Astrophysical Fluid Dynamics
Professor Gordon Ogilvie

Mathematical Tripos, Part III
Michaelmas Term 2016

Example Sheet 1

1. *Validity of a fluid approach*

The Coulomb cross-section for 'collisions' (i.e. large-angle scatterings) between electrons and protons is $\sigma \approx 1 \times 10^{-4}(T/\mathrm{K})^{-2}\,\mathrm{cm}^2$. Why does it depend on the inverse square of the temperature?

Using the numbers quoted in lectures (or elsewhere), estimate the order of magnitude of the mean free path and the collision frequency in (i) the centre of the Sun, (ii) the solar corona, (iii) a molecular cloud and (iv) the hot phase of the interstellar medium. Comment on your answers.

2. *Vorticity equation*

Show that the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ of an ideal fluid without a magnetic field satisfies the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \boldsymbol{\nabla} p \times \boldsymbol{\nabla} v,$$

where $v = 1/\rho$ is the specific volume. Explain why the last term, which acts as a source of vorticity, can also be written as $\boldsymbol{\nabla} T \times \boldsymbol{\nabla} s$. Under what conditions does this 'baroclinic' source term vanish, and in what sense(s) can the vorticity then be said to be 'conserved'?

Show that the (Rossby–Ertel) *potential vorticity* $\frac{1}{\rho}\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s$ is conserved, as a material invariant, even when the baroclinic term is present.

3. *Homogeneous expansion or contraction*

(This question explores a very simple fluid flow in which compressibility and self-gravity are important.)

A homogeneous perfect gas of density $\rho = \rho_0(t)$ occupies the region $|\boldsymbol{x}| < R(t)$, surrounded by a vacuum. The pressure is $p = p_0(t)(1 - |\boldsymbol{x}|^2/R^2)$ and the velocity field is $\boldsymbol{u} = A(t)\boldsymbol{x}$, where $A = \dot{R}/R$.

Using either Cartesian or spherical polar coordinates, show that the equations of Newtonian gas dynamics and the boundary conditions are satisfied provided that

$$\rho_0 \propto R^{-3}, \qquad p_0 \propto R^{-3\gamma}, \qquad \ddot{R} = -\frac{4\pi G \rho_0 R}{3} + \frac{2p_0}{\rho_0 R}.$$

Deduce the related energy equation

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi G \rho_0 R^2}{3} + \frac{2p_0}{3(\gamma - 1)\rho_0} = \text{constant},$$

and interpret the three contributions. Discuss the dynamics qualitatively in the two cases $\gamma > 4/3$ and $1 < \gamma < 4/3$.[1]

4. *Dynamics of ellipsoidal bodies*

(This question uses Cartesian tensor notation and the summation convention.)

A fluid body occupies a time-dependent ellipsoidal volume centred on the origin. Let $f(\boldsymbol{x}, t) = 1 - S_{ij}x_ix_j$, where $S_{ij}(t)$ is a symmetric tensor with positive eigenvalues, such that the body occupies the region $0 < f \leqslant 1$ with a free surface at $f = 0$. The velocity field is $u_i = A_{ij}x_j$, where $A_{ij}(t)$ is a tensor that is not symmetric in general. Assume that the gravitational potential inside the body has the form $\Phi = B_{ij}x_ix_j + \text{constant}$, where $B_{ij}(t)$ is a symmetric tensor.

Show that the equations of Newtonian gas dynamics and the boundary conditions are satisfied if the density and pressure are of the form

$$\rho = \rho_0(t)\hat{\rho}(f), \qquad p = \rho_0(t)T(t)\hat{p}(f),$$

where the dimensionless functions $\hat{p}(f)$ and $\hat{\rho}(f)$ are related by $\hat{p}'(f) = \hat{\rho}(f)$ with the normalization $\hat{\rho}(1) = 1$ and the boundary condition $\hat{p}(0) = 0$, provided that the coefficients evolve according to

$$\begin{aligned}
&\dot{S}_{ij} + S_{ik}A_{kj} + S_{jk}A_{ki} = 0,\\
&\dot{A}_{ij} + A_{ik}A_{kj} = -2B_{ij} + 2TS_{ij},\\
&\dot{\rho}_0 = -\rho_0 A_{ii},\\
&\dot{T} = -(\gamma - 1)TA_{ii}.
\end{aligned}$$

Examples of the spatial structure are the homogeneous body: $\hat{\rho} = 1$, $\hat{p} = f$, and the polytrope of index n: $\hat{\rho} = f^n$, $\hat{p} = f^{n+1}/(n+1)$. Show that Poisson's equation cannot be satisfied if the body is inhomogeneous.[2]

Show how the results of the previous question are recovered in the case of a homogeneous, spherically symmetric body.

Please send any comments and corrections to gio10@cam.ac.uk

[1] This flow is similar in form to the cosmological 'Hubble flow' and can be seen as a homogeneous expansion or contraction centred on any point, if a Galilean transformation is made. In the limit $R \to \infty$ (for $\gamma > 4/3$), or if the pressure is negligible, the equations derived here correspond to the Friedmann equations for a 'dust' universe (i.e. negligible relativistic pressure $p \ll \rho c^2$) with a scale factor $a \propto R$,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{3}, \qquad \frac{\dot{a}^2 + \text{constant}}{a^2} = \frac{8\pi G\rho_0}{3}.$$

See Bondi, *Cosmology* (Cambridge University Press) for a discussion of Newtonian cosmology.

[2] It can be shown that the self-gravity of a homogeneous ellipsoid generates an interior gravitational potential of the assumed form. The behaviour of self-gravitating, homogeneous, incompressible ellipsoids was investigated by many great mathematicians, including Maclaurin, Jacobi, Dirichlet, Dedekind, Riemann and Poincaré, illustrating the equilibrium and stability of rotating and tidally deformed astrophysical bodies. See Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press).