Example Sheet 3

1. *Integral relations for the shearing box*

A homogeneous incompressible fluid in the shearing sheet satisfies the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}\Phi_\mathrm{t} - \frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u}\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0\,,$$

where

$$\mathbf{u} = -Sx\,\mathbf{e}_y + \mathbf{v}$$

is the total velocity, $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ is the angular velocity of the frame of reference, $\Phi_\mathrm{t} = -\Omega S x^2$ is the tidal potential (neglecting vertical gravity) and ν is the kinematic viscosity. The velocity perturbation $\mathbf{v}$ therefore satisfies

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\mathbf{v} - Sv_x\,\mathbf{e}_y + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} = -\boldsymbol{\nabla}\psi + \nu\nabla^2\mathbf{v}\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,,$$

where $\psi = p/\rho$.

The *shearing box* is a rectangular domain

$$0 < x < L_x\,, \qquad 0 < y < L_y\,, \qquad 0 < z < L_z\,,$$

on which the following boundary conditions are applied, where f stands for ψ or any component of $\mathbf{v}$:

$$\begin{aligned} f(0,y,z,t) &= f(L_x,(y - SL_xt) \bmod L_y, z, t)\,,\\ f(x,0,z,t) &= f(x,L_y,z,t)\,,\\ f(x,y,0,t) &= f(x,y,L_z,t)\,. \end{aligned} \tag{1}$$

Interpret these boundary conditions, and show that they are compatible with solutions in the form of shearing waves in which

$$f = \mathrm{Re}\left\{\tilde{f}(t)\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\}\,,$$

provided that the wavevector lies on the shearing lattice

$$k_x = \frac{2\pi n_x}{L_x} + Sk_yt\,, \qquad k_y = \frac{2\pi n_y}{L_y}\,, \qquad k_z = \frac{2\pi n_z}{L_z}\,,$$

where n_x, n_y and n_z are integers.

Let $\langle\cdot\rangle$ denote a volume average over the box. Show that

$$\left\langle\frac{\partial f}{\partial x}\right\rangle=\left\langle\frac{\partial f}{\partial y}\right\rangle=\left\langle\frac{\partial f}{\partial z}\right\rangle=0\,,$$

where f is any quantity satisfying the boundary conditions (1), but not necessarily a shearing wave; this result is useful for integration by parts.

Show that

$$\frac{d}{dt}\langle\mathbf{v}\rangle=S\langle v_x\rangle\,\mathbf{e}_y-2\boldsymbol{\Omega}\times\langle\mathbf{v}\rangle\,,$$

and deduce that the mean velocity executes an epicyclic oscillation, but if initially zero will remain so.

Show further that

$$\frac{d}{dt}\left(\frac{1}{2}\langle v_x^2\rangle\right)=2\Omega\langle v_xv_y\rangle-\nu\langle|\boldsymbol{\nabla}v_x|^2\rangle+\left\langle\psi\frac{\partial v_x}{\partial x}\right\rangle\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}\langle v_y^2\rangle\right)=-(2\Omega-S)\langle v_xv_y\rangle-\nu\langle|\boldsymbol{\nabla}v_y|^2\rangle+\left\langle\psi\frac{\partial v_y}{\partial y}\right\rangle\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}\langle v_z^2\rangle\right)=-\nu\langle|\boldsymbol{\nabla}v_z|^2\rangle+\left\langle\psi\frac{\partial v_z}{\partial z}\right\rangle\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}\langle|\mathbf{v}|^2\rangle\right)=S\langle v_xv_y\rangle-\nu\langle|\boldsymbol{\nabla}\times\mathbf{v}|^2\rangle\,.$$

Deduce that, if hydrodynamic turbulence is to be maintained (without external forcing) against viscous dissipation in a Keplerian shear flow ($S/\Omega=3/2$), then $\langle v_xv_y\rangle$ must be positive (corresponding to outward transport of angular momentum) and the pressure–strain correlation $\langle\psi\,\partial v_i/\partial x_j\rangle$ must play an important role.

2. *Magnetic fields in the shearing sheet*

The induction equation in an incompressible fluid of uniform magnetic diffusivity η is

$$\frac{\partial\mathbf{B}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{B}=\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u}+\eta\nabla^2\mathbf{B}\,.$$

Supposing that the velocity retains purely the form of a linear shear flow, $\mathbf{u}=-Sx\,\mathbf{e}_y$, show that the induction equation has solutions in the form of shearing waves,

$$\mathbf{B}=\mathrm{Re}\left\{\tilde{\mathbf{B}}(t)\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\}\,,$$

provided that the wavevector evolves in time according to

$$\frac{d\mathbf{k}}{dt}=Sk_y\,\mathbf{e}_x\,.$$

Solve for $\mathbf{k}(t)$, and interpret the result geometrically.

Deduce the equations satisfied by the components of the wave amplitude $\tilde{\mathbf{B}}(t)$, and find their general solution. Show that the magnetic energy typically experiences a phase of growth but ultimately decays.

Verify that $\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{B} = \mathbf{0}$ for this solution, and confirm that the magnetic field has no influence on the flow. Given that any magnetic field can be considered as a superposition of such shearing waves, explain how a non-zero Lorentz force can result.

3. *Mechanical analogue of the magnetorotational instability*

In the local approximation, the dynamics of two particles of mass m connected by a spring of spring constant $k = \beta m$ is described by the equations

$$\begin{aligned}
\ddot{x}_1 - 2\Omega\dot{y}_1 - 2\Omega S x_1 &= \beta(x_2 - x_1)\,, \\
\ddot{y}_1 + 2\Omega\dot{x}_1 &= \beta(y_2 - y_1)\,, \\
\ddot{z}_1 + \Omega_z^2 z_1 &= \beta(z_2 - z_1)\,,
\end{aligned}$$

together with similar equations in which the suffixes 1 and 2 are interchanged.

Assume that the quantities β, Ω, q, $\Omega_r^2 = 2\Omega(2\Omega - S)$ and Ω_z^2 are positive. Show that relative motions of the two particles in the (x, y) plane proportional to $\exp(\lambda t)$ are possible, where

$$\lambda^4 + (\Omega_r^2 + 4\beta)\lambda^2 + 4\beta(\beta - \Omega S) = 0\,.$$

Determine the range of β for which instability occurs. For fixed Ω and q, find the maximum growth rate of the instability and the value of β for which this occurs. Write down the explicit form of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ for this optimal solution.

Please send any comments and corrections to `gio10@cam.ac.uk`
Answers to questions 2 and 3 may be submitted for marking.