Example Sheet 2

1. Define $\delta_\epsilon(x)$ for $\epsilon > 0$ by

$$\delta_\epsilon(x) = \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right) .$$

(a) Evaluate

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\,\mathrm{d}x \quad \text{given that} \quad \int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2} .$$

(b) Argue that for a 'good' function f and a constant ξ

$$\lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \delta_\epsilon(x-\xi) f(x)\,\mathrm{d}x = f(\xi\,).$$

(c) Sketch $\delta_\epsilon(x)$ and comment.

2. (a) Starting from the definition that $\delta(x)$ is the generalized function such that for all 'good' functions $f(x)$

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(x)\,\mathrm{d}x = f(\xi)\,,$$

show that, for constant $a \neq 0$,

$$x\delta(x) = 0 \quad \text{and} \quad \delta(ax) = \frac{1}{|a|}\delta(x)\,.$$

(b) Evaluate

$$\int_{-\infty}^{\infty} |x|\delta(x^2-a^2)\,\mathrm{d}x\,,$$

where a is a non-zero constant. *Hint: the answer is not* $2a$. If keen, discuss the case $a = 0$.

3. The differential equation

$$y'' + y = H(x) - H(x-\epsilon)\,,$$

where H is the Heaviside step function and ϵ is a positive parameter, represents a simple harmonic oscillator subject to a constant force for a finite time. By solving the equation in the three intervals of x separately and applying appropriate matching conditions, show that the solution that vanishes for $x < 0$ is

$$y = \begin{cases} 0\,, & x < 0\,, \\ 1 - \cos x\,, & 0 < x < \epsilon\,, \\ \cos(x-\epsilon) - \cos x\,, & x > \epsilon\,. \end{cases}$$

Hence show that the solution of

$$y'' + y = \frac{H(x) - H(x-\epsilon)}{\epsilon}$$

that vanishes for $x < 0$ agrees, in the limit $\epsilon \to 0$, with the appropriate solution of $y'' + y = \delta(x)$, namely $y = H(x)\sin x$.

4. The function $G(x,\xi)$ is defined by

$$G(x,\xi) = \begin{cases} x(\xi - 1)\,, & 0 \leqslant x \leqslant \xi\,, \\ \xi(x-1)\,, & \xi \leqslant x \leqslant 1\,. \end{cases}$$

If $f(x)$ is continuous for $0 \leqslant x \leqslant 1$, and

$$y(x) = \int_0^1 f(\xi) G(x,\xi)\,\mathrm{d}\xi\,,$$

show that $y''(x) = f(x)$ and find $y(0)$ and $y(1)$.

Hint: use the definition of $G(x,\xi)$ to write $y(x)$ as the sum of two integrals, one with $\xi \leqslant x$ and the other with $x \leqslant \xi$.

5. Solve

$$y'' - y = \delta(x-a)\,,$$

subject to the boundary condition that y is bounded as $x \to \pm\infty$. Hence show that the solution of

$$y'' - y = f(x)\,,$$

subject to the same boundary condition, and with $f(x) \to 0$ as $x \to \pm\infty$, is

$$y = -\frac{1}{2}\int_{-\infty}^{\infty} f(a)\exp(-|x-a|)\,\mathrm{d}a\,.$$

6. Use the method of Green's function to solve

(a)

$$y'' - y = x^2 \qquad \text{with} \quad y(0) = y(1) = 0\,,$$

(b)

$$y'' + \omega^2 y = x \qquad \text{with} \quad y'(0) = y(\pi/\omega) = 0\,,$$

(c)

$$y'''' = f(x) \qquad \text{with} \quad y(0) = y'(0) = y''(0) = y'''(0) = 0\,.$$

7. Use the method of Green's function to find the *general* solution of

$$y'' - 2y' + y = 2x\,\mathrm{e}^{x}\,.$$

Hint: invent any convenient boundary conditions to obtain a particular solution, then deduce the general solution.

8. Show that the equation

$$y'' + py' + qy = f(x)\,,$$

where p and q are constants, can be written in the form

$$z' - az = f\,, \qquad y' - by = z\,,$$

for suitable choices of the constants a and b. Solve these first-order equations using integrating factors, subject to the initial conditions $y(0) = y'(0) = 0$, to obtain the solution

$$y(x) = \mathrm{e}^{bx} \int_0^x \int_0^\eta f(\xi)\,\mathrm{e}^{-a\xi}\,\mathrm{e}^{(a-b)\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

By changing the order of integration and carrying out the integration with respect to η, show that

$$y(x) = \frac{1}{a-b} \int_0^x f(\xi)\left[\mathrm{e}^{a(x-\xi)} - \mathrm{e}^{b(x-\xi)}\right]\,\mathrm{d}\xi$$

and interpret this result.

9. Let α and β be positive constants, and let $H(x)$ denote the Heaviside step function. Find the Fourier transforms of

(a) the odd function $f_\mathrm{o}(x)$, where f_o is defined for $x > 0$ by

$$f_\mathrm{o}(x) = \begin{cases} 1\,, & 0 < x \leqslant 1\,, \\ 0\,, & x > 1\,. \end{cases}$$

(b) the even function $f_\mathrm{e}(x) = \mathrm{e}^{-|x|}$.

(c) the even function $g(x)$, where

$$g(x) = \begin{cases} 1\,, & |x| < \alpha\,, \\ 0\,, & |x| \geqslant \alpha\,. \end{cases}$$

(d) the function

$$h(x) = H(x)\sinh(\alpha x)\,\mathrm{e}^{-\beta x}\,, \quad \text{where} \quad \alpha < \beta\,.$$

10. Show that, if a function f and its Fourier transform $\tilde{f}$ are both real, then f is even. Show also that, if a function f is real and its Fourier transform $\tilde{f}$ is purely imaginary, then f is odd.

11. (a) Use Parseval's theorem and the result of question 9a to show that

$$\int_{-\infty}^{\infty} \left(\frac{1-\cos x}{x}\right)^2 \mathrm{d}x = \pi\,.$$

(b) Use Parseval's theorem and the result of question 9b to evaluate the integral

$$\int_0^{\infty} \frac{\mathrm{d}k}{(1+k^2)^2}\,.$$

12. For $g(x)$ as given in question 9c define

$$G(x) = \int_{-\infty}^{\infty} g(x-\xi)\, g(\xi)\, \mathrm{d}\xi\,.$$

Find an expression for $G(x)$. Explicitly demonstrate that the Fourier transforms of $G(x)$ and $g(x)$ satisfy the convolution theorem.

This example sheet is available at `http://www.damtp.cam.ac.uk/user/examples/`
Please send any comments and corrections to `gio10@cam.ac.uk`