Example Sheet 3

1. **Integral relations for the shearing box**

A homogeneous incompressible fluid in the shearing sheet satisfies the Navier–Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla \Phi_t - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

where

\[
\mathbf{u} = -S_x \mathbf{e}_y + \mathbf{v}
\]

is the total velocity, \( \Omega = \Omega \mathbf{e}_z \) is the angular velocity of the frame of reference, \( \Phi_t = -\Omega S_x^2 \) is the tidal potential (neglecting vertical gravity) and \( \nu \) is the kinematic viscosity. The velocity perturbation \( \mathbf{v} \) therefore satisfies

\[
\left( \frac{\partial}{\partial t} - S_x \frac{\partial}{\partial y} \right) \mathbf{v} - S v_x \mathbf{e}_y + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \times \mathbf{v} = -\nabla \psi + \nu \nabla^2 \mathbf{v},
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

where \( \psi = p/\rho \).

The **shearing box** is a rectangular domain

\[
0 < x < L_x, \quad 0 < y < L_y, \quad 0 < z < L_z,
\]

on which the following boundary conditions are applied, where \( f \) stands for \( \psi \) or any component of \( \mathbf{v} \):

\[
\begin{align*}
  f(0, y, z, t) &= f(L_x, (y - SL_x t) \mod L_y, z, t), \\
  f(x, 0, z, t) &= f(x, L_y, z, t), \\
  f(x, y, 0, t) &= f(x, y, L_z, t).
\end{align*}
\]

Interpret these boundary conditions, and show that they are compatible with solutions in the form of shearing waves in which

\[
\begin{align*}
  f &= \text{Re} \left\{ \tilde{f}(t) \exp[i \mathbf{k}(t) \cdot \mathbf{x}] \right\},
\end{align*}
\]

provided that the wavevector lies on the shearing lattice

\[
\begin{align*}
  k_x &= \frac{2\pi n_x}{L_x} + S k_y, \\
  k_y &= \frac{2\pi n_y}{L_y}, \\
  k_z &= \frac{2\pi n_z}{L_z},
\end{align*}
\]
where \( n_x, n_y \) and \( n_z \) are integers.

Let \( \langle \cdot \rangle \) denote a volume average over the box. Show that

\[
\langle \frac{\partial f}{\partial x} \rangle = \langle \frac{\partial f}{\partial y} \rangle = \langle \frac{\partial f}{\partial z} \rangle = 0,
\]

where \( f \) is any quantity satisfying the boundary conditions (1), but not necessarily a shearing wave; this result is useful for integration by parts.

Show that

\[
\frac{d}{dt} \langle v \rangle = S \langle v_x e_y \rangle - 2\Omega \times \langle v \rangle,
\]

and deduce that the mean velocity executes an epicyclic oscillation, but if initially zero will remain so.

Show further that

\[
\frac{d}{dt} \left( \frac{1}{2} \langle v_x^2 \rangle \right) = 2\Omega \langle v_x v_y \rangle - \nu \langle |\nabla v_x|^2 \rangle + \langle \psi \frac{\partial v_x}{\partial x} \rangle,
\]
\[
\frac{d}{dt} \left( \frac{1}{2} \langle v_y^2 \rangle \right) = -(2\Omega - S) \langle v_x v_y \rangle - \nu \langle |\nabla v_y|^2 \rangle + \langle \psi \frac{\partial v_x}{\partial y} \rangle,
\]
\[
\frac{d}{dt} \left( \frac{1}{2} \langle v_z^2 \rangle \right) = -\nu \langle |\nabla v_z|^2 \rangle + \langle \psi \frac{\partial v_z}{\partial z} \rangle,
\]
\[
\frac{d}{dt} \left( \frac{1}{2} \langle |v|^2 \rangle \right) = S \langle v_x v_y \rangle - \nu \langle |\nabla \times v|^2 \rangle.
\]

Deduce that, if hydrodynamic turbulence is to be maintained (without external forcing) against viscous dissipation in a Keplerian shear flow \((S/\Omega = 3/2)\), then \( \langle v_x v_y \rangle \) must be positive (corresponding to outward transport of angular momentum) and the pressure–strain correlation \( \langle \psi \frac{\partial v_i}{\partial x_j} \rangle \) must play an important role.

2. Magnetic fields in the shearing sheet

The induction equation in an incompressible fluid of uniform magnetic diffusivity \( \eta \) is

\[
\frac{\partial B}{\partial t} + u \cdot \nabla B = B \cdot \nabla u + \eta \nabla^2 B.
\]

Supposing that the velocity retains purely the form of a linear shear flow, \( u = -S x e_y \), show that the induction equation has solutions in the form of shearing waves,

\[
B = \text{Re} \left\{ \tilde{B}(t) \exp[i \mathbf{k}(t) \cdot \mathbf{x}] \right\},
\]

provided that the wavevector evolves in time according to

\[
\frac{d \mathbf{k}}{dt} = S k_y e_x.
\]
Solve for \( k(t) \), and interpret the result geometrically.

Deduce the equations satisfied by the components of the wave amplitude \( \tilde{B}(t) \), and find their general solution. Show that the magnetic energy typically experiences a phase of growth but ultimately decays.

Verify that \( \mathbf{B} \cdot \nabla \mathbf{B} = 0 \) for this solution, and confirm that the magnetic field has no influence on the flow. Given that any magnetic field can be considered as a superposition of such shearing waves, explain how a non-zero Lorentz force can result.

3. Mechanical analogue of the magnetorotational instability

In the local approximation, the dynamics of two particles of mass \( m \) connected by a spring of spring constant \( k = \beta m \) is described by the equations

\[
\begin{align*}
\ddot{x}_1 - 2\Omega \dot{y}_1 - 2\Omega S x_1 &= \beta (x_2 - x_1), \\
\dot{y}_1 + 2\Omega x_1 &= \beta (y_2 - y_1), \\
\ddot{z}_1 + \Omega^2 z_1 &= \beta (z_2 - z_1),
\end{align*}
\]

together with similar equations in which the suffixes 1 and 2 are interchanged.

Assume that the quantities \( \beta, \Omega, q, \Omega^2_r = 2\Omega(2\Omega - S) \) and \( \Omega^2_z \) are positive. Show that relative motions of the two particles in the \((x, y)\) plane proportional to \( \exp(\lambda t) \) are possible, where

\[
\lambda^4 + (\Omega^2_r + 4\beta)\lambda^2 + 4\beta(\beta - \Omega S) = 0.
\]

Determine the range of \( \beta \) for which instability occurs. For fixed \( \Omega \) and \( q \), find the maximum growth rate of the instability and the value of \( \beta \) for which this occurs. Write down the explicit form of \( x_1(t) \) and \( x_2(t) \) for this optimal solution.

*Please send any comments and corrections to gio10@cam.ac.uk*

*Answers to questions 2 and 3 may be submitted for marking.*