Example Sheet 1

1. In one spatial dimension, two frames of reference S and S' have coordinates (x, t) and (x', t') respectively. The coordinates are related by

$$x' = f(x,t) \qquad \text{and} \qquad t' = t\,.$$

Viewed in frame S, a particle follows a trajectory $x = x(t)$. It has velocity $v = \mathrm{d}x/\mathrm{d}t$ and acceleration $a = \mathrm{d}^2x/\mathrm{d}t^2$. Viewed in S', the trajectory is $x' = f(x(t), t)$. Using the chain rule, show that the velocity and acceleration of the particle in S' are given by

$$v' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\,, \qquad a' = \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = a\frac{\partial f}{\partial x} + v^2\frac{\partial^2 f}{\partial x^2} + 2v\frac{\partial^2 f}{\partial x\,\partial t} + \frac{\partial^2 f}{\partial t^2}\,.$$

Suppose now that both S and S' are inertial frames. Explain why the function f must obey $\partial^2 f/\partial x^2 = \partial^2 f/\partial x\,\partial t = \partial^2 f/\partial t^2 = 0$. What is the most general form of f with these properties? Interpret this result.

2. A particle of mass m experiences a force field

$$\boldsymbol{F}(\boldsymbol{r}) = \left(-\frac{a}{r^2} + \frac{2b}{r^3}\right)\hat{\boldsymbol{r}}\,,$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is a unit vector in the radial direction and a and b are positive constants. Show, by finding a potential energy $V(r)$ such that $\boldsymbol{F} = -\boldsymbol{\nabla}V$, that $\boldsymbol{F}$ is conservative. (You will need the result $\boldsymbol{\nabla}r = \hat{\boldsymbol{r}}$.)

Sketch $V(r)$ and describe qualitatively the possible motions of the particle moving in the radial direction, considering different initial positions and velocities. If the particle starts at $r = 2b/a$, what is the minimum speed that it must have in order to escape to infinity?

3. A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius. Treating the Earth as a point of mass M, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = C\left(\frac{R^3}{GM}\right)^{1/2}\,,$$

where G is the gravitational constant and C is a dimensionless constant. (You will need the fact that the acceleration due to the Earth's gravitational field at a distance r from the centre of the Earth is GM/r^2.)

What is the equation of energy conservation for the satellite? By solving this differential equation, show that $C = \pi/2\sqrt{2}$.

4. A long time ago in a galaxy far, far away, a Death Star was constructed. Its surrounding force field caused any particle at position $\boldsymbol{r}$ relative to the centre of the Death Star to experience an acceleration

$$\ddot{\boldsymbol{r}} = \lambda\, \boldsymbol{r} \times \dot{\boldsymbol{r}}\,,$$

where λ is a constant. Show that the particle moves in this force field with constant speed. Show also that the magnitude of its acceleration is constant.

(a) A particle is projected *radially* with speed v from a point $\boldsymbol{r} = R\,\hat{\boldsymbol{r}}$ on the surface of the Death Star. Show that its trajectory is given by

$$\boldsymbol{r} = (R + vt)\,\hat{\boldsymbol{r}}\,.$$

(b) By considering the second derivative of $\boldsymbol{r}\cdot\boldsymbol{r}$ show that, for any particle moving in the force field, the distance r from the centre of the Death Star is given by

$$r^2 = v^2(t - t_0)^2 + r_0^2\,,$$

where t_0 and r_0 are constants and v is the speed of the particle. Obtain an expression for $\boldsymbol{r}\cdot\dot{\boldsymbol{r}}$ and show that $|\ddot{\boldsymbol{r}}| = \lambda r_0 v$.

5. A particle of mass m, charge q and position $\boldsymbol{x}(t)$ moves in both a uniform magnetic field $\boldsymbol{B}$, which points in a horizontal direction, and a uniform gravitational field $\boldsymbol{g}$, which points vertically downwards. Write down the equation of motion and show that it is invariant under translations $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{x}_0$. Show that

$$\dot{\boldsymbol{x}} = \omega\,\boldsymbol{x} \times \boldsymbol{n} + \boldsymbol{g}\,t + \boldsymbol{a}\,,$$

where $\omega = qB/m$ is the gyrofrequency, $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{B}$, and $\boldsymbol{a}$ is a constant vector. Show also that, with a suitable choice of origin, $\boldsymbol{a}$ can be written in the form $\boldsymbol{a} = a\,\boldsymbol{n}$.

By choosing suitable axes, show that the particle undergoes a helical motion together with a constant horizontal drift perpendicular to $\boldsymbol{B}$.

Suppose that you now wish to eliminate the drift by imposing a uniform electric field $\boldsymbol{E}$. Determine the direction and magnitude of $\boldsymbol{E}$.

6. At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\boldsymbol{v}$, while a wind blows with constant velocity $\boldsymbol{u}$. The gravitational acceleration is $\boldsymbol{g}$ and the air exerts a drag force on the insect equal to mk times the velocity of the wind *relative to the insect.*

(a) Show that the path of the insect is given by

$$\boldsymbol{x} = \left(\boldsymbol{u} + \frac{\boldsymbol{g}}{k}\right) t + \frac{1 - \mathrm{e}^{-kt}}{k}\left(\boldsymbol{v} - \boldsymbol{u} - \frac{\boldsymbol{g}}{k}\right)\,.$$

(b) In the case where the insect jumps vertically in a horizontal wind, show that the time T that elapses before it returns to the ground (which is also horizontal) satisfies

$$1 - \mathrm{e}^{-kT} = \frac{kT}{1+\lambda},$$

where $\lambda = kv/g$. Find an expression for the horizontal range R in terms of λ, u and T. (Here $v = |\boldsymbol{v}|$, $g = |\boldsymbol{g}|$ and $u = |\boldsymbol{u}|$.)

7. A ball of mass m moves, under gravity, in a resistive medium that produces a frictional force of magnitude kv^2, where v is the ball's speed. If the ball is projected vertically upwards with initial speed u, show by dimensional analysis that when the ball returns to its point of projection, its speed w can be written in the form

$$w = uf(\lambda),$$

where $\lambda = ku^2/mg$.

Integrate the equation of motion to show that $f(\lambda) = (1+\lambda)^{-1/2}$. [Hint: Thinking about v as a function of time may not be the easiest approach.]. Discuss what happens in the two extremes $\lambda \gg 1$ and $\lambda \ll 1$.

8*. The temperature $\theta(x,t)$ in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2},$$

where D is a constant (the *thermal diffusivity* of the rod). At time $t = 0$, the point $x = 0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$Q = \int_{-\infty}^{\infty} \theta(x,t)\,\mathrm{d}x$$

is constant. Use dimensional analysis to show that $\theta(x,t)$ can be written in the form

$$\theta(x,t) = \frac{Q}{\sqrt{Dt}}\,F(z),$$

where $z = x/\sqrt{Dt}$, and show further that

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \frac{z}{2}\frac{\mathrm{d}F}{\mathrm{d}z} + \frac{1}{2}F = 0.$$

Integrate this equation once directly to obtain a first-order differential equation. Evaluate the constant of integration by considering either the symmetry of the problem or the behaviour of the solution as $z \to \pm\infty$. Hence show that, for $t > 0$,

$$\theta(x,t) = \frac{Q}{\sqrt{4\pi Dt}}\,\exp\left(-\frac{x^2}{4Dt}\right).$$

Please send any comments and corrections to `gio10@cam.ac.uk`