Dynamics of Astrophysical Discs — Mathematical Tripos, Part III
Professor Gordon Ogilvie — Lent Term 2020

Example Sheet 2

1. *Inertial waves in a rotating shear flow*

Consider the evolution of linear shearing waves in a rotating, three-dimensional, incompressible shearing sheet. The wave amplitudes $(\tilde{\mathbf{v}}, \tilde{\psi})$ of the velocity and modified pressure perturbations satisfy the equations (see lectures)

$$
\begin{aligned}
\frac{d\hat{v}_x}{dt} - 2\Omega\hat{v}_y &= -ik_x\hat{\psi}\,, \\
\frac{d\hat{v}_y}{dt} + (2\Omega - S)\hat{v}_x &= -ik_y\hat{\psi}\,, \\
\frac{d\hat{v}_z}{dt} &= -ik_z\hat{\psi}\,, \\
i\mathbf{k}\cdot\hat{\mathbf{v}} &= 0\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&(\tilde{\mathbf{v}}, \tilde{\psi}) = (\hat{\mathbf{v}}, \hat{\psi})\,\exp\left(-\int \nu k^2\,dt\right)\,, \\
&k^2 = k_x^2 + k_y^2 + k_z^2\,, \\
&\frac{dk_x}{dt} = Sk_y\,.
\end{aligned}
$$

Eliminate variables in favour of $\hat{v}_x$ to obtain the second-order ODE

$$
\frac{d^2}{dt^2}\left(k^2\hat{v}_x\right) + \Omega_r^2 k_z^2 \hat{v}_x = 0\,,
$$

where $\Omega_r^2 = 2\Omega(2\Omega - S)$ is the square of the epicyclic frequency.

The quantity $|\tilde{\mathbf{v}}|^2$ is proportional to the energy of the disturbance, and the quantity $|\hat{\mathbf{v}}|^2$ is the equivalent in the absence of viscosity. Determine whether $|\hat{\mathbf{v}}|^2$ and $|\tilde{\mathbf{v}}|^2$ grow or decay as $t \to +\infty$, starting from a generic initial condition, depending on whether $0 < S/\Omega < 2$ or $S/\Omega > 2$. Treat separately the cases of axisymmetric ($k_y = 0$) and non-axisymmetric ($k_y \neq 0$) disturbances. [Note that, in the case $k_y \neq 0$, $\hat{v}_x$ behaves as t^σ as $t \to +\infty$, where the (possibly complex) index σ satisfies a quadratic indicial equation.]

2. *Transient growth in a non-rotating shear flow*

In the special case of a non-rotating shear flow $\mathbf{u}_0 = -Sx\,\mathbf{e}_y$, the wave amplitudes satisfy

$$
\begin{aligned}
\frac{d\hat{v}_x}{dt} \qquad &= -ik_x\hat{\psi}\,, \\
\frac{d\hat{v}_y}{dt} - S\hat{v}_x &= -ik_y\hat{\psi}\,, \\
\frac{d\hat{v}_z}{dt} \qquad &= -ik_z\hat{\psi}\,, \\
i\mathbf{k}\cdot\hat{\mathbf{v}} &= 0\,,
\end{aligned}
$$

where

$$\frac{dk_x}{dt} = Sk_y\,.$$

Show that, for generic non-axisymmetric disturbances ($k_y \neq 0$), the inviscid wave amplitudes satisfy

$$\hat{v}_x \propto t^{-2}\,, \qquad \hat{v}_y \propto \text{constant}\,, \qquad \hat{v}_z \propto \text{constant}$$

in the limit $t \to +\infty$.

Show that generic axisymmetric disturbances ($k_y = 0$) experience algebraic growth in the absence of viscosity. When viscosity is included, show that the kinetic energy of the disturbance is proportional to

$$(1 + T^2 \cos^2\theta)\,\exp\left(-\frac{2T}{\mathrm{Re}}\right),$$

where $T = St$ is the dimensionless time measured from the instant when $\hat{v}_y = 0$, θ is the angle between the wavevector and the vertical, and $\mathrm{Re} = S/\nu k^2$ is the Reynolds number of the disturbance.

Hence show that, for large Reynolds number, an energy amplification of

$$\left(\frac{\cos\theta}{e}\right)^2 (\mathrm{Re})^2 + O(1)$$

can be achieved, taking a time

$$\Delta t = \frac{\mathrm{Re}}{S} + O(\mathrm{Re})^{-1}\,.$$

3. *Elliptical vortices*

(a) In a two-dimensional inviscid incompressible fluid, an elliptical vortex patch of semimajor axis a, semiminor axis b and uniform vorticity ζ_0 is surrounded by an irrotational flow with velocity tending to zero as $|\mathbf{r}| \to \infty$. Show that the velocity field induced by the vortex causes it to rotate with angular velocity

$$\frac{ab\,\zeta_0}{(a+b)^2}\,.$$

[This result, due to Kirchhoff, can be derived in several different ways. A standard method is as follows. Introduce elliptical coordinates (ξ, η) such that

$$x = c\cosh\xi\cos\eta\,, \qquad y = c\sinh\xi\sin\eta\,, \qquad \xi > 0\,, \qquad 0 \leqslant \eta < 2\pi\,,$$

and choose $c > 0$ such that the boundary of the vortex patch at a particular instant corresponds to the curve $\xi = \xi_0 = \text{constant}$. Verify that the coordinates are orthogonal and have scale factors $h_\xi = h_\eta = c(\sinh^2\xi + \sin^2\eta)^{1/2}$. Solve Poisson's equation

$\nabla^2\psi = -\zeta$ for the streamfunction by separation of variables, choosing a solution for which the velocity is finite at the singular points of the coordinate system. Then show that the normal component of the instantaneous velocity on the boundary of the vortex is the same as that of a rigid rotation with the angular velocity specified above. Note that the normal velocity component is determined by the variation of the streamfunction along the boundary.]

(b) Determine the effect of a uniform shear flow, $\mathbf{u} = -Sx\,\mathbf{e}_y$, on the shape of an elliptical material curve, such as the boundary of an elliptical vortex patch. If the ellipse has semiaxes $a(t)$ and $b(t)$, with the former being directed at an angle $\theta(t)$ (measured in a positive sense) with respect to the y axis, show that these parameters evolve in time according to the equations

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta\,,$$

$$\dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}\,.$$

[In this part of the question, neglect the velocity due to the vortex itself.]

4. *Viscous overstability*

A two-dimensional, compressible, non-self-gravitating, viscous shearing sheet has surface density $\Sigma(x,y,t)$ and two-dimensional velocity $\mathbf{u}(x,y,t)$, governed by the equation of mass conservation,

$$\frac{\partial\Sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\Sigma\mathbf{u}) = 0\,,$$

and the equation of motion,

$$\Sigma\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u}\right) = -\Sigma\boldsymbol{\nabla}\Phi_{\mathrm{t}} - \boldsymbol{\nabla}P + \boldsymbol{\nabla}\cdot\mathbf{T}\,.$$

Here $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ is the angular velocity of the rotating frame of reference, $\Phi_{\mathrm{t}} = -\Omega S x^2$ is the tidal potential, P is the vertically integrated pressure and

$$\mathbf{T} = 2\bar{\nu}\Sigma\,\mathbf{S} + \bar{\nu}_{\mathrm{b}}\Sigma(\boldsymbol{\nabla}\cdot\mathbf{u})\,\mathbf{I}$$

is the vertically integrated viscous stress tensor. Also

$$\mathbf{S} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\top}\right] - \frac{1}{3}(\boldsymbol{\nabla}\cdot\mathbf{u})\,\mathbf{I}$$

is the traceless shear tensor, $\mathbf{I}$ is the unit tensor, $\bar{\nu}$ is the mean kinematic shear viscosity and $\bar{\nu}_{\mathrm{b}}$ is the mean kinematic bulk viscosity. Assume that P, $\bar{\nu}$, $\bar{\nu}_{\mathrm{b}}$ are known functions of Σ, with $dP/d\Sigma = v_{\mathrm{s}}^2$ being the square of the sound speed.

The basic state of the sheet corresponds to the homogeneous solution in which $\mathbf{u} = -Sx\,\mathbf{e}_y$, while Σ, P and $\mathbf{T}$ are uniform, with $T_{xy} = -\bar{\nu}\Sigma S$.

(i) Formulate the linearized equations for perturbations Σ', $\mathbf{v}$, etc., on this background. Assume that the perturbations are axisymmetric (i.e. independent of y) and depend on x and t through the factor $\exp(st + ik_x x)$, where s is a complex growth rate. Hence obtain the equations

$$s\Sigma' = -\Sigma\, ik_x v_x \,,$$

$$sv_x = 2\Omega v_y - ik_x v_{\rm s}^2 \frac{\Sigma'}{\Sigma} - (\bar{\nu}_{\rm b} + \tfrac{4}{3}\bar{\nu}) k_x^2 v_x \,,$$

$$sv_y = -(2\Omega - S)v_x - \bar{\nu} k_x^2 v_y - ik_x S \frac{d(\bar{\nu}\Sigma)}{d\Sigma} \frac{\Sigma'}{\Sigma} \,,$$

where unprimed quantities represent their values in the basic state. (Note that the crucial final term comes from the fact the shear stress T_{xy} is affected by the density perturbation in the wave as well as the perturbed velocity gradient.)

(ii) Deduce that the dispersion relation is

$$s^3 + (\bar{\nu}_{\rm b} + \tfrac{7}{3}\bar{\nu}) k_x^2 s^2 + [\Omega_r^2 + v_{\rm s}^2 k_x^2 + \bar{\nu}(\bar{\nu}_{\rm b} + \tfrac{4}{3}\bar{\nu}) k_x^4] s + 2\Omega S \frac{d(\bar{\nu}\Sigma)}{d\Sigma} k_x^2 + v_{\rm s}^2 \bar{\nu} k_x^4 = 0 \,,$$

where $\Omega_r^2 = 2\Omega(2\Omega - S)$ is the square of the epicyclic frequency (assumed positive). Verify that this reduces to the expected result in the case of an inviscid disc.

(iii) Consider the limit of long wavelengths, $k_x \to 0$. Show that one root of the dispersion relation behaves as

$$s = -\frac{2\Omega S}{\Omega_r^2} \frac{d(\bar{\nu}\Sigma)}{d\Sigma} k_x^2 + O(k_x^4)$$

in this limit. This root yields exponential growth (*viscous instability*) when

$$\frac{d(\bar{\nu}\Sigma)}{d\Sigma} < 0 \,.$$

(iv) Show that the other two roots behave in this limit as

$$s = \pm i\Omega_r \left(1 + \frac{v_{\rm s}^2 k_x^2}{2\Omega_r^2}\right) + \frac{1}{2}\left[-(\bar{\nu}_{\rm b} + \tfrac{7}{3}\bar{\nu}) + \frac{2\Omega S}{\Omega_r^2} \frac{d(\bar{\nu}\Sigma)}{d\Sigma}\right] k_x^2 + O(k_x^4) \,.$$

These roots yield exponentially growing oscillations (*viscous overstability*) when

$$\frac{d(\bar{\nu}\Sigma)}{d\Sigma} > (\bar{\nu}_{\rm b} + \tfrac{7}{3}\bar{\nu}) \frac{\Omega_r^2}{2\Omega S} \,.$$

Show that this condition is satisfied even when $\bar{\nu}$ is independent of Σ, $\bar{\nu}_{\rm b} = 0$ and the disc is Keplerian.

Please send any comments and corrections to `gio10@cam.ac.uk`
Answers to questions 1 and 3 may be submitted for marking.