How Mathematics can help you sleep at night...

Overview of research in the Mathematical Analysis in Acoustics SIG

Georg Maierhofer, Anastasia Kisil, Dave Hewett, Ed Brambley

Laboratoire Jacques-Louis Lions

Sorbonne Université

acoustics.ac.uk

THE UK ACOUSTICS NETWORK

Outline

Our goal

Bring together researchers developing and applying tools from the mathematical fields of applied, numerical and asymptotic analysis in the context of acoustics.

- Fundamental physics and the language of Mathematics
- Complex analysis techniques and wavefield properties
- Multiscale problems and rigorous analysis of computational techniques

Lighthill's acoustic analogy

Lighthill's acoustic analogy

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

where

$$T_{ij} = \rho u_i u_j + \mathrm{e}_{ij} + (p - \rho c_0^2)\delta_{ij} = \text{Lighthill stress tensor},$$

$$\mathrm{e}_{ij} = \text{viscous stress tensor}.$$

Lighthill's acoustic analogy

Lighthill's acoustic analogy

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

where

$$T_{ij} = \rho u_i u_j + \mathrm{e}_{ij} + (p - \rho c_0^2)\delta_{ij} = \text{Lighthill stress tensor},$$
$$\mathrm{e}_{ij} = \text{viscous stress tensor.}$$

Lighthill's acoustic analogy (integral form)

$$\rho(\mathbf{x}, t) - \rho_0 = \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} G(\mathbf{x} - \mathbf{y}, t - \tau) \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}(\mathbf{y}, \tau)\, \mathrm{d}\tau\, \mathrm{d}^3 y.$$

Green's function: $G(\mathbf{x}, t) = \dfrac{\delta\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi c_0^2 |\mathbf{x}|}.$

Lighthill's eighth power law

Far-field observer:

$$\rho'(\mathbf{x},t) \sim \frac{1}{4\pi c_0^2 |\mathbf{x}|} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} T_{ij} \left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) \mathrm{d}^3 y$$

Source restricted to compact region:

$$\rho'(\mathbf{x},t) \sim \frac{1}{4\pi c_0^4 |\mathbf{x}|} \hat{x}_i \hat{x}_j \ddot{S}_{ij}\left(t - \frac{|\mathbf{x}|}{c_0}\right), \quad S_{ij} = \int T_{ij}(\mathbf{y},t)\,\mathrm{d}^3 y.$$

Typical scale of T_{ij} is $\rho_0 u_0^2$ thus $\rho' \sim \rho_0 \frac{l}{|\mathbf{x}|} m^4$ and

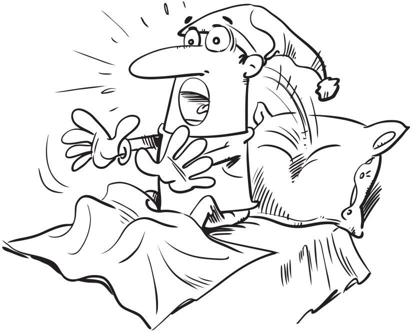
Lighthill's eight power law

Noise produced by a compact region of turbulence has intensity

$$|I|^2 \propto m^8.$$

Modern research topics

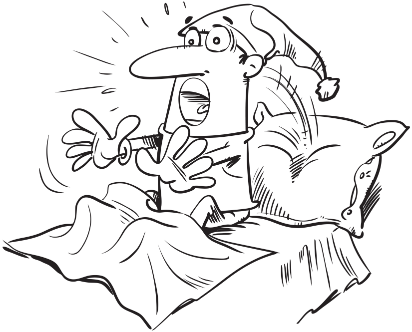
Modern research topics



Modern research topics

Our members investigate:

- Theoretical Aeroacoustics (Jet noise reduction, . . .),
- Boundary layer instabilities
- Complex boundary conditions (Impedance, . . .),
- Adiabatic invariants and similar conserved quantities in general flows,
- Fundamental theoretical research about wave propagation (resonances, . . .).



A classical problem: Scattering by a half-plane

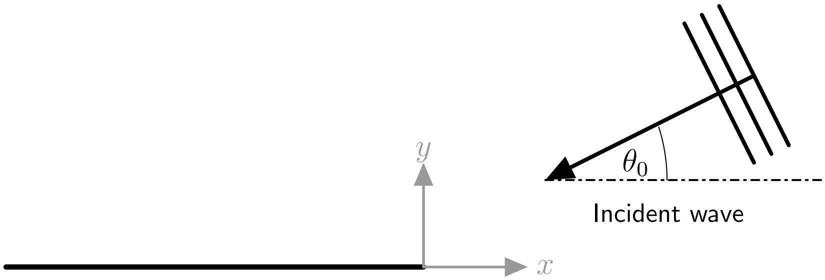


Figure: Sketch of cascade of blades with incident upstream acoustic wave.

Solution of Sommerfeld half-plane problem

Wiener–Hopf technique (complex analysis) $\rightarrow$ Scattered field is ϕ is

$$\phi(x,y) = \frac{\mathrm{sgn}\,(y) k_0 \sin\theta_0}{2\pi\gamma^+(-k_0\cos\theta_0)} \int_{\mathcal{C}} \frac{\mathrm{e}^{-ikx-\gamma(k)|t|}}{\gamma^-(k)(k+k_0\cos\theta_0)} \mathrm{d}k.$$

Asymptotic results: Far-field radiation (diffracted field)

$$\phi_\mathrm{d} \sim \frac{1}{4}\left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} \mathrm{e}^{-i\frac{\pi}{4}} \frac{\sin\frac{\theta_0}{2}\sin\frac{\theta}{2}}{\cos\theta+\cos\theta_0} \mathrm{e}^{-ik_0 r}$$

Pole contribution: reflected field and shadow: if $\theta > \pi - \theta_0$ or $\theta < \theta_0 - \pi$

$$\phi_g = \begin{cases} \exp(ik_0\cos\theta_0 x - ik_0\sin\theta_0 y), y > 0 \\ -\exp(+ik_0\cos\theta_0 x + ik_0\sin\theta_0 y), y < 0. \end{cases}$$

can also do uniform asymptotic expansion and understand the shadow boundary.

Modern research topics

Our members investigate:

- Wiener–Hopf method used for noise reduction on wind turbines, biological acoustics, . . .

Modern research topics

Our members investigate:

- Wiener–Hopf method used for noise reduction on wind turbines, biological acoustics, . . .

Modern research topics

Our members investigate:

- Wiener–Hopf method used for noise reduction on wind turbines, biological acoustics, . . .
- Canonical scattering problems: Quarter plane, concave boundaries, . . .
- Theoretical study of Wiener–Hopf technique, Matrix factorisation problems,
- Structural and underwater acoustics (canonical problems for the design of underwater structures).

Ray Theory

Time-harmonic pressure waves in slowly varying medium

$$\nabla_{\mathbf{x}}^2 p'(\mathbf{x}) + \rho_0(\epsilon\mathbf{x}) \boldsymbol{\nabla}_{\mathbf{x}} p' \cdot \boldsymbol{\nabla} \left(\frac{1}{\rho_0(\epsilon\mathbf{x})} \right) + \frac{p'}{c_0(\epsilon\mathbf{x})^2} = 0.$$

where $\epsilon \ll 1$ (wavelength $\ll$ typical length of medium).

Introduce slow variable $\mathbf{s} = \epsilon\mathbf{x}$ and fast scalar variable $\theta(\mathbf{x})$, then

$$p' = A_0(\mathbf{s}) \exp\left(-\frac{i\theta(\mathbf{x})}{c_0(\mathbf{s})|\boldsymbol{\nabla}_{\mathbf{x}}\theta|} \right) + \mathcal{O}(\epsilon).$$

Eikonal equation

$$|\boldsymbol{\nabla}_{\mathbf{x}}\theta|^2 = \frac{1}{c_0(\epsilon\mathbf{x})^2}$$

Bending waves

Waves travel along rays (curved in general).

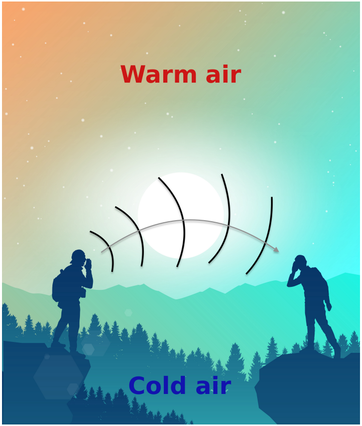
Let σ arclength along such rays then

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{c_0}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma}\right) = \boldsymbol{\nabla}\left(\frac{1}{c_0}\right).$$

At night $c_0 = c_0(z)$ increases in z.

Thus $\boldsymbol{\nabla}\left(\frac{1}{c_0}\right)$ points down.

$\Longrightarrow$ rays bend downwards.

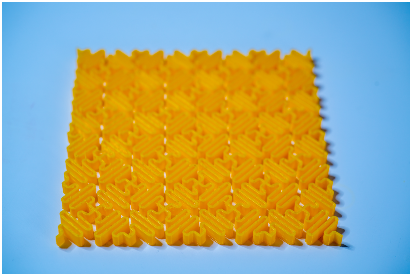


Sound travels further at night.

Multiscale methods

Our members investigate:

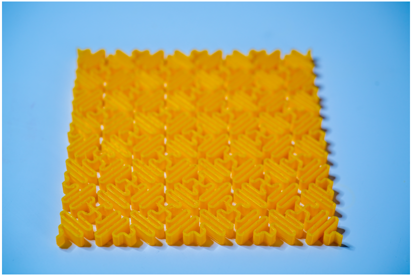
- Theoretical analysis and design of acoustic metamaterials,



Multiscale methods

Our members investigate:

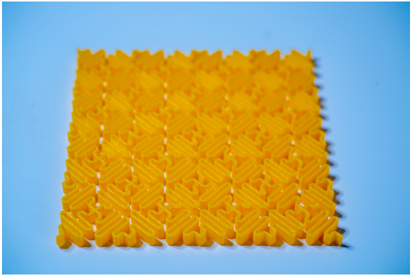
- Theoretical analysis and design of acoustic metamaterials,



Multiscale methods

Our members investigate:

- Theoretical analysis and design of acoustic metamaterials,
- Hybrid basis functions for high-frequency wave scattering based on asymptotic results,
- Noise propagation in varying domains (turbines, musical instruments,. . .),
- . . .



Outlook and exciting future research

Wavinar ICMS online seminar series

First Tuesday of every month (next on 7th December at 4pm) - two virtual talks about waves in complex continua.

Fundamental physics:

- Deeper mathematical understanding of noise propagation
- Suppression of waves in a given 3D volume (it is not the same as cloaking).

Complex analysis techniques:

- Systematic complexification of time-harmonic acoustics problems,
- Three-dimensional diffraction problems.

Multiple scales problems:

- Multi-scale boundaries/scattering structures, high-contrast heterogeneous media, "micro-resonators"

Thank you for listening!

Please get in touch if you would like to learn more, explore, or collaborate:
https://acoustics.ac.uk/sigs/mathematical-analysis-in-acoustics/

acoustics.ac.uk
THE UK ACOUSTICS NETWORK