad g_m^2 \quad = \quad \frac{1}{\frac{1}{g_M^2} + b \ln\left(\frac{m^2}{M^2}\right)}$$

The solution has a pole at

$$m_0 \quad =: \quad \Lambda \quad = \quad M \exp\left(-\frac{b}{2g^2}\right)$$

which is the natural scale of the theory. For $m \to \infty$, get asymptotic freedom as long as $b > 0$, i.e.

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$\lim_{m\to\infty} g_m = 0$. This is the case in QCD. If $b < 0$, however, have a Landau - pole which is an upper - bound for the energy - scales where we can trust the theory. QED breaks down in that way.

4.4 Supergravity

4.4.1 Supergravity as a Gauge - Theory

We have seen that a superfield Φ transforms under supersymmetry like

$$\delta\Phi \quad = \quad i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\Phi \ .$$

The questions arises if we can make ϵ a function of spacetime - coordinates $\epsilon(x)$, i.e. extend SUSY to a local symmetry. The answer is yes, the corresponding theory is supergravity.

How did we deal with local $\alpha(x)$ in internal - symmetries? We introduced a gauge - field A_μ coupling to a current J^μ via interaction - term $A_\mu J^\mu$. That current J^μ is conserved and the corresponding charge constant

$$Q \quad = \quad \int d^3x \ J^0 \quad = \quad const \ .$$

For spacetime - symmetries, local Poincaré - parameters imply the equivalence - principle which is connected with gravity. The metric $g_{\mu\nu}$ as a gauge - field couples to "current" $T^{\mu\nu}$ via $g_{\mu\nu}T^{\mu\nu}$. Conservation $\partial_\mu T^{\mu\nu} = 0$ implies constant total - momentum

$$P^\mu \quad = \quad \int d^3x \ T^{\mu 0} \quad = \quad const \ .$$

Now consider local SUSY. The gauge - field of that supergravity is the gravitino ψ_α^μ with associated super - current $\psi_\alpha^\mu J_\mu^\alpha$ and SUSY - charge

$$Q_\alpha \quad = \quad \int d^3x \ J_\alpha^0 \ .$$

The supergravity - action given by

$$\mathcal{S} \quad = \quad \int d^4x \, \sqrt{-g}\Big(\underbrace{R}_{\text{Einstein}} + \underbrace{(\psi_\mu \sigma_\nu D_\rho \bar{\psi}_\sigma - \bar{\psi}_\mu \bar{\sigma}_\nu D_\rho \psi_\sigma)\epsilon^{\mu\nu\rho\sigma}}_{\text{Rarita - Schwinger}}\Big)$$

is invariant under

$$\begin{aligned} \delta e_\mu^a &= \frac{i}{2}(\psi_\mu \sigma^a \bar{\epsilon} - \epsilon \sigma^a \bar{\psi}_\mu) \\ \delta \psi_\mu^\alpha &= D_\mu \epsilon^\alpha \ . \end{aligned}$$

Historically, the first supergravity actions were constructed by S. Ferrara, D. Freedman and P. van Niewenhuizen, followed closely by deser and Zumino, in 1976. We do not provide details of these calculations that are beyond the scope of these lectures.

4.4.2 N = 1 - Supergravity Coupled to Matter

Here we will provide,, without proof, some properties of $N = 1$ supergravity actions coupled to matter. The total Lagrangian, a sum of supergravity - contribution and the SUSY - Lagrangian discussed before,

$$\mathcal{L} \quad = \quad \mathcal{L}_{SUGRA} + \mathcal{L}(K, W, f, \xi) \ .$$

where the second term is understood to be covariantized (to be invariant under general coordinate transformations.).

- This action has a so - called Kähler - invariance:

$$\begin{aligned} K &\longmapsto K + h(\Phi) + h^*(\Phi^*) \\ W &\longmapsto \exp\big(h(\Phi)\big)W \end{aligned}$$

- There is a modification to the scalar potential of global supersymmetry V_F

$$V_F \quad = \quad \exp\left(\frac{K}{M_P^2}\right)\left\{K_{i\bar{\jmath}}^{-1} D_i W D_{\bar{\jmath}} W^* - 3\frac{|W|^2}{M_P^2}\right\} \ , \qquad D_i W \quad = \quad \partial_i W + \partial_i K \frac{W}{M_P^2} \ .$$

  In the $M_P \to \infty$ - limit, gravity is decoupled and $V_F = K_{i\bar{\jmath}} \partial_i X \partial_{\bar{\jmath}} W^*$ which is the global supersymmetric potential. Notice that for finite values of the Planck mass, the potential V_F above is no longer positive. The extra (negative) factor proportional to $-3|W|^2$ comes from the auxiliary fields of the gravity multplet.

Chapter 5

Supersymmetry - Breaking

5.1 Basics

We know that fields φ_i of gauge - theories transform like

$$\varphi_i \quad \longmapsto \quad \big(\exp(i\alpha^a T^a)\big)_i{}^j \varphi_j$$

under finite group elements. The infinitesimal case is

$$\delta\varphi_i \quad = \quad i\alpha^a (T^a)_i{}^j \varphi_j \ .$$

Symmetry is broken if the vacuum - state $(\varphi_{vac})_i$ transforms in a non - trivial way, i.e.

$$(\alpha^a T^a)_i{}^j (\varphi_{vac})_j \quad \neq \quad 0 \ .$$

In $U(1)$, let $\varphi = \rho \exp(i\vartheta)$ in complex polar - coordinates, then infinitesimally

$$\delta\varphi \quad = \quad i\alpha\varphi \quad \Longrightarrow \quad \delta\rho \quad = \quad 0 \ , \qquad \delta\vartheta \quad = \quad \alpha \ ,$$

the last of which corresponds to a Goldstone - boson.

Similarly, we speak of broken SUSY if the vacuum - state $|\mathrm{vac}\rangle$ satisfies

$$Q_\alpha |\mathrm{vac}\rangle \quad \neq \quad 0 \ .$$

Let's consider the anticommutation - relation $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$ multiplied by $(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}$,

$$(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} \Big\{ Q_\alpha \ , \ \bar{Q}_{\dot{\beta}} \Big\} \quad = \quad 2(\bar{\sigma}^\nu)^{\dot{\beta}\alpha} (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \quad = \quad 4\eta^{\mu\nu} P_\mu \quad = \quad 4P^\nu \ ,$$

especially the $(\nu = 0)$ - component using $\bar{\sigma}^0 = \mathbb{1}$:

$$(\bar{\sigma}^0)^{\dot{\beta}\alpha} \Big\{ Q_\alpha \ , \ \bar{Q}_{\dot{\beta}} \Big\} \quad = \quad \sum_{\alpha=1}^{2} (Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha) \quad = \quad 4P^0 \quad = \quad 4E$$

This has two very important implications:

- $E \geq 0$ for any state, since $Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha$ is positive definite
- $\langle \text{vac}|Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha|\text{vac}\rangle > 0$, so in broken SUSY, the energy is strictly positive, $E > 0$.

5.2 F- and D - Breaking

5.2.1 F - Term

Consider the transformation - laws under SUSY for components of chiral superfields Φ,

$$\begin{aligned} \delta\varphi &= \sqrt{2}\epsilon\psi \\ \delta\psi &= \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\varphi \\ \delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ . \end{aligned}$$

If one of $\delta\varphi$, $\delta\psi$, $\delta F \neq 0$, then SUSY is broken. But to preserve Lorentz - invariance, need

$$\langle\psi\rangle \;=\; \langle\partial_\mu\varphi\rangle \;=\; 0$$

as they would both transform under some representation of Lorentz - group. So our SUSY - breaking - condition simplifies to

$$SU\!\!\!/SY \quad \Longleftrightarrow \quad \langle F\rangle \;\neq\; 0 \ .$$

Only the fermionic part of Φ will change,

$$\delta\varphi \;=\; \delta F \;=\; 0 \ , \qquad \delta\psi \;=\; \sqrt{2}\epsilon\langle F\rangle \;\neq\; 0 \ ,$$

so call ψ "Goldstone - fermion" or "goldstino". Remember that the F - term of the scalar - potential is given by

$$V_{(F)} \;=\; K_{ij}^{-1}\frac{\partial W}{\partial\varphi_i}\left(\frac{\partial W}{\partial\varphi_j}\right)^* \ ,$$

so SUSY - breaking is equivalent to a positive vacuum - expectation - value

$$SU\!\!\!/SY \quad \Longleftrightarrow \quad \langle V_{(F)}\rangle \;>\; 0 \ .$$

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5.2.2 O'Raifertaigh - Model

The O'Raifertaigh - model involves a triplet of chiral superfields Φ_1, Φ_2, Φ_3 for which Kähler - and super - potentials are given by

$$K = \Phi_i^\dagger \Phi_i \ , \qquad W = g\Phi_1(\Phi_3^2 - m^2) + M\Phi_2\Phi_3 \ , \qquad M >> m \ .$$

From the F - equations of motion, if follows that

$$\begin{aligned} -F_1^* &= \frac{\partial W}{\partial \varphi_1} = g(\varphi_3^2 - m^2) \\ -F_2^* &= \frac{\partial W}{\partial \varphi_2} = M\varphi_3 \\ -F_3^* &= \frac{\partial W}{\partial \varphi_3} = 2g\varphi_1\varphi_3 + M\varphi_2 \ . \end{aligned}$$

We cannot have $F_i^* = 0$ for $i = 1, 2, 3$ simultaneously, so that form of W indeed breaks SUSY. Now, determine the spectrum:

$$V = \left(\frac{\partial W}{\partial \varphi_i}\right)\left(\frac{\partial W}{\partial \varphi_j}\right)^* = g^2|\varphi_3^2 - m^2|^2 + M^2|\varphi_3|^2 + |2g\varphi_1\varphi_3 + M\varphi_2|^2$$

If $m^2 < \frac{M^2}{2g^2}$, then the minimum is at

$$\langle\varphi_2\rangle = \langle\varphi_3\rangle = 0 \ , \qquad \langle\varphi_1\rangle \text{ arbitrary} \ .$$

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$$\Longrightarrow \quad \langle V\rangle = g^2 m^4 > 0 \ .$$

This arbitrariness of φ_1 implies zero - mass, $m_{\varphi_1} = 0$. For simplicity, set $\langle\varphi_1\rangle = 0$ and compute the spectrum of fermions and scalars. Consider the mass - term

$$\left\langle \frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\right\rangle \psi_i\psi_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix} \psi_i\psi_j$$

in the Lagrangian, which gives ψ_i - masses

$$m_{\psi_1} = 0 \ , \qquad m_{\psi_2} = m_{\psi_3} = M \ .$$

ψ_1 turns out to be the goldstino (due to $\delta\psi_1 \propto \langle F_1\rangle \neq 0$ and zero - mass). To determine scalar - masses, look at the quadratic terms in V:

$$V_{quad} = -m^2 g^2(\varphi_3^2 + \varphi_3^{*2}) + M^2|\varphi_3|^2 + M^2|\varphi_2|^2 \quad \Longrightarrow \quad m_{\varphi_1} = 0 \ , \qquad m_{\varphi_2} = M$$

Regard φ_3 as a complex field $\varphi_3 = a + ib$ where real- and imaginary - part have different masses,

$$m_a^2 = M^2 - 2g^2m^2 \ , \qquad m_b^2 = M^2 + 2g^2m^2 \ .$$

This gives the following spectrum:

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We generally get heavier and lighter superpartners, the "supertrace" of M (treating bosonic and fermionic parts differently) vanishes. This is generic for tree - level of broken SUSY. Since W is not renormalized to all orders in perturbation theory, we have an important result: If SUSY is unbroken at tree - level, then it also unbroken to all orders in perturbation theory. This means that in order to break supersymmetry we need to consider non-perturbative effects:

$$\Longrightarrow \quad \not{SUSY} \text{ non - perturbatively}$$

5.2.3 D - Term

Consider a vector superfield $V = (\lambda\ ,\ A_\mu\ ,\ D)$,

$$\delta\lambda \quad \propto \quad \epsilon D \quad \Longrightarrow \quad \langle D \rangle \quad \neq \quad 0 \quad \Longrightarrow \quad \not{SUSY}\ .$$

λ is a goldstino (which is NOT the fermionic partner of any goldstone boson). More on that in the examples.

5.3 Supersymmetry - Breaking in N = 1 - Supergravity

- Supergravity multiplet adds new auxiliary - fields F_g with nonzero $\langle F_g \rangle$ for broken SUSY.
- The F - term is proportional to
$$F \quad \propto \quad DW \quad = \quad \partial W + \partial K \frac{W}{M_P^2}\ .$$
- Scalar potential $V_{(F)}$ has a negative gravitational term,
$$V_{(F)} \quad = \quad \exp\left(\frac{K}{M_P^2}\right)\left\{K_{ij}^{-1} D_i W (D_j W)^* - 3\frac{|W|^2}{M_P^2}\right\}\ .$$
That is why both $\langle V \rangle = 0$ and $\langle V \rangle \neq 0$ are possible after SUSY - breaking in supergravity, whereas broken SUSY in the global case required $\langle V \rangle > 0$. This is very important for the cosmological constant problem (which is the lack of understadning of why the vacuum energy today is almost zero). The vacuum energy essentially corresponds to the value of the scalar potential at the minimum. In global supersymmetry, we know that the breaking of supersymmetry implies this vacuum energy to be large. In supergravity it is possible to break supersymmetry at a physically allowed scale and still to keep the vacuum energy zero. This does not solve the cosmological constant problem, but it makes supersymmetri theories still viable.

- *The super-Higgs effect.* Spontaneosuly broken gauge theories realize the Higgs mechanism in which the corresponding Goldstone boson is 'eaten' by the corresponding gauge field to get a mass. A similar phenomenon happens in supersymmetry. The goldstino field joins the originally massless gravitino field (which is the gauge field of $N = 1$ supergravity) and gives it a mass, in this sense the gravitino 'eats' the goldstino to get a mass. A massive gravitino (keeping a massless graviton) illustrates the breaking of supersymmetry. The super-Higgs effect should not be confused with the supersymmetric extension of the standard Higgs effect in which a massless vector superfield, eats a chiral superfield to receive a mass making it into a superssymetric massive multiplet.

Chapter 6

The MSSM

6.1 Basic Ingredients

6.1.1 Particles

First of all, we have vector fields transforming under $SU(3)_c \times SU(2)_L \times U(1)_Y$, secondly there are chiral superfields representing

- quarks

$$\underbrace{Q_i \;=\; \left(3\ ,\ 2\ ,\ -\frac{1}{6}\right)}_{\text{left - handed}}\ ,\qquad \underbrace{\bar{u}_i^c \;=\; \left(\bar{3}\ ,\ 1\ ,\ \frac{2}{3}\right)\ ,\qquad \bar{d}_i^c \;=\; \left(\bar{3}\ ,\ 1\ ,\ -\frac{1}{3}\right)}_{\text{right - handed}}$$

- leptons

$$\underbrace{L_i \;=\; \left(1\ ,\ 2\ ,\ \frac{1}{2}\right)}_{\text{left - handed}}\ ,\qquad \underbrace{\bar{e}_i^c \;=\; (1\ ,\ 1\ ,\ -1)\ ,\qquad \bar{\nu}_i^c \;=\; (1\ ,\ 1\ ,\ 0)}_{\text{right - handed}}$$

- higgses

$$H_1 \;=\; \left(1\ ,\ 2\ ,\ \frac{1}{2}\right)\ ,\qquad H_2 \;=\; \left(1\ ,\ 2\ ,\ -\frac{1}{2}\right)$$

  the second of which is a new particle, not present in the stanrd model. It is needed in order to avoid anomalies, like the one shown below.

The sum of Y^3 over all the MSSM - particles must vanish (i.e. multiply the third quantum number with the product of the first two to cover all the distinct particles).

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6.1.2 Interactions

- $K = \Phi^\dagger \exp(qV)\Phi$ is renormalizable.
- $f_a = \tau_a$ where $\Re\{\tau_a\} = \frac{4\pi}{g_a^2}$ determines the gauge coupling constants. These coupling constants change with energy as mentioned before. The precise way they run is determined by the low energy spectrum of the matter fields in the theory. We know from precision tests of the standard model, that with its spectrum, the running of the three gauge couplings is such that they do not meet at a single point at higher energies, signalling a gauge coupling unification. However with the matter field spectrum of the MSSM, the three different couplings evolve in such a way that they meet at a large energy E. This is considered to be the main phenomenological success of supersymmetric theories and it hints to a supersymmetric grand unified theory at large energies.

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- Fayet-Iliopoulos term: need $\xi = 0$, otherwise break charge and colour.
- The superpotential W is given by

$$W = y_1 Q H_2 \bar{u}^c + y_2 Q H_1 \bar{d}^c + y_3 L H_1 \bar{e}^c + \mu H_1 H_2 + W_{\not{B}\not{L}} \ ,$$

$$W_{\not{B}\not{L}} = \lambda_1 L L \bar{e}^c + \lambda_2 L Q \bar{d}^c + \lambda_3 \bar{u}^c \bar{d}^c \tilde{d}^c + \mu' L H_2$$

The first three terms in W correspond to standard Yukawa couplings giving masses to up quarks, down quarks and leptons. The four term is a mass term for the two Higgs fields. But each $\not{B}L$ - term breaks baryon- or lepton - number. These couplings are not present in the standard model that automatically preserves baryon and lepton number (as accidental symmetries), but this is not the case in supersymmetry. The shown interaction would allow proton - decay $p \to e^+ + \pi^0$ within seconds.

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In order to forbid those couplings an extra symmetry should be imposed. The simplest one that works is R - *partiy* R defined as

$$R := (-1)^{3(B-L)+2S} = \begin{cases} +1 & : \text{ all observed particles} \\ -1 & : \text{ superpartners} \end{cases} .$$

It forbids all the terms in $W_{\not{B}\not{L}}$.

The possible existence of R-parity would have important physical implications:

- The lightest superpartner (LSP) is stable.
- Usually, LSP is neutral (higgsino, photino), the neutralino is best candidate for dark matter (WIMP).
- In colliders, super - particles are produced in pairs, decay to LSP and give a signal of ”missing energy”.

6.1.3 Supersymmetry - Breaking

Recall the two sectors of the Standard Model:

$$\begin{pmatrix} \text{observable} \\ \text{sector (quarks)} \end{pmatrix} \overset{\text{Yukawa}}{\longleftrightarrow} \begin{pmatrix} \text{symmetry -} \\ \text{breaking (Higgs)} \end{pmatrix}$$

Supersymmetry has an additional ”messenger” - sector

$$\begin{pmatrix} \text{observable} \\ \text{sector} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{messenger -} \\ \text{sector} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{SUSY -} \\ \text{breaking} \end{pmatrix}$$

involving three types of mediation

- gravity - mediation

  The inverse Planck - mass M_{pl} is the natural scale of gravity. We must include some mass - square to get the right dimension for the mass - splitting in the observable sector. That will be the square of SUSY - breaking - mass $M_{\not{S}USY}$:

  $$\Delta m \ = \ \frac{M_{\not{S}USY}^2}{M_{pl}} \ .$$

  We want $\Delta m \sim$ TeV and know $M_{pl} \sim 10^{18}$GeV, so

  $$M_{\not{S}USY} \ = \ \sqrt{\Delta m \cdot M_{pl}} \ \approx \ 10^{11} \text{ GeV} \ .$$

  The gravitino gets a mass $m_{3/2}$ of Δm - order TeV. Note that gravitino eating goldstino to get mass is called superhiggs - effect.

- gauge - mediation

  $$G \ = \ \big(SU(3) \times SU(2) \times U(1)\big) \times G_{\not{S}USY} \ =: \ G_0 \times G_{\not{S}USY}$$

  Matter fields are charged under both G_0 and $G_{\not{S}USY}$ which gives a $M_{\not{S}USY}$ of order Δm, i.e. TeV. In that case, the gravitino mass $m_{3/2}$ is given by $\frac{M_{\not{S}USY}^2}{M_{pl}} \sim 10^{-3}$ eV

- anomaly - mediation

  Auxiliary fields of supergravity get a vacuum expectation - value. The effects are all present but suppressed by loop - effects.

In any case, the Lagrangian for the observable sector has contributions

$$\mathcal{L} \;=\; \mathcal{L}_{SUSY} + \mathcal{L}_{\not{S}USY} \;=\; \mathcal{L}_{SUSY} + \left(\underbrace{M_\lambda \lambda \cdot \lambda}_{\text{gaugino - masses}} + h.c. \right) + \underbrace{m_0^2 \varphi^* \cdot \varphi}_{\text{scalar - masses}} + (A\varphi\varphi\varphi + h.c.)$$

M_λ, m_0^2, A are called "soft - breaking terms". They determine the amount by which supersymmetry is expected to be broken in the observable sector and are the main parameters to follow in the attempts to identify supersymmetric theories with potential experimental observations.

6.1.4 Hierarchy - Problem

In high energy physics there are at least two fundamental scales the Planck mass $M_{planck} \sim 10^{19}$ GeV defining the scale of quantum gravity and the electroweak scale $M_{EW} \sim 10^2$ GeV, defining the symmetry breaking scale of the standard model. Undesrstanding why these two scales are so different is the hierarchy problem. Actually the problem can be formulated in two parts:

1. Why $M_{EW} << M_{Planck}$? which is the proper hierarchy problem.
2. Is this hierarchy stable under quantum corrections? This is the 'naturalness' part of thehierarchy problem which is the one that presents a bigger challenge.

Let us try to understand the naturalness part of thehierarchy problem.

In the Standard Model we know that:

- Gauge particles are massless due to gauge - invariance, that means, a direct mass term for the gauge particles $MA_\mu A^\mu$ is not allowed by gauge invariance ($A_\mu \to A_\mu + \partial_\mu \alpha$ for a $U(1)$ field).
- Fermions: Also gauge invariance forbids $m\psi\psi$ for all quarks and leptons. Recall these particles receive a mass only thorugh the Yukawa couplings to the Higgs ($H\psi\psi$ gives a mass to ψ after H gets a nonzero value).
- Scalars: only the Higgs in the standard model. They are the only ones that can have a mass term in the Lagrangian $m^2 H\bar{H}$. So there is not a symmetry that protects the scalars from becoming very heavy. Actually, if the standard model is valid up to a fixed cut-off scale Λ (for instance $\Lambda \sim M_{Planck}$ as an extreme case), it is known that loop corrections to the scalar mass m^2 induce values of order Λ^2 to the scalar mass. These corrections come from both bosons and fermions running in the loop. These would make the Higgs to be as heavy as Λ. This is unnatural since Λ can be much larger than the electroweak scale $\sim 10^2$ GeV. Therefore even if we start with a Higgs mass of order the electroweak scale, loop corrrections would bring it up to the highest scale in the theory, Λ. This would ruin the hierarchy between large and small scales. It is possible to adjust or 'fine tune' the loop corrections such as to keep the Higgs light, but this would require adjustments to many decimal figures on each ordere of perturbation theory. This fine tuning is considered unnatural and an explanation of why the Higgs mass (and the whole electroweak scale)

can be naturally maintained to be hierarchically smaller than the Planck scale or any other large cut-off scale Λ is required.

In SUSY, bosons have the same masses as fermions, so no problem about hierarchy for all squarks and sleptons since the fermions have their mass protected by gauge invariance. Secondly, we have seen that explicit computation of loop diagrams cancel boson against fermion loops due to the fact that the couplings defining the vertices on each case are determined by the same quantity (g in the Yukawa coupling of fermions to scalar and g^2 in the quartic couplings of scalars as was mentioned in the discussion of the WZ model). These “miraculous cancellations” protect the Higgs mass from becoming arbitrarily large. See the discussion and diagram at the end of subsection 4.1.1. Another way to see this is that even though a mass term is still allowed for the Higgs by the coupling in the superpotential $\mu H_1 H_2$, the non-renormalization of the superpotential guarantees that the, as long as supersymmetry is not broken, the mass parameter μ will not be corrected by loop effects.

Therefore if supersymmetry were exact the fermions and bosons would be degenerate but if supersymmetry breaks at a scale close to the electroweak scale then it will protect the Higgs from becoming too large. This is the main reason to expect supersymmetry to be broken at low energies of order $10^2 - 10^3$ GeV to solve the naturalness part of the hierarchy problem.

Furthemore, the fact that we expect supersymmetry to be broken by non-perturbative effects (of order e^{-1/g^2}) is very promising as a way to explain the existence of the hierarchy (first part of the hierarchy problem). That is that if we start at a scale $M >> M_{EW}$ ($M \sim M_{Planck}$ in string theory or GUT's), the supersymmetry breaking scale can be generated as $M_{susy} \sim Me^{-1/g^2}$, for a small gauge coupling, say $g \sim 0.1$, this would naturally explain why $M_{susy} << M$.

6.1.5 Cosmological Constant - Problem

This is probably a more difficult problem as explained in section 1.2. The recent evidence of an accelrating universe indicates a new scale in physics which is the cosmological constant scale M_Λ, with $M_\Lambda / M_{EW} \sim M_{EW}/M_{Planck} \sim 10^{-15}$. Explaining why M_Λ is so small is the cosmological constant problem. Again it can expressed in two parts, why the ratio is so small and (more difficult) why this ratio is stable under quantum corrections.

Supersymmetry could in principle solve this problem, since it is easy to keep the vacuum energy Λ to be zero in a supersymmetric theory. However keeping it so small would require a supersymmetry breaking scale of order $\Delta m \sim M_\Lambda \sim 10^{-3}$ eV. But that would imply that the superpartner of the electron would be essentially of the same mass as the electron and should have been seen experimentally long ago. Therefore the best supersymetry can do is to keep the cosmological constant Λ small until it breaks. If it breaks at the elctroweak scale M_{EW} that would lead to $M_\Lambda \sim M_{EW}$ which is not good enough.

Can we address both the hierarchy- and the cosmological constant - problem at the same time? Some attempts are rcently put forward in terms of the string theory ‘landscape’ in which our universe is only one of a set of a huge number of solutions (or vacua) of the theory. This number being greater than 10^{500} would indicate that a few of these universes will have the value of the cosmological constant we have today, and we happen to live in one of those (in the same way that there are many galaxies and planets in the universe and we just happen to live in one). This is still very controversial, but has lead to speculations that if this is a way of solving the cosmological constant problem, it would indicate a similar solution of the hierarchy problem and the role of supersymmetry would be diminished in explaining the hierarchy problem. This would imply that the scale of supersymmetry breaking could be much larger. It is fair to say that there is not at present a satisfactory approach to both the hierarchy and cosmological constant problems. It is important to keep in mind that even though low-energy supersymmetry solves the hierarchy problem in a very elegant way, tyhe fact that it does not address the cosmological constant problem is worrisome in the sense that any solution of the cosmological constant problem could affect our understanding of low energy physics to change the nature of the hierarchy problem and then the importance of low-energy supersymmetry. This is a very active area of research at the moment.

Chapter 7

Extra Dimensions

It is important to look for alternative ways to address the problems that supersymmetry solves and also to address other problems of the standard model. We mentioned in the first lecture that supersymmetry and extra dimensions are the natural extensions of spacetime symmetries that may play an important role in our understanding of nature. here we will start the discussion of physics in extra dimensions.

7.1 Basics of Kaluza - Klein - Theories

7.1.1 History

- In 1914 Nordstrom and 1919 - 1921 Kaluza independently tried to unify gravity and electromagnetism. Nordstrom was attempting an unsuccessful theory of gravity in terms of scalar fields, prior to Einstein. Kaluza used Einstein's theory extended to five dimensions. His concepts were based on Weyl's ideas.
- 1926 Klein: cylindric universe with 5th dimension of small radius R
- after 1926 Several people developed the KK ideas (Einstein, Jordan, Pauli, Ehrenfest,...)

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- 1960's: B. de Witt ($D > 5$) obtaining Yang-Mills in 4d. Also strings with $D = 26$.
- In 1970's and 1980's. Superstrings required $D = 10$. Developments in supergravity required extra dimensions and possible maximum numbers of dimensions for SUSY were discussed: $D = 11$ turned out to be the maximum number of dimensions (Nahm). Witten examined the coset

$$G/H = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}, \qquad \dim(G/H) = (8+3+1) - (3+1+1) = 7$$

  which implied $D = 11$ also to be the minimum. 11 dimensions, however, do not admit chirality since in odd dimensions, there is no analogue of Dirac γ - matrices.

- 1990's: Superstrings, revived $D = 11$ (M - theory) and brane - world - scenario (large extra dimensions).

7.1.2 Scalar Field in 5 Dimensions

Before discusing the Kaluza-Klein ideas of gravity in extra dimensions, we will start with the simpler cases of scalar fields in extra dimensions, followed by vector fields and other bosonic fields of helicity $\lambda \leq 1$. This will illustrate in simple terms the effects of having extra dimensions. We will be building up on the level of complexity to reach gravitational theories in five and higher dimensions. In the next chapter we extend the discusiion to include fermionic fields.

Consider a massless 5D scalar field $\varphi(x^M)$, $M = 0, 1, ..., 4$ with action

$$\mathcal{S}_{5D} \quad = \quad \int d^5x \; \partial^M \varphi \partial_M \varphi \; .$$

Set the extra dimension $x^4 = y$ defining a circle of radius r with $y \equiv y + 2\pi r$. Our spacetime is now $\mathbb{M}_4 \times S^1$. Periodicity in y - direction implies Fourier - expansion

$$\varphi(x^\mu, y) \quad = \quad \sum_{n=-\infty}^{\infty} \varphi_n(x^\mu) \exp\left(\frac{iny}{r}\right) \; .$$

Notice that the Fourier coefficients are functions of the standrd 4D coordinates and therefore are (an infinite number of) 4D scalar fields. The equations of motion for the Fourier - modes are wave - equations

$$\begin{aligned} \partial^M \partial_M \varphi \quad = \quad 0 \quad &\Longrightarrow \quad \sum_{n=-\infty}^{\infty} \left(\partial^\mu \partial_\mu - \frac{n^2}{r^2} \right) \varphi_n(x^\mu) \exp\left(\frac{iny}{r}\right) \quad = \quad 0 \\ &\Longrightarrow \quad \partial^\mu \partial_\mu \varphi_n(x^\mu) - \frac{n^2}{r^2} \varphi_n(x^\mu) \quad = \quad 0 \; . \end{aligned}$$

These are then an infinite number of Klein-Gordon equations for massive 4D fields. This means that each Fourier mode φ_n is a 4D particle with mass, $m_n^2 = \frac{n^2}{r^2}$. Only the zero - mode ($n = 0$) is massless. Visualize the states as an infinite tower of massive states (with increasing mass proportional to n). This is called "Kaluza-Klein" - tower and the massive states ($n \neq 0$) are called Kaluza-Klein or momentum states, since they come from the momentum in the extra dimension:

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In order to obtain the effective action in 4D for all these particles, let us plug the mode - expansion of φ into the original 5D action,

$$\begin{aligned} \mathcal{S}_{5D} \quad &= \quad \int d^4x \int dy \sum_{n=-\infty}^{\infty} \left(\partial^\mu \varphi_n(x^\mu) \partial_\mu \varphi_n(x^\mu)^* - \frac{n^2}{r^2} |\varphi_n|^2 \right) \\ &= \quad 2\pi r \int d^4x \; (\partial^\mu \varphi_0(x^\mu) \partial_\mu \varphi_0(x^\mu)^* + ...) \quad = \quad \mathcal{S}_{4D} + ... \; . \end{aligned}$$

This means that the 5D action reduces to one 4D action for a massless scalar field plus an infinite sume of massive scalars in 4D. If we are interested only about energies smaller than $1/r$ we may concentrate only on the 0-mode action. If we keep only the 0 - mode (like Kaluza did), then $\varphi(x^M) = \varphi(x^\mu)$. This would be equivalent to just 'truncating' all the massive fields. In this case speak of 'dimensional reduction'. More generally, if we keep all the massive modes we talk about "compactification", meaning that the extra dimension is compact and its existence is taken into account as long as the Fourier modes are included.

7.1.3 Vector - Field in 5 Dimensions

Let us now move to the next simpler case of an abelian vector field in 5D, similar to electromagnetic field in 4D. We can split a massless vector - field $A_M(x^M)$ into

$$A_M = \begin{cases} A_\mu & \text{(vector in 4 dimensions)} \\ A_4 =: \rho & \text{(scalar in 4 dimensions)} \end{cases} .$$

Each component has a Fourier - expansion

$$A_\mu = \sum_{n=-\infty}^{\infty} A_\mu^n \exp\left(\frac{iny}{r}\right) , \qquad \rho = \sum_{n=-\infty}^{\infty} \rho_n \exp\left(\frac{iny}{r}\right) .$$

Consider the action

$$\mathcal{S}_{5D} = \int d^5x \, \frac{1}{g_{5D}^2} F_{MN} F^{MN}$$

with field - strength

$$F_{MN} := \partial_M A_N - \partial_N A_M$$

implying

$$\partial^M \partial_M A_N - \partial^M \partial_N A_M = 0 .$$

Choose a gauge, e.g. transverse

$$\partial^M A_M = 0 , \qquad A_0 = 0 \implies \partial^M \partial_M A_N = 0 ,$$

therefore this becomes equivalent to the scalar field case (for each component A_M) indicating an infinite tower of massive states for each massless state in 5D. In order to find the 4D effective action we can plug this into the 5D action:

$$\mathcal{S}_{5D} \longmapsto \mathcal{S}_{4D} = \int d^4x \left(\frac{2\pi r}{g_{5D}^2} F_{(0)}{}^{\mu\nu} F_{(0)\mu\nu} + \frac{2\pi r}{g_{5D}^2} \partial_\mu \rho_0 \partial^\mu \rho_0 + \dots \right) ,$$

Therefore we have a 4D theory of a gauge particle (massless), a massless scalar and infinite towers of massive vector and scalar fields. Notice that the gauge couplings of 4 - dimensional and 5 - dimensional actions (coefficients of $F_{MN}^F{}^{MN}$ and $F_{\mu\nu}F^{\mu\nu}$) are related by

$$\frac{1}{g_{4D}^2} = \frac{2\pi r}{g_{5D}^2} .$$

In D spacetime - dimensions, this genealizes to

$$\frac{1}{g_4^2} = \frac{V_{D-4}}{g_D^2}$$

where V_n is the volume of the n - dimensional sphere of radius r.

Higher dimensional electromagnetic fields have further interesting issues that we pass to discuss:

Electric (and Gravitational) Potential

Gauss' law implies for the electric field $\vec{E}$ and its potential Φ of a point - charge Q:

$$\oint_{S^2} \vec{E} \cdot d\vec{S} = Q \implies \|\vec{E}\| \propto \frac{1}{R^2} , \qquad \Phi \propto \frac{1}{R} \qquad \text{4 dimensions}$$

$$\oint_{S^3} \vec{E} \cdot d\vec{S} = Q \implies \|\vec{E}\| \propto \frac{1}{R^3} , \qquad \Phi \propto \frac{1}{R^2} \qquad \text{5 dimensions}$$

So in D spacetime - dimensions

$$\|\vec{E}\| \propto \frac{1}{R^{D-2}} , \qquad \Phi \propto \frac{1}{R^{D-3}} .$$

If one dimension is compactified (radius r) like in $\mathbb{M}_4 \times S^1$, then

$$\|\vec{E}\| \propto \begin{cases} \frac{1}{R^3} & : \ R < r \\ \frac{1}{R^2} & : \ R >> r \end{cases} .$$

Analogues arguments hold for graviational fields and their potentials.

Comments on Spin and Number of Degrees of Freedom

We know that in 4D a gauge particle has spin one and carries two degrees of freedom. We may ask what is the generalization of these results to a higher dimensionalgauge field.

Recall Lorentz - algebra in 4 dimension

$$\left[M^{\mu\nu} \ , \ M^{\rho\sigma} \right] = i(\eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\rho} M^{\nu\sigma})$$

$$J_i = \epsilon_{ijk} M_{jk} , \qquad J \propto M_{23} .$$

For massless representations in D dimensions, $O(D-2)$ is little group:

$$P^\mu = (E \ , \ E \ , \underbrace{0 \ , \ ... \ , \ 0}_{O(D-2)})$$

The Lorentz - algebra is just like in 4 dimensions, replace μ, ν, ... by M, N, ..., so M_{23} commutes with M_{45} and M_{67} for example. Define the spin to be the maximum eigenvalue of any $M^{i(i+1)}$. The number of degrees of freedom in 4 dimensions is 2 ($A_\mu \mapsto A_i$ with $i = 2, 3$) corresponding to the 2 photon - polarizations and $(D-2)$ in D dimension, $A_M \mapsto A_i$ where $i = 1, 2, ..., D-2$.

7.1.4 Duality and Antisymmetric Tensor Fields

So far we considered scalar- and vector - fields:

	scalar	vector	index - range
$D = 4$	$\varphi(x^\mu)$	$A_\mu(x^\mu)$	$\mu = 0, 1, 2, 3$
$D > 4$	$\varphi(x^M)$	$A_M(x^M)$	$M = 0, 1, ..., D-1$

We will see now that in extra dimensions there are further fields corresponding to bosonic particles of helicity $\lambda \leq 1$. These are antisymmetric tensor fields, which in 4D are just equivalent to scalars or vector fields by a symmetry known as 'duality' but in extra dimensions these will be new types of particles (that play an important role in string thoery for instance).

In 4 dimensions, define a dual field - strength to the Faraday - tensor $F^{\mu\nu}$ via

$$\tilde{F}^{\mu\nu} \quad := \quad \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ,$$

then Maxwell's equations in vacuum read:

$$\partial^\mu F_{\mu\nu} \quad = \quad 0 \qquad \text{(field - equations)}$$
$$\partial^\mu \tilde{F}_{\mu\nu} \quad = \quad 0 \qquad \text{(Bianchi - identities)}$$

The exchange $F \leftrightarrow \tilde{F}$ corresponding to $\vec{E} \leftrightarrow \vec{B}$ swaps field - equations and Bianchi - identities (EM - duality).

In 5 dimensions, one could define in analogy

$$\tilde{F}^{MNP} \quad = \quad \epsilon^{MNPQR} F_{QR} \ .$$

One can generally start with an antisymmetric $(p+1)$ - tensor $A_{M_1 \ldots M_{p+1}}$ and derive a field strength

$$F_{M_1 \ldots M_{p+2}} \quad = \quad \partial_{[M_1} A_{M_2 \ldots M_{p+2}]}$$

and its dual (with $D - (p+2)$ indices)

$$\tilde{F}_{M_1 \ldots M_{D-p-2}} \quad = \quad \epsilon_{M_1 \ldots M_D} F^{M_{D-p-1} \ldots M_D} \ .$$

Consider for example

- $D = 4$

  $$F_{\mu\nu\rho} \quad = \quad \partial_{[\mu} B_{\nu\rho]} \quad \Longrightarrow \quad \tilde{F}_\sigma \quad = \quad \epsilon_{\sigma\mu\nu\rho} F^{\mu\nu\rho} \quad = \quad \partial_\sigma a$$

  The dual potentials that yield field strengths $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu}$ have different number of indices, 2 - tensor $B_{\nu\rho} \leftrightarrow a$ (scalar potential).

- $D = 6$

  $$F_{MNP} \quad = \quad \partial_{[M} B_{NP]} \quad \Longrightarrow \quad \tilde{F}_{QRS} \quad = \quad \epsilon_{MNPQRS} F^{MNP} \quad = \quad \partial_{[Q} \tilde{B}_{RS]}$$

  Here the potentials $B_{NP} \leftrightarrow \tilde{B}_{RS}$ are of the same type.

Antisymmetric tensors carry spin 1 or less, in 6 dimensions:

$$B_{MN} \quad = \quad \begin{cases} B_{\mu\nu} & : \text{ scalar in 4 dimensions} \\ B_{\mu 5} \ , \ B_{\mu 6} & : \text{ 2 vectors in 4 dimensions} \\ B_{56} & : \text{ scalar in 4 dimensions} \end{cases}$$

To see the number of degrees of freedom, consider little group

$$B_{M_1 \ldots M_{p+1}} \quad \longmapsto \quad B_{i_1 \ldots i_{p+1}} \ , \qquad i_k \quad = \quad 1, ..., (D-2) \ .$$

These are $\begin{pmatrix} D-2 \\ p+1 \end{pmatrix}$ independent components. Note that under duality,

$$\mathcal{L} \quad = \quad \frac{1}{g^2} (\partial_{[M_1} B_{M_2 \ldots M_{p+2}]})^2 \quad \longleftrightarrow \quad g^2 (\partial_{[M_1} \tilde{B}_{M_2 \ldots M_{D-(p+2)}]})^2$$

p branes

Electromagnetic fields couple to the worldline of particles via

$$\int A_\mu \ dx^\mu \ ,$$

This can be seen as follows: the electromagnetic field couples to a conserved current in four dimensions as $\int d^4 x A_\mu J^\mu$ ($J^\mu = \bar{\psi} \gamma^\mu \psi$ for an electron field for instance). For a particle of charge q, the current can be written as an integral over the world line of the particle $J^\mu = q \int d\xi^\mu \delta^4 (x - \xi)$ such that $\int J^0 d^3 x = q$ and so the coupling becomes $\int d^4 x J^\mu A_\mu = q \int d\xi^\mu A_\mu$.

We can extend this idea for higher dimensional objects. For a potential $B_{[\mu\nu]}$ with two indices, the analogue is

$$\int B_{\mu\nu} \ dx^\mu \wedge dx^\nu \ ,$$

i.e. need a string with 2 dimensional worldsheet to couple. Further generalizations are

$$\int B_{\mu\nu\rho} \ dx^\mu \wedge dx^\nu \wedge dx^\rho \qquad\qquad \text{(membrane)}$$
$$\int B_{M_1 \ldots M_{p+1}} \ dx^{M_1} \wedge \ldots \wedge dx^{M_{p+1}} \qquad\qquad (p - \text{brane})$$

Therefore we can see that antisymmetric tensors of higher rank coupled naturally to extended objects. This leads to n introduction of the concept of a p-brane as a generalisation of a particle that couples to antisymmetric tensors of rank $p+1$. A particle carries charge under a vector field, such as electromagnetism. In the same sense, p branes carry a *new* kind of charge with respect to a higher rank antisymmetric tensor.

7.1.5 Gravitation: Kaluza-Klein Theory

After discussing scalar-, vector- and antisymmetric tensor - fields

	spin	deg. of freedom
scalar φ	0	$1+1$
vector A_M	0 , 1	$D-2$
antisymmetric tensor $A_{M_1 \dots M_{p+1}}$	0 , 1	$\begin{pmatrix} D-2 \\ p+1 \end{pmatrix}$

we are now ready to consider the graviton G_{MN} of Kaluza - Klein - theory in D dimensions

$$G_{MN} \;=\; \begin{cases} G_{\mu\nu} & \text{graviton} \\ G_{\mu n} & \text{vectors} \\ G_{mn} & \text{scalars} \end{cases}$$

where $\mu, \nu = 0, 1, 2, 3$ and $m, n = 4, ..., D-1$.

The background - metric appears in the 5 - dimensional Einstein - Hilbert - action

$$\mathcal{S} \;=\; \int d^5x \ \sqrt{|G|}^{(5)} R \ , \qquad {}^{(5)}R_{MN} \;=\; 0 \ .$$

One possible solution is 5 dimensional Minkowski - metric $G_{MN} = \eta_{MN}$, another one is of four-dimensional Minkowski spacetime M_4 times a circle S^1, *i.e.* the metric is of the $\mathbb{M}_4 \times S^1$ - type

$$ds^2 \;=\; W(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

where $\mathbb{M}_3 \times S^1 \times S^1$ is equally valid. $W(y)$ is a "warped factor" that is allowed by the symmetries of the background and y is restricted to the interval $[0, 2\pi r]$. For somplicity we will set the wwarp factor to a constant but will consider it later where it will play an important role.

Consider excitations in addition to the background - metric

$$G_{MN} \;=\; \phi^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi A_\mu A_\nu & -\kappa\phi A_\mu \\ -\kappa\phi A_\nu & \phi \end{pmatrix}$$

in Fourier - expansion

$$G_{MN} \;=\; \underbrace{\phi^{(0)-\frac{1}{3}} \begin{pmatrix} g^{(0)}_{\mu\nu} - \kappa^2 \phi^{(0)} A^{(0)}_\mu A^{(0)}_\nu & -\kappa\phi^{(0)} A^{(0)}_\mu \\ -\kappa\phi^{(0)} A^{(0)}_\nu & \phi^{(0)} \end{pmatrix}}_{\text{Kaluza - Klein - ansatz}} + \infty \text{ tower of massive modes}$$

and plug the zero - mode - part into the Einstein - Hilbert - action:

$$\mathcal{S}_{4D} \;=\; \int d^4x \ \sqrt{|g|} \left\{ M_{pl}^2 \, {}^{(4)}R - \frac{1}{4}\phi^{(0)} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \frac{1}{6} \frac{\partial^\mu \phi^{(0)} \partial_\mu \phi^{(0)}}{(\phi^{(0)})^2} + ... \right\}$$

This is the unified theory of gravity, electromagnetism and scalar fields! Its symmetries will be discussed in the next section.

Symmetries

- General 4 - dimensional coordinate - transformations

$$x^\mu \longmapsto x'^\mu(x^\nu)\ , \qquad g_{\mu\nu}^{(0)}\ \text{(graviton)}\ , \qquad A_\mu^{(0)}\ \text{(vector)}$$

- y - transformation

$$y \longmapsto y' = F(x^\mu, y)$$

Notice that

$$ds^2 = \phi^{(0)-\frac{1}{3}}\left\{g_{\mu\nu}^{(0)}dx^\mu dx^\nu - \phi^{(0)}(dy - \kappa A_\mu^{(0)}dx^\mu)^2\right\}$$

so, in order to leave ds^2 invariant, need

$$F(x^\mu, y) = y + f(x^\mu) \implies dy' = dy + \frac{\partial f}{\partial x^\mu}dx^\mu\ , \qquad A_\mu^{\prime(0)} = A_\mu^{(0)} + \frac{1}{\kappa}\frac{\partial f}{\partial x^\mu}$$

which is the gauge - transformation for a massless field $A_\mu^{(0)}$! This is the way to understand that standard gauge symmetries can be derived from general coordinate transformations in extra dimensions, explaining the Kaluza-Klein programme of unifying all the interactions by means of extra dimensions.

- overall scaling

$$y \longmapsto \lambda y\ , \qquad A_\mu^{(0)} \longmapsto \lambda A_\mu^{(0)}\ , \qquad \phi^{(0)} \longmapsto \frac{1}{\lambda^2}\phi^{(0)} \implies ds^2 \longmapsto \lambda^{\frac{2}{3}}ds^2$$

$\phi^{(0)}$ is a massless "modulus - field", a flat direction in the potential. $\langle\phi^{(0)}\rangle$ and therefore the size of the 5th dimension is arbitrary. $\phi^{(0)}$ is called breathing mode, radion or dilaton. This is a major problem for these theories. It looks like *all* the values of the radius (or volume in general) of the extra dimensions are equally good and the theory does not provide a way to fix this size. It is a manifestation of the problem that the theory cannot prefer a flat 5D Minkowski space (infinite radius) over $M_4 \times S^1$ (or $M_3 \times S^1 \times S^1$, etc.). This is the 'moduli' problem of extra dimensional theories. String theories share this problem. Recent developments in string theory allows to fix the value of the volume and shape of the extra dimension, leading to a large but discrete set of solutions. This is the so-called 'landscape' of string solutions (each one describing a different universe and ours is only one among a huge number of them).

Comments

- The Planck - mass $M_{pl}^2 = M_*^3 \cdot 2\pi r$ is a derived quantity. We know experimentally taht $M_{pl} \approx 10^{19}$ GeV, therefore we can adjust M_* and r to give the right result. But there is no other constraint to fix M_* and r.

- Generalization to more dimensions

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 A^i_\mu A^j_\nu h_{ij} & -\kappa\gamma_{mn} K^n_i A^i_\mu \\ -\kappa\gamma_{mn} K^m_i A^i_\nu & \gamma_{mn} \end{pmatrix}$$

The K^m_i are Killing - vectors of an internal manifold M_{D-4} with metric γ_{mn}. The theory corresponds to Yang - Mills in 4 dimensions. Note that the Planck - mass now behaves like

$$M_{pl}^2 = M_*^{D-2} V_{D-4} \propto M_*^{D-2} r^{D-4} = M_*^2 (M_* r)^{D-4} \ .$$

In general we know that the highest energies explored so far require $M_* > 1$ TeV and $r < 10^{-16}$ cm since no signature of extra dimensions has been seen in any experiment. In Kaluza-Klein theories there is no reason to expect a large value of the volume and it has been usually assumed that $M_* \approx M_{pl}$.

7.2 The Brane - World - Scenario

So far we have been discussion the standard Kaluza-Klein theory in which our universe is higher dimensional. We have not seen the extra dimensions because they are very small (smaller than the smallest scale that can be probed experimenatlly at colliders which is 10^{-16} cm).

We will introduce now a different and more general higher dimensional scenario. The idea here is that our universe is a p brane, or a surface inside a higher dimenional 'bulk' spacetime. A typical example of this is as follows: all the standard model particles (quarks, leptons but also gauge fields) are trapped on a three-dimensional spatial surface (the brane) inside a high dimension spacetime (the bulk). Gravity on the other hand lives on the full bulk spacetime and therefore only gravity probes the extra dimensions.

Therefore we have to distinguish the D - dimensional "Bulk" - space (background spacetime) from the $(p+1)$ world - volume - coordinates of a p - brane. Matter lives in the $d(=4)$ dimensions of the brane, whereas gravity takes place in the D Bulk - dimensions. This scenario seems very ad-hoc at first sight but it is naturally realised in string theory where matter tends to live on D-branes (a particular class of p-branes corresponding to surfaces where ends of open strings are attached to). Whereas gravity, coming from closed strings can leave in the full higher dimensional (10) spacetime. Then the correspondence is as follows:

$$\begin{array}{rcl} \text{gravity} & \longleftrightarrow & \text{closed strings} \\ \text{matter} & \longleftrightarrow & \text{open strings} \end{array}$$

heightwidthdepthSUSY18.png

For phenomenological purposes we can distinguish two different classes of brane world scenarios.

1. *Large Extra Dimensions.* Let us first consider an unwarped compactification, that is a constant warp factor $W(y)$. We have remarked that the fundamental higher dimensional scale M_* is limited to be $M_* \geq 1$ TeV in order to not contradict experimental observations which can probe up to that energy. By the same argument we have constrained the size of the extra dimensions r to be $r < 10^{-16}$ cm because this is the length associated to the TeV scale of that accelerators can probe. However, in the brane world scenario, if only gravity feels the extra dimensions, we have to use the constraints for gravity only. Since gravity is so weak, it is difficult to test experimentally and so far the best experiments can only test it to scales of larger than 0.1 mm. This is much larger than the 10^{-16} cm of the standard model. Therefore, in the brane world scenario it is possibe to have extra dimensions as large as 0.1 mm without contradicting any experiment!

   This has an important implication also as to the value of M_* (which is usually taken to be of order M_{pl} in Kaluza-Klein theories. From the Einstein-Hilbert action, the Planck - mass M_{pl} is still given by

   $$M_{pl}^2 = M_*^{D-2} V_{D-4} \ .$$

   with $V_{D-4} \sim r^{D-4}$ the volume of the extra dimensions. But now we can have a much smaller fundamental scale M_* if we allow the volume to be large enough. We may even try to have the fundamental scale to be of order $M_* \sim 1$ TeV. In five dimensions, this will require a size of the extra dimension to be of order $r \sim 10^8$ Km in order to have a Planck mass of the observed value $M_{pl} \sim 10^{18}$ GeV (where we have used $r = M_{pl}^2/M_*^3$). This is clearly ruled out by experiments. However, starting with a six-dimensional spacetime we get $r^2 = M_{pl}^2/M_*^4$, which gives $r \sim 0.1$mm for $M_* = 1$ TeV. This is then consistent with all gravitational experiments as well as standard model tests. Higher dimensions would give smaller values of r and will also be consistent. The inetersting thing about the six-dimensional case is that it is possible to be tested by the next round of experiments in both, the accelerator experiments probing scales of order TeV and gravity experiments, studying deviations of the squared law at scales smaller than 0.1mm.

   Notice that this set up changes the nature of the hierarchy problem because now the small scale (*i.e.* $M_{EW} \sim M_* 1$ Tev is fundamental whereas the large Planck scale is a derived quantity. The hierarchy problem now is changed to explain why the size of the extra dimensions is so large to generate the Planck scale of 10^{18} GeV starting from a small scale $M_* \sim 1$ TeV. This changes the nature of the hierarchy problem, becasue it turns it into a dynamical question of how to fix the size of the extra dimensions. Notice that this will require exponentially large extra dimensions (in units of the inverse fundamental scale M_*). The hierarchy problem then becomes the problem of finding a mechanism that gives rise to exponentially large sizes of the extra dimensions.

2. *Warped Compactifications* This is the so-called Randall-Sundrum scenario. The simplest case is again a five-dimensional theory but with the following properties. Instead of the extra dimension being a circle S^1, it is now an interval I (which can be defined as an *orbifold* of S^1 by identifying the points $y = -y$, if the original circle had length $2\pi r$, the interval I will have half that size, πr). The surfaces at each end of the interval play a role similar to a brane, being three-dimensional surfaces inside a five-dimensional spacetime. The second important ingredient is that the warp factor $W(y)$ is not asumed to be a constant but to be determined by solving Einstein's equations in this background. We then have warped geometries with a y - dependent warp - factor $\exp\big(W(y)\big)$,

in 5 dimensions

$$ds^2 \quad = \quad \exp\big(W(y)\big)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ .$$

The volume V_{D-4} has a factor

$$\int_{-\pi}^{+\pi} dy \ \exp\big(W(y)\big) \ .$$

Consider then the two 'branes', one at $y = 0$ ('the Planck brane') and one at $y = \pi r$ ('the standard model brane'), the total action has contributions from the two branes and the Bulk itself:

heightwidthdepthSUSY19.png

$$\mathcal{S} \quad = \quad \mathcal{S}_{y=0} + \mathcal{S}_{y=\pi r} + \mathcal{S}_{\text{bulk}}$$

Einstein's equations imply $W(y) \propto e^{-|ky|}$ with k a constant (see hep-ph 9905221 and example sheet 4), so the metric changes from $y = 0$ to $y = \pi r$ via $\eta_{\mu\nu} \longmapsto \exp(-k\pi r)\eta_{\mu\nu}$. This means that all the length and energy scales change by changing y. If the fundamental scale is $M_* \sim M_{pl}$, the $y = 0$ - brane carries physics at M_{pl}, but as long as we move away from this end of the interval, all the energy scales will be 'red-shifted' by the factor $e^{-|ky|}$ until we reach the other end of the interval in which $y = \pi r$. This exponential changes of scales is appropriate for the hierarchy problem. If the fundamental scale is the Planck scale, at $y = 0$ the physics will be governed by this scale but at $y = r$ we will have an exponentially smaller scale. In particular we can have the electroweak scale $M_{ew} \sim M_{pl}e^{-\pi kr} \propto 1$ TeV if r is only slightly bigger than the Planck length $r \geq 50 l_{pl}$. This is a more elegant way to 'solve' thehierarchy problem. We only need to find a mechanism to fix the value of r of order $50 l_{pl}$! Notice that in this scenario five-dimesions are compatible with experiment (unlike the unwarped case that required a radius many kilometers large).

Notice that in both scenarios the problem of solving thehierarchy problem has been turned into the problem of fixing the size of the extra dimensions. It is worth remarking that both mechanisms have been found to be realised in string theory (putting them on firmer grounds). Studying mechanisms to fix the 'moduli' that determines the size and shape of extra dimensions is one of the most active areas of reserach within string theory.

Chapter 8

Supersymmetry in Higher Dimensions

So far we have been discussed the possible bosonic fields in extra dimensions (scalars, vectors, antisymmetric tensors and metrics).

What about fermionic fields in extra dimensions?

8.1 Spinors in Higher Dimensions

For a theory of fermions in more than 4 dimensions, need some analogue of the 4 - dimensional Dirac γ - matrices, i.e. representations of

$$\Big\{\Gamma^M \ , \ \Gamma^N\Big\} \ = \ 2\eta^{MN} \ , \qquad \Sigma^{MN} \ = \ \frac{i}{4}\Big[\Gamma^M \ , \ \Gamma^N\Big] \ ,$$

where the Σ^{MN} are the generators of $SO(1, D-1)$.

- Representations in even dimensions $D = 2n$:

  Define

$$\begin{aligned} a_i \ &= \ \frac{i}{2}(\Gamma_{2i-1} + i\Gamma_{2i}) \ , \qquad i = 1, ..., n \\ \Longrightarrow \quad \Big\{a_i \ , \ a_j^\dagger\Big\} \ &= \ \delta_{ij} \ , \qquad \Big\{a_i \ , \ a_j\Big\} \ = \ \Big\{a_i^\dagger \ , \ a_j^\dagger\Big\} \ = \ 0 \ . \end{aligned}$$

  Let $|0\rangle$ denote the vacuum such that $a_i|0\rangle = 0$, then there are states

$$\begin{array}{c|ccccc} \text{states} & |0\rangle & a_i^\dagger|0\rangle & a_i^\dagger a_j^\dagger|0\rangle & \cdots & (a_n^\dagger a_{n-1}^\dagger ... a_1^\dagger)|0\rangle \\ \text{number} & 1 & n & \begin{pmatrix} n \\ 2 \end{pmatrix} & \cdots & 1 \end{array}$$

  of total number

$$1 + n + \begin{pmatrix} n \\ 2 \end{pmatrix} + ... + 1 \ = \ \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ = \ 2^n \ = \ 2^{\frac{D}{2}} \ .$$

The spinor - representation is given by $s_i = \pm\frac{1}{2}$

$$|s_1...s_n\rangle \;=\; a_1^{\dagger(s_1+\frac{1}{2})}...a_n^{\dagger(s_n+\frac{1}{2})}|0\rangle \;.$$

Note that the generators $\Sigma_{2i,2i-1}$ commute with each other. Consider

$$S_i \;:=\; \Sigma_{2i,2i-1} \;=\; a_i^{\dagger}a_i - \frac{1}{2} \;,$$

then the $|s_1...s_n\rangle$ defined above are simultaneous eigenstates of all the S_i's,

$$S_i|s_1...s_n\rangle \;=\; s_i|s_1...s_n\rangle \;,$$

call those $|s_1...s_n\rangle$ Dirac - spinors. In $D = 4$ dimensions, e.g., $n = 2$, the states $|\pm\frac{1}{2}, \pm\frac{1}{2}\rangle$ form a 4 component - spinor. Recall $\Sigma_{03} = K_3$ and $\Sigma_{21} = J_3$.

Representations in even dimensions are reducible, since the generalization of γ_5,

$$\Gamma_{2n+1} \;=\; i^n\Gamma_1\Gamma_2...\Gamma_{2n}$$

satisfies

$$\Big\{\Gamma_{2n+1} \;,\; \Gamma_M\Big\} \;=\; 0 \;, \qquad \Big[\Gamma_{2n+1} \;,\; \Sigma_{MN}\Big] \;=\; 0 \;, \qquad \Gamma_{2n+1}^2 \;=\; \mathbb{1} \;.$$

All the $|s_1...s_n\rangle$ are eigenstates to Γ_{2n+1}

$$\Gamma_{2n+1}|s_1...s_n\rangle \;=\; \pm|s_1...s_n\rangle$$

with eigenvalue $+1$ for even numbers of $s_i = +\frac{1}{2}$ and -1 for odd ones. This property is called chirality, the spinors are "Weyl - spinors" in that case. Note that

$$\Gamma_{2n+1} \;=\; 2^n S_1S_2...S_n$$

- Representations in odd dimensions $D = 2n + 1$:

  Just add Γ_{2n+1} to the Γ_M - matrices, there is no extra a_i. So the representation is the same as for $D = 2n$, but now irreducible. Since odd dimensions don't have a "γ_5", there is no chirality. The spinor - representation's dimension is $2^{\frac{D-1}{2}}$.

- Majorana - spinors

  Can define a charge - conjugation C such that

$$C\Gamma^M C^{-1} \;=\; \pm(\Gamma^M)^T \;.$$

The + defines a reality condition for "Majorana - spinors". If $D = 8k+2$, then spinors can be both Majorana and Weyl.

8.2 Supersymmetry - Algebra

The SUSY - algebra in D dimensions consists of generators M_{MN}, P_M, Q_α last of which are spinors in D dimensions. The algebra has the same structure as in 4 dimensions, with the bosonic generators defining a standard Poincaré algebra in higher dimensions and

$$\Big\{Q_\alpha \ , \ Q_\beta\Big\} \ \ = \ \ a^M_{\alpha\beta} P_M + Z_{\alpha\beta}$$

where $a^M_{\alpha\beta}$ are constants and the central charges $Z_{\alpha\beta}$ now can also include brane charges. This is the $D > 4$ Coleman - Mandula- or H - L - S - generalization of te 4d algebra. The arguments for the proof are identical to those in 4d and we will skip them here.

A new feature of the Poincaré algebra is that all the generators $M_{2i,2i+1}$ commute with each other and can be simultaneously diagonalised as we have seen in the discussion of the higher dimensional spinorial representation. Then we can have several 'spins' defined as the eigenvalues of these operators. Of particular relevance is the generator M_{01}. This is used to define a *weight* w of an operator $\mathcal{O}$ by

$$[M_{01}, \mathcal{O}] = -iw\, \mathcal{O}$$

(notice that $\mathcal{O}$ and $\mathcal{O}^*$ have the same weight).

8.2.1 Representations of Supersymmetry - Algebra in Higher Dimensions

Consider massless states $P^\mu = (E \ , \ E \ , \ 0 \ , \ \dots \ , \ 0)$ with little - group $SO(D-2)$. We define the spin to be the maximum eigenvalue of M_{MN} in the representation. Notice that for the momentum of a massless particle $P_1 - P_0 = 0$ and that

$$[M_{01}, P_1 \pm P_0] = \mp i(P_1 \pm P_0)$$

Therefore the weight of $P_1 \pm P_0$ is $w = \pm 1$. Therefore in the anticommutators we only need to consider combinations of $\{Q, Q\}$ in which both Q's have weight $w = +1/2$ (so the anticommutator gives weight $w = +1$ since the weight $w = -1$ combination $P_1 - P_0$ vanishes).

So starting with arbitrary spinors Q_α of the form

$$| \pm 1/2, \pm 1/2, \cdots, \pm 1/2 >$$

which number $\mathcal{N} = 2^n = 2^{D/2}, 2^{(D-1)/2}$ for even and odd dimensionality respectively, having weight $+1/2$ it means that Q_α is of the form:

$$| + 1/2, \pm 1/2, \cdots, \pm 1/2 >$$

(recall that each entry is an eigenvalue of M_{MN} and the first one is the eigenvalue of M_{01} which is the weight.) leading to half of the number of components of Q_α: $\mathcal{N}/\in$.

Furthermore, we can separate the Q's into Q^+ and Q^- according to eigenvalues of M_{23} (standard spin in 4d). Since $P_1 + P_0$ has M_{23} eigenvalue equal to 0 as it can be easily seen from the M_{MN}, P_Q algebra, then the Q^+ and Q^- satisfy an algebra of the form $\{Q^+, Q^+\} = \{Q^-, Q^-\} = 0$ and $\{Q^+, Q^-\} \neq 0$ which is again the algebra of creation and annihilation operators.

This implies that a supersymmetric multiplet can be constructed starting from a 'vacuum' state of helicity λ annihilated by the Q^- operators: $Q^-|\lambda >= 0$ and the rest of the states in the multplet are generated by acting on Q^+. Therefore they will be of the form

$$| + 1/2, +1/2, \pm 1/2, \cdots, \pm 1/2 >$$

and the total number will be $\mathcal{N}/\triangle$. Since $M_{23}(Q^+|\lambda >) = (\lambda - 1/2)\,(Q^+|\lambda >)$ then the states will be

$$|\lambda >, |\lambda - 1/2 >, \cdots, |\lambda - 1/2\,(\mathcal{N}/4) >$$

Therefore

$$\lambda_{max} - \lambda_{min} = \lambda - (\lambda - \mathcal{N}/8) = \mathcal{N}/8$$

Imposing $|\lambda| < 2$ this implies that $\mathcal{N} < 2^5 = 32$ but remembering that $\mathcal{N} = 2^{D/2}, 2^{(D-1)/2}$ for even and odd dimensionality this implies a maximum number of spacetime dimensions $D = 10, 11$!!!.

Notice the similraity of this argument with the previous proof that the maximum number of supersymmetries in 4-dimensions was $N = 8$. We will see later that precisely $N = 8$ supergravity is obtained from the supersymmetric theories in $D = 10$ and $D = 11$.

Let's take a closer look at the spectrum of $D = 11$ and $D = 10$:

- D = 11

  Only $N = 1$ - SUSY is possible. The only multiplet consists of

$$\underbrace{g_{MN}}_{\text{graviton}}\ , \quad \underbrace{\psi^\alpha_M}_{\text{gravitino}}\ , \quad \underbrace{A_{MNP}}_{\text{antisymmetric tensor (non - chiral)}}$$

  For the counting of degrees of freedom for each field we have to recall performing the analysis using the little group $O(D-2)$. The graviton in D dimensions carry $(D-2)(D-1)/2-1$, corresponding to a symmetric tensor in $D-2$ dimensions minus the trace, which is in this case ($D = 11$) $45 - 1 = 44$. The antisymmetric tensor of rank $p+1$ in D dimensions has $\begin{pmatrix} D-2 \\ p+1 \end{pmatrix}$ degrees of freedom, in this case is $\begin{pmatrix} 9 \\ 3 \end{pmatrix} = 84$, whereas for the gravitino, the spinor has $2^{(D-2)/2} \times (D-2) - 2^{(D-2)/2}$ the first factor is the product of the spinor components times the vector components of the gravitino (since it carries both indices), the subtraction of the degrees of freedom of a spin 1/2 component is similar to the subtraction of the trace for the graviton). In this case we this gives 128 which matches the number of bosonic degrees of freedom $84 + 44$.

- D = 10

  This allows $N = 2$:

IIA	g_{MN}	$2\psi^\alpha_M$	B_{MN}	ϕ	A_{MNP}	λ
IIB	g_{MN}	$2\psi^\alpha_M$	$2B_{MN}$	2ϕ	$A^\dagger_{MNPQ}$	λ
I	$(g_{MN}$	B_{MN}	ϕ	$\psi^\alpha_M)$	$(A_M$	$\lambda)$ (chiral)

About antisymmetric tensors $A_{M_1 \ldots M_{p+1}}$ of spin 0 or 1, we know:

- A_M couples to a particle $\int A^M \, dx_M$, where dx_M refers to the world - line
- A_{MN} couples to a string $\int A^{MN} \, dx_M \wedge dx_N$ (world - sheet)
- A_{MNP} to a membrane ...
- $A_{M_1 \ldots M_{p+1}}$ to a p - brane

The coupling is dependent of the object's charges:

object	charge	couples to
particle	q	A_M
string	q_M	A_{MN}
$p-$ brane	$q_{M_1 \ldots M_p}$	$A_{M_1 \ldots M_{p+1}}$

Charges are new examples of central - charges in SUSY - algebra:

$$\left\{ Q \, , \, Q \right\} \quad \propto \quad aP + b^{M_1 \ldots M_p} q_{M_1 \ldots M_p}$$

8.3 Dimensional Reduction

Let's review the general procedure of reducing any number of dimensions bigger than 4 to $4D$. We start with 5 dimensions (one of which has radius R):

$$\mathbb{M}_5 \;=\; \mathbb{M}_4 \times S^1 \quad \Longrightarrow \quad \varphi(x^M) \;=\; \varphi(x^\mu, x^5 = y) \;=\; \sum_{n=-\infty}^{\infty} \varphi_n(x^\mu) \exp\left(\frac{iny}{R}\right)$$

and replace one field in 5 dimensions by ∞ many fields in 4 D. If φ is massless,

$$(\partial_M \partial^M)_5 \varphi \;=\; 0 \quad \Longrightarrow \quad (\partial_\mu \partial^\mu)_4 \varphi_n - \frac{n^2}{R^2} \varphi_n \;=\; 0 \; ,$$

then φ_n has a mass of $\frac{n}{R}$.

For dimensional reduction, take the $n = 0$ - mode,

$$\begin{aligned}
\varphi(x^M) \;&\longmapsto\; \varphi(x^\mu) \\
A_M(x^M) \;&\longmapsto\; A_\mu(x^\mu) \; , \qquad \underbrace{A_m(x^\mu)}_{\text{scalars}} \; , \qquad m \;=\; 4, ..., D \\
B_{MN} \;&\longmapsto\; B_{\mu\nu} \; , \qquad \underbrace{B_{\mu n}}_{\text{vectors}} \; , \qquad \underbrace{B_{mn}}_{\text{scalars}} \\
\underbrace{\psi}_{2^n} \;&\longmapsto\; \underbrace{\psi}_{\frac{1}{4}2^n \;\, 4D-\text{spinors}} \quad .
\end{aligned}$$

Consider e.g. the reduction of 11 D to 4 D: The fundamental fields are garviton g_{MN} that carries $9 \times 10/2 - 1 = 44$ degrees of freedom (using the Little group $O(9)$ and the subtraction of -1 corresponds to the overall trace of the symmetric tensor that is an extra scalar field degree of freedom. The second field is the gravitino ψ_M^α carrrying $9 \times 2^{(9-1)/2} - 2^{(9-1)/2} = 8 \times 16 = 128$. Again the subtraction is an extra spinor degree of freedom. The final field is an antisymmetric tensor A_{MNP} that carries $9!/3!6! = 84$degrees of freedom. Notice we have 128 bosonic degrees of freedom and 128 fermionic degrees of freedom. Dimensional reduction to 4D leads to:

$$\begin{array}{lllll} g_{MN} & \longmapsto & \underbrace{g_{\mu\nu}}_{\text{graviton}} \; , & \underbrace{g_{\mu m}}_{7 \text{ vectors}} \; , & \underbrace{g_{mn}}_{\frac{7\cdot 8}{2}=28 \text{ scalars (symmetry!)}} \\ A_{MNP} & \longmapsto & A_{\mu\nu\rho} \; , & \underbrace{A_{\mu\nu m}}_{7 \text{ tensors}} \; , \quad \underbrace{A_{\mu mn}}_{21 \text{ vectors}} \; , & \underbrace{A_{mnp}}_{\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}=35 \text{ scalars (antisymmetry!)}} \\ \psi_M^\alpha & \longmapsto & \underbrace{\psi_M^\alpha}_{\frac{32}{4}=8} \; , & \underbrace{\psi_m^\alpha}_{7\cdot 8=56 \text{ fermions}} & \end{array}$$

Recall here that a three index antisymmetric tensor in 4 dimensions carries no degrees of freedom and that two-index antisymmetric tensors are dual to scalars. The spectrum is the same as the $N = 8$ supergravity in 4 dimensions (one graviton, 8 gravitini, 35 vectors, 70 scalars and 56 fermions).

There is a theory of $N = 8$ - supergravity based on the g_{MN} and A_{MNP}. Reducing the dimension from 11 to 4 has an effect of $N = 1 \mapsto N = 8$. This $N = 8$ - model is non - chiral, but by other compactifications and p - branes in a 10 - dimensional string - theory can provide chiral $N = 1$ - models close to the MSSM. Notice that the statement of why the maximum dimensionality of supersymmetric theories is 11 is identical to the statement that the maximum number of supersymmetries in four-dimensions is $N = 8$. Since both thoeries are related by dimensional reduction. Actually, the explicit construction of extended supergravity theories was originally done by going to the simpler theory in extra dimensions and dimensionally reduce it.

8.4 Summary

This is the end of these lectures. We have seen that both supersymmetry and extra dimensions provide the natural way to extend the spacetime symmetries of standard field theories.

They both have a set of beautiful formal properties, but they also address important unsoved physical questions, like the hierarchy problem.

For supersymmetry we can say that it is a very elegant extension of spacetime - symmetry:

- It may be realized at low energies, the energy of SUSY - breaking of 1 TeV is within experimental reach (hierarchy, unification, dark matter)
- It may be essential ingredient of fundamental theory (M - theory, strings)
- It is a powerful tool to understand QFTs, especially non - perturbatively (S-duality, seiberg-Witten, AdS/CFT).

Both supersymmetry and extra dimensions may be tested soon in experiments. They are both basic ingredients of string theory and may be relevant only at large energies.