

### Example Sheet 3

1. A square hoop  $ABCD$  is made of fine smooth wire and has side length  $2a$ . The hoop is horizontal and rotating with constant angular speed  $\omega$  about a vertical axis through  $A$ . A small bead that can slide on the wire is initially at rest at the midpoint of the side  $BC$ . Choose axes fixed relative to the hoop, and let  $x$  be the distance of the bead from the vertex  $B$  on the side  $BC$ . Write down the position vector of the bead in the rotating frame.

Using the standard expression for acceleration in a rotating frame, show that

$$\ddot{x} - \omega^2 x = 0.$$

Hence show that the time that the bead takes to reach a corner of the hoop is  $\omega^{-1} \cosh^{-1} 2$ . Using dimensional analysis, explain why this time is independent of  $a$ .

Obtain an expression for the magnitude of the force exerted by the hoop on the bead.

2. In a system of particles, the  $i$ th particle has mass  $m_i$  and position vector  $\mathbf{r}_i$  with respect to a fixed origin. The centre of mass of the system is at  $\mathbf{R}$ . Show that  $\mathbf{L}$ , the total angular momentum of the system about the origin, and  $\mathbf{L}_c$ , the total angular momentum of the system about the centre of mass, are related by

$$\mathbf{L}_c = \mathbf{L} - \mathbf{R} \times \mathbf{P},$$

where  $\mathbf{P}$  is the total linear momentum of the system.

Given that  $d\mathbf{P}/dt = \mathbf{F}$ , where  $\mathbf{F}$  is the total external force, and  $d\mathbf{L}/dt = \mathbf{G}$  where  $\mathbf{G}$  is the total external torque about the origin, show that

$$\frac{d\mathbf{L}_c}{dt} = \mathbf{G}_c,$$

where  $\mathbf{G}_c$  is the total external torque about the centre of mass.

3. A system of particles with masses  $m_i$  and position vectors  $\mathbf{r}_i$  ( $i = 1, \dots, n$ ) moves under its own mutual gravitational attraction alone. Write down the equation of motion for  $\mathbf{r}_i$ . Show that a possible solution of the equations of motion is given by  $\mathbf{r}_i = t^{2/3} \mathbf{a}_i$ , where the vectors  $\mathbf{a}_i$  are constant vectors satisfying

$$\mathbf{a}_i = \frac{9G}{2} \sum_{j \neq i} \frac{m_j (\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

4. Two particles of masses  $m_1$  and  $m_2$  move under their mutual gravitational attraction. Show from first principles that the quantity

$$\frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{GM}{r}$$

is constant, where  $\mathbf{r}$  is the position vector of one particle relative to the other and  $M = m_1 + m_2$ .

The particles are released from rest a long way apart and fall towards each other. Show that the position of their centre of gravity is fixed, and that when they are a distance  $r$  apart their relative speed is  $\sqrt{2GM/r}$ .

(\*) When the particles are a distance  $a$  apart, they are given equal and opposite impulses (changes of momentum), each of magnitude  $I$ , and each perpendicular to the direction of motion. Show that subsequently  $r^2\omega = aI/\mu$ , where  $\omega$  is the angular speed of either particle relative to the centre of mass and  $\mu$  is the reduced mass of the system.

Show further that the minimum separation,  $d$ , of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2d.$$

5. A rocket, moving vertically upwards, ejects gas vertically downwards at speed  $u$  relative to the rocket. Derive the equation of motion

$$m \frac{dv}{dt} = -u \frac{dm}{dt} - mg,$$

where  $v$  and  $m$  are the speed and total mass of the rocket (including fuel) at time  $t$ . If  $u$  is constant and the rocket starts from rest with total mass  $m_0$ , show that

$$m = m_0 e^{-(gt+v)/u}.$$

6. A firework of initial mass  $m_0$  is fired vertically upwards from the ground. Fuel is burnt at a constant rate  $-dm/dt = \alpha$  and the exhaust is ejected at constant speed  $u$  relative to the firework. Show that the speed of the firework at time  $t$ , where  $0 < t < m_0/\alpha$ , is

$$v(t) = -gt - u \log \left( 1 - \frac{\alpha t}{m_0} \right),$$

and that this is positive provided  $u > m_0 g/\alpha$ .

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha} \left( u - \frac{m_0 g}{2\alpha} \right).$$

7.

- (a) Thin circular discs of radius  $a$  and  $b$  are made of uniform materials with mass per unit area  $\rho_a$  and  $\rho_b$ , respectively. They lie in the same plane. Their centres  $A$  and  $B$  are connected by a light rigid rod of length  $c$ . Find the moment of inertia of the system about an axis through  $B$  perpendicular to the plane of the discs.
- (b) A thin uniform circular disc of radius  $a$  and centre  $A$  has a circular hole cut in it of radius  $b$  and centre  $B$ , where  $AB = c < a - b$ . The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius  $b$  passing through the hole. Using the result of part (a), with  $\rho_b$  suitably chosen, show that the period of small oscillations is  $2\pi\sqrt{l/g}$ , where

$$l = c + \frac{a^4 - b^4}{2a^2c}.$$

8. A yo-yo consists of two uniform discs, each of mass  $M$  and radius  $R$ , connected by a short light axle of radius  $a$  around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point  $P$ . The yo-yo is held with its centre of mass vertically below  $P$  and then released.

Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. Find the tension in the string as the yo-yo falls.

If the string has length  $L$ , what is the speed of the yo-yo just before it reaches the end? Explain what happens next. What is the impulse due to the tension in the string at this time?

9. A uniform circular cylinder of mass  $M$  and radius  $a$  is free to turn about its axis which is horizontal. A thin uniform cylindrical shell of mass  $M/2$  and radius  $a$  is fitted over the cylinder. At time  $t = 0$  the angular velocity of the cylinder is  $\Omega$ , while the shell is at rest. The shell exerts a frictional torque on the cylinder of magnitude  $k(\omega - \bar{\omega})$ , where  $\omega(t)$  and  $\bar{\omega}(t)$  are the angular velocities of the cylinder and shell, respectively, at time  $t$  about the axis. Prove that

$$\omega(t) = \frac{1}{2}\Omega \left(1 + e^{-4kt/Ma^2}\right),$$

and find the corresponding expression for  $\bar{\omega}(t)$ .

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