Example Sheet 1

1. *Revision of Keplerian orbits*

The equation of motion of a test particle in the gravitational field of a point mass M is

$$\ddot{\boldsymbol{r}} = -\frac{GM\boldsymbol{r}}{|\boldsymbol{r}|^3}.$$

Show that the motion is confined to a plane containing the central mass. Introduce polar coordinates (r, ϕ) in the plane and deduce that

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2},$$

$$r^2\dot{\phi} = h = \text{constant}.$$

Find an equation for the shape $r(\phi)$ of the orbit and show that the general solution is

$$r = \frac{\lambda}{1 + e\cos(\phi - \omega)},$$

where e and ω are arbitrary constants, and

$$\lambda = \frac{h^2}{GM}.$$

Sketch the orbit for the cases $e = 0$, $0 < e < 1$, $e = 1$ and $e > 1$.

Show that the orbit of least energy for a given angular momentum is a circular orbit.

2. *Linear diffusion equation*

The Green-function solution for a Keplerian accretion disc with constant kinematic viscosity $\bar{\nu}$ and with vanishing torque at $r = 0$ is (see lectures)

$$\Sigma(r, t) = \int_0^\infty G(r, s, t)\Sigma(s, 0)\,\mathrm{d}s,$$

where

$$G(r, s, t) = r^{-1/4}s^{5/4}\frac{1}{6\bar{\nu}t}\exp\left[-\frac{(r^2 + s^2)}{12\bar{\nu}t}\right] I_{1/4}\left(\frac{rs}{6\bar{\nu}t}\right).$$

Write down the solution in the case of an initial condition

$$\Sigma(r, 0) = \frac{1}{2\pi}\delta(r - 1),$$

in which a unit mass is concentrated into a narrow ring of unit radius, and verify directly that it satisfies the diffusion equation

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(r^{1/2}\bar{\nu}\Sigma)\right].$$

By considering the limit $r \to 0$, show that the accretion rate at the centre is

$$\dot{M} = \frac{3\bar{\nu}}{\Gamma(\frac{5}{4})}(12\bar{\nu}t)^{-5/4}\exp\left(-\frac{1}{12\bar{\nu}t}\right),$$

and interpret this behaviour. [For properties of the modified Bessel function I_ν, see the sheet of useful mathematical results.]

3. Nonlinear diffusion equation

A gas pressure-dominated Keplerian accretion disc with Thomson opacity and an alpha viscosity has the viscosity law (see lectures)

$$\bar{\nu} = Ar\Sigma^{2/3},$$

where A is a constant. By means of a dimensional analysis, identify the similarity variable ξ appropriate for a solution with vanishing torque at $r = 0$. Derive the similarity solution.

4. Nonlinear diffusion equation

(i) Consider a disc with the viscosity law

$$\bar{\nu} = Ar^2\Sigma,$$

where A is a constant. Show that solutions of the form

$$\Sigma = \sigma(t)\left\{\left[\frac{R(t)}{r}\right]^a - 1\right\}, \qquad r \leq R(t),$$

exist for only two non-zero values of the parameter a, namely $a = 1$ and $a = 5/4$. In each case, solve for $\sigma(t)$ and $R(t)$, assuming that $R(0) = 0$.

(ii) For the solution with $a = 1$, show that the mean radial velocity in the disc is

$$\bar{u}_r = -\frac{(R - 5r)}{10t}.$$

Determine the trajectories of fluid elements moving with the mean radial velocity, and deduce that almost every fluid element is accreted in a finite time. For the solution with $a = 5/4$, show that $\bar{u}_r$ is strictly positive.

(iii) Investigate whether mass and angular momentum are globally conserved in either of the two solutions. Comment on the likely significance of these special solutions among solutions of the nonlinear diffusion equation as an initial-value problem with various initial and boundary conditions.

5. Vertical structure with radiation pressure

The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and a mixture of gas and radiation is governed by the equations

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z,$$

$$\frac{\partial F}{\partial z} = \mu\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2,$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z},$$

$$p = \frac{k\rho T}{\mu_\mathrm{m} m_\mathrm{p}} + \frac{4\sigma T^4}{3c}.$$

(i) Show that there is only one possible value of the effective viscosity μ of an accretion disc in which the gas pressure is negligible compared to the radiation pressure. How does this value compare with the viscosity of water?

(ii) According to one theory, whatever the ratio of gas and radiation pressures, the effective viscosity is always related to the gas pressure p_g by $\mu = \alpha p_\mathrm{g}/\Omega$, where α is a constant. Show that the temperature then satisfies an equation of the form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial T^4}{\partial z}\right) + A\Omega T = 0,$$

where A is a constant to be determined. By means of a suitable change of variables, find an explicit expression for the mean kinematic viscosity $\bar{\nu}(r,\Sigma)$ of the disc. You may assume that the 'zero boundary conditions' apply, and that there exists a unique non-trivial solution $t(\zeta)$ of the boundary-value problem

$$\frac{\mathrm{d}^2 t^4}{\mathrm{d}\zeta^2} + t = 0, \qquad t'(0) = t(1) = 0.$$

Your answer may involve an integral of $t(\zeta)$, which need not be evaluated.

6. Steady alpha disc

Write down the relation that must hold between $\bar{\nu}$ and Σ in a steady, Keplerian accretion disc with inner radius $r_{\rm in}$ and accretion rate $\dot{M}$. Combine this with an order-of-magnitude treatment of the vertical structure of a gas pressure-dominated disc with Thomson opacity to deduce that

$$H \sim \alpha^{-1/10}\Omega^{-7/10}(f\dot{M})^{1/5}\left(\frac{\mu_{\rm m}m_{\rm p}}{k}\right)^{-2/5}\left(\frac{\sigma}{\kappa}\right)^{-1/10},$$

where $f = 1 - (r_{\rm in}/r)^{1/2}$. [Notice how insensitive this result is to the poorly known value of α.] Hence show that the disc is very slightly flared, in the sense that H/r increases very slowly with increasing r.

Repeat the calculation for Kramers opacity to obtain

$$H \sim \alpha^{-1/10}\Omega^{-3/4}(f\dot{M})^{3/20}\left(\frac{\mu_{\rm m}m_{\rm p}}{k}\right)^{-3/8}\left(\frac{\sigma}{C_\kappa}\right)^{-1/20}.$$

Please send comments and corrections to `gio10@cam.ac.uk`