Example Sheet 2

1. Magnetic fields in the shearing sheet

The induction equation in an incompressible fluid of constant magnetic diffusivity η is

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \eta \nabla^2 \boldsymbol{B}.$$

Why is the induction equation invariant when rewritten in the uniformly rotating frame of reference of the shearing sheet?

Supposing that the velocity retains the form of a linear shear flow, $\boldsymbol{u} = -2Ax\,\boldsymbol{e}_y$, show that the induction equation has solutions of the form

$$\boldsymbol{B} = \mathrm{Re}\left\{\tilde{\boldsymbol{B}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\},$$

provided that the Fourier wavevector evolves in time according to

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = 2Ak_y\,\boldsymbol{e}_x.$$

Solve for $\boldsymbol{k}(t)$, and interpret the result geometrically.

Deduce the equations satisfied by the components of the Fourier amplitude $\tilde{\boldsymbol{B}}(t)$, and find their general solution. Show that the magnetic energy typically experiences a phase of growth but ultimately decays.

Verify that $\boldsymbol{B} \cdot \nabla \boldsymbol{B} = \boldsymbol{0}$ for this solution, and confirm that the magnetic field has no influence on the flow. Given that any magnetic field can be considered as a superposition of such Fourier modes, explain how a non-zero Lorentz force can result.

2. Amplification of hydrodynamic disturbances in a non-rotating shear flow

Consider the evolution of sheared plane-wave disturbances in a non-rotating shearing sheet. The wave amplitudes $(\tilde{\boldsymbol{v}}, \tilde{\psi})$ of the velocity and pressure perturbations satisfy the equations (see lectures)

$$\begin{aligned}
\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} \qquad\qquad &= -\mathrm{i}k_x\hat{\psi},\\
\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} - 2A\hat{v}_x &= -\mathrm{i}k_y\hat{\psi},\\
\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} \qquad\qquad &= -\mathrm{i}k_z\hat{\psi},\\
\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} &= 0,
\end{aligned}$$

where

$$(\tilde{\boldsymbol{v}}, \tilde{\psi}) = (\hat{\boldsymbol{v}}, \hat{\psi}) \exp\left(-\int \nu k^2 \, \mathrm{d}t\right).$$

Show that, for generic non-axisymmetric disturbances ($k_y \neq 0$), the inviscid wave amplitudes satisfy

$$\hat{v}_x \propto t^{-2}, \qquad \hat{v}_y \propto \text{constant}, \qquad \hat{v}_z \propto \text{constant},$$

in the limit $t \to \infty$.

Show that generic axisymmetric disturbances ($k_y = 0$) experience algebraic growth in the absence of viscosity. When viscosity is included, show that the kinetic energy of the disturbance is proportional to

$$(1 + T^2 \cos^2\theta) \exp\left(-\frac{T}{\mathrm{Re}}\right),$$

where $T = 2At$ is the dimensionless time measured from the instant when $\hat{v}_y = 0$, θ is the angle between the wavevector and the vertical at that instant, and $\mathrm{Re} = A/\nu k^2$ is the Reynolds number of the disturbance.

Hence show that, for large Reynolds number, an energy amplification of

$$\left(\frac{2\cos\theta}{\mathrm{e}}\right)^2 (\mathrm{Re})^2 + O(1)$$

can be achieved, taking a time

$$\Delta t = \frac{\mathrm{Re}}{A} + O(\mathrm{Re})^{-1}.$$

3. Magnetorotational dispersion relation

Consider the magnetorotational dispersion relation derived in the lectures,

$$\left[(\omega + \mathrm{i}\nu k^2)(\omega + \mathrm{i}\eta k^2) - \omega_{\mathrm{A}}^2\right]^2 - 4\Omega(\Omega - A)(\omega + \mathrm{i}\eta k^2)^2 - 4\Omega A \omega_{\mathrm{A}}^2 = 0,$$

in the generic case $0 \neq \nu \neq \eta \neq 0$. As the parameters are varied, instability first sets in at a bifurcation where $\mathrm{Im}(\omega)$ passes through zero. If also $\mathrm{Re}(\omega) = 0$ at this point, it is a stationary bifurcation. Otherwise it is an oscillatory bifurcation.

By analysing the dispersion relation for a real value of ω, but without solving the equation, show that an oscillatory bifurcation cannot occur, and deduce that the instability always sets in as a monotonically growing mode.

4. Mechanical analogy for the magnetorotational instability

Two particles of mass m are orbiting in a gravitational potential $\Phi(\boldsymbol{r})$ and are connected by a Hookean spring of natural length ℓ and spring constant k. The Lagrangian of the system is

$$L(\boldsymbol{r}_1, \boldsymbol{r}_2, \dot{\boldsymbol{r}}_1, \dot{\boldsymbol{r}}_2) = \tfrac{1}{2}m|\dot{\boldsymbol{r}}_1|^2 + \tfrac{1}{2}m|\dot{\boldsymbol{r}}_2|^2 - m\Phi(\boldsymbol{r}_1) - m\Phi(\boldsymbol{r}_2) - \tfrac{1}{2}k\left(|\boldsymbol{r}_1 - \boldsymbol{r}_2| - \ell\right)^2.$$

Write out the Lagrangian using cylindrical polar coordinates (r, ϕ, z) for motion confined to the mid-plane $z = 0$, where the potential is $\Phi_{\mathrm{m}}(r)$. Deduce the equations of motion.

Show that there is a uniformly rotating solution with

$$r_1 = r_2 = r = \frac{\ell}{2\sin\alpha}, \qquad \phi_1 = \Omega t + 2\alpha, \qquad \phi_2 = \Omega t,$$

where α is a constant and $\Omega(r)$ is the orbital frequency at radius r, given by

$$r\Omega^2 = \frac{\mathrm{d}\Phi_{\mathrm{m}}}{\mathrm{d}r}.$$

Derive the linearized equations for small perturbations about this solution. Show that, in the limit of small α, the linearized system becomes

$$\delta\ddot{r}_1 - 2r\Omega\,\delta\dot{\phi}_1 = 4\Omega A\,\delta r_1,$$

$$r^2\delta\ddot{\phi}_1 + 2r\Omega\,\delta\dot{r}_1 = -\frac{kr^2}{m}(\delta\phi_1 - \delta\phi_2),$$

with symmetrical equations for the second particle, where $A(r)$ is the Oort parameter. Show that the symmetric modes of the system are either neutral or oscillate at the epicyclic frequency, while the frequencies of antisymmetric modes satisfy the equation

$$\omega^4 - \omega^2\left[4\Omega(\Omega - A) + \frac{2k}{m}\right] - 4\Omega A\left(\frac{2k}{m}\right) = 0.$$

Compare the properties of this equation with those of the ideal magnetorotational dispersion relation. What is the maximum possible growth rate, if the properties of the spring may be varied?

[*Recall Lagrange's equation of motion for coordinate* q,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.]$$

5. Shearing box

The shearing box is a local model of a differentially rotating disc. The velocity perturbation $\boldsymbol{v}$ and magnetic field $\boldsymbol{B}$ in an incompressible shearing box of uniform density ρ, kinematic viscosity ν and magnetic diffusivity η satisfy the same equations as in the shearing sheet,

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{v} - 2Av_x\,\boldsymbol{e}_y + 2\Omega\,\boldsymbol{e}_z \times \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\psi + \frac{1}{\mu_0\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \nu\nabla^2\boldsymbol{v},$$

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\boldsymbol{B} + 2AB_x\,\boldsymbol{e}_y + \boldsymbol{v}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{v} + \eta\nabla^2\boldsymbol{B},$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0.$$

However, unlike the shearing sheet, these equations are solved in a finite box

$$0 < x < L_x, \qquad 0 < y < L_y, \qquad 0 < z < L_z.$$

If f denotes ψ or any component of $\boldsymbol{v}$ or $\boldsymbol{B}$, the boundary conditions are

$$\begin{aligned} f(L_x, y, z, t) &= f(0, y', z, t), \\ f(x, L_y, z, t) &= f(x, 0, z, t), \\ f(x, y, L_z, t) &= f(x, y, 0, t), \end{aligned} \tag{$*$}$$

where

$$y' = (y + 2AL_x t) \bmod L_y.$$

(i) Let angle brackets denote an average over the volume of the box, i.e.

$$\langle f\rangle(t) = \frac{1}{L_xL_yL_z}\int_0^{L_z}\int_0^{L_y}\int_0^{L_x} f(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

If $Q(x,y,z,t)$ denotes any quantity satisfying the boundary conditions (*), show that

$$\left\langle\frac{\partial Q}{\partial x}\right\rangle = \left\langle\frac{\partial Q}{\partial y}\right\rangle = \left\langle x\frac{\partial Q}{\partial y}\right\rangle = \left\langle\frac{\partial Q}{\partial z}\right\rangle = 0.$$

(ii) Derive the volume-averaged equations

$$\frac{\mathrm{d}\langle\boldsymbol{v}\rangle}{\mathrm{d}t} - 2A\langle v_x\rangle\,\boldsymbol{e}_y + 2\Omega\,\boldsymbol{e}_z \times \langle\boldsymbol{v}\rangle = \boldsymbol{0},$$

$$\frac{\mathrm{d}\langle\boldsymbol{B}\rangle}{\mathrm{d}t} + 2A\langle B_x\rangle\,\boldsymbol{e}_y = \boldsymbol{0}.$$

Deduce that the box as a whole can undergo epicyclic oscillations. Explain why it is impossible for an accretion flow to develop in the shearing box.

(iii) Derive the energy-like equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{1}{2}v^2+\frac{B^2}{2\mu_0\rho}\right\rangle=2A\left\langle v_xv_y-\frac{B_xB_y}{\mu_0\rho}\right\rangle-\nu\langle|\nabla\times\boldsymbol{v}|^2\rangle-\frac{\eta}{\mu_0\rho}\langle|\nabla\times\boldsymbol{B}|^2\rangle.$$

Let $L=\max(L_x,L_y,L_z)$, and define the magnetic Reynolds number

$$\mathrm{Rm}=\frac{L^2|A|}{\eta}.$$

Show that, if $\nu=\eta$ and $\langle\boldsymbol{v}\rangle=\langle\boldsymbol{B}\rangle=\boldsymbol{0}$, turbulence cannot be sustained in the box if $\mathrm{Rm}<4\pi^2$.

Please send comments and corrections to `gio10@cam.ac.uk`