Example Sheet 2

1. Magnetic fields in the shearing sheet

The induction equation in an incompressible fluid of constant magnetic diffusivity $\eta$ is

$$ \frac{D\mathbf{B}}{Dt} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}. $$

Why is the induction equation invariant when rewritten in the uniformly rotating frame of reference of the shearing sheet?

Supposing that the velocity retains the form of a linear shear flow, $\mathbf{u} = -2Ax \mathbf{e}_y$, show that the induction equation has solutions of the form

$$ B = \text{Re} \left\{ \tilde{B}(t) \exp[i \mathbf{k}(t) \cdot \mathbf{x}] \right\}, $$

provided that the Fourier wavevector evolves in time according to

$$ \frac{d\mathbf{k}}{dt} = 2A k_y \mathbf{e}_x. $$

Solve for $\mathbf{k}(t)$, and interpret the result geometrically.

Deduce the equations satisfied by the components of the Fourier amplitude $\tilde{B}(t)$, and find their general solution. Show that the magnetic energy typically experiences a phase of growth but ultimately decays.

Verify that $\mathbf{B} \cdot \nabla \mathbf{B} = 0$ for this solution, and confirm that the magnetic field has no influence on the flow. Given that any magnetic field can be considered as a superposition of such Fourier modes, explain how a non-zero Lorentz force can result.

2. Amplification of hydrodynamic disturbances in a non-rotating shear flow

Consider the evolution of sheared plane-wave disturbances in a non-rotating shearing sheet. The wave amplitudes $(\tilde{\mathbf{v}}, \hat{\psi})$ of the velocity and pressure perturbations satisfy the equations (see lectures)

$$ \frac{d\tilde{v}_x}{dt} = -ik_x \hat{\psi}, $$

$$ \frac{d\tilde{v}_y}{dt} - 2A \tilde{v}_x = -ik_y \hat{\psi}, $$

$$ \frac{d\tilde{v}_z}{dt} = -ik_z \hat{\psi}, $$

$$ i\mathbf{k} \cdot \tilde{\mathbf{v}} = 0, $$

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where

\[
(\mathbf{v}, \mathbf{\psi}) = (\mathbf{\tilde{v}}, \mathbf{\tilde{\psi}}) \exp \left( -\int \nu k^2 \, dt \right).
\]

Show that, for generic non-axisymmetric disturbances \((k_y \neq 0)\), the inviscid wave amplitudes satisfy

\[
\mathbf{\hat{v}}_x \propto t^{-2}, \quad \mathbf{v}_y \propto \text{constant}, \quad \mathbf{\hat{v}}_z \propto \text{constant},
\]
in the limit \(t \to \infty\).

Show that generic axisymmetric disturbances \((k_y = 0)\) experience algebraic growth in the absence of viscosity. When viscosity is included, show that the kinetic energy of the disturbance is proportional to

\[
(1 + T^2 \cos^2 \theta) \exp \left( -\frac{T}{\text{Re}} \right),
\]

where \(T = 2At\) is the dimensionless time measured from the instant when \(\mathbf{\hat{v}}_y = 0\), \(\theta\) is the angle between the wavevector and the vertical at that instant, and \(\text{Re} = A/\nu k^2\) is the Reynolds number of the disturbance.

Hence show that, for large Reynolds number, an energy amplification of

\[
\left( \frac{2 \cos \theta}{e} \right)^2 (\text{Re})^2 + O(1)
\]
can be achieved, taking a time

\[
\Delta t = \frac{\text{Re}}{A} + O(\text{Re}^{-1}).
\]

3. Magnetorotational dispersion relation

Consider the magnetorotational dispersion relation derived in the lectures,

\[
\left[ (\omega + i\nu k^2)(\omega + i\eta k^2) - \omega_0^2 \right]^2 - 4\Omega(\Omega - A)(\omega + i\eta k^2)^2 - 4\Omega A \omega_0^2 = 0,
\]
in the generic case \(0 \neq \nu \neq \eta \neq 0\). As the parameters are varied, instability first sets in at a bifurcation where \(\text{Im}(\omega)\) passes through zero. If also \(\text{Re}(\omega) = 0\) at this point, it is a stationary bifurcation. Otherwise it is an oscillatory bifurcation.

By analysing the dispersion relation for a real value of \(\omega\), but without solving the equation, show that an oscillatory bifurcation cannot occur, and deduce that the instability always sets in as a monotonically growing mode.
4. Mechanical analogy for the magnetorotational instability

Two particles of mass \(m\) are orbiting in a gravitational potential \(\Phi(r)\) and are connected by a Hookean spring of natural length \(\ell\) and spring constant \(k\). The Lagrangian of the system is

\[
L(r_1, r_2, \dot{r}_1, \dot{r}_2) = \frac{1}{2} m |\dot{r}_1|^2 + \frac{1}{2} m |\dot{r}_2|^2 - m \Phi(r_1) - m \Phi(r_2) - \frac{1}{2} k (|r_1 - r_2| - \ell)^2.
\]

Write out the Lagrangian using cylindrical polar coordinates \((r, \phi, z)\) for motion confined to the mid-plane \(z = 0\), where the potential is \(\Phi_m(r)\). Deduce the equations of motion.

Show that there is a uniformly rotating solution with

\[
r_1 = r_2 = r = \frac{\ell}{2 \sin \alpha}, \quad \phi_1 = \Omega t + 2\alpha, \quad \phi_2 = \Omega t,
\]

where \(\alpha\) is a constant and \(\Omega(r)\) is the orbital frequency at radius \(r\), given by

\[
r\Omega^2 = \frac{d \Phi_m}{dr}.
\]

Derive the linearized equations for small perturbations about this solution. Show that, in the limit of small \(\alpha\), the linearized system becomes

\[
\ddot{\delta r}_1 - 2r\Omega \dot{\phi}_1 = 4\Omega A \delta r_1,
\]

\[
r^2 \ddot{\delta \phi}_1 + 2r\Omega \dot{\delta r}_1 = -\frac{k r^2}{m} (\delta \phi_1 - \delta \phi_2),
\]

with symmetrical equations for the second particle, where \(A(r)\) is the Oort parameter. Show that the symmetric modes of the system are either neutral or oscillate at the epicyclic frequency, while the frequencies of antisymmetric modes satisfy the equation

\[
\omega^4 - \omega^2 \left[ 4\Omega(\Omega - A) + \frac{2k}{m} \right] - 4\Omega A \left( \frac{2k}{m} \right) = 0.
\]

Compare the properties of this equation with those of the ideal magnetorotational dispersion relation. What is the maximum possible growth rate, if the properties of the spring may be varied?

[Recall Lagrange’s equation of motion for coordinate \(q\),

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.
\]

]
5. Shearing box

The shearing box is a local model of a differentially rotating disc. The velocity perturbation $\mathbf{v}$ and magnetic field $\mathbf{B}$ in an incompressible shearing box of uniform density $\rho$, kinematic viscosity $\nu$ and magnetic diffusivity $\eta$ satisfy the same equations as in the shearing sheet,

$$
\left( \frac{\partial}{\partial t} - 2A x \frac{\partial}{\partial y} \right) \mathbf{v} - 2Av_x \mathbf{e}_y + 2\Omega \mathbf{e}_z \times \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \psi + \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{v},
$$

$$
\left( \frac{\partial}{\partial t} - 2A x \frac{\partial}{\partial y} \right) \mathbf{B} + 2AB_x \mathbf{e}_y + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{B},
$$

$$
\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0.
$$

However, unlike the shearing sheet, these equations are solved in a finite box

$$
0 < x < L_x, \quad 0 < y < L_y, \quad 0 < z < L_z.
$$

If $f$ denotes $\psi$ or any component of $\mathbf{v}$ or $\mathbf{B}$, the boundary conditions are

$$
f(L_x, y, z, t) = f(0, y', z, t),
$$

$$
f(x, L_y, z, t) = f(x, 0, z, t),
$$

$$
f(x, y, L_z, t) = f(x, y, 0, t),
$$

where

$$
y' = (y + 2AL_xt) \text{ mod } L_y.
$$

(i) Let angle brackets denote an average over the volume of the box, i.e.

$$
\langle f \rangle(t) = \frac{1}{L_x L_y L_z} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} f(x, y, z, t) \, dx \, dy \, dz.
$$

If $Q(x, y, z, t)$ denotes any quantity satisfying the boundary conditions (*), show that

$$
\langle \frac{\partial Q}{\partial x} \rangle = \langle \frac{\partial Q}{\partial y} \rangle = \langle \frac{\partial Q}{\partial y} \rangle = \langle \frac{\partial Q}{\partial z} \rangle = 0.
$$

(ii) Derive the volume-averaged equations

$$
\frac{d}{dt} \langle \mathbf{v} \rangle - 2A \langle v_x \rangle \mathbf{e}_y + 2\Omega \mathbf{e}_z \times \langle \mathbf{v} \rangle = 0,
$$

$$
\frac{d}{dt} \langle \mathbf{B} \rangle + 2A \langle B_x \rangle \mathbf{e}_y = 0.
$$

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Deduce that the box as a whole can undergo epicyclic oscillations. Explain why it is impossible for an accretion flow to develop in the shearing box.

(iii) Derive the energy-like equation

\[ \frac{d}{dt} \left( \frac{1}{2} v^2 + \frac{B^2}{2 \mu_0 \rho} \right) = 2 A \left\langle v_x v_y - \frac{B_x B_y}{\mu_0 \rho} \right\rangle - \nu \langle |\nabla \times v|^2 \rangle - \frac{\eta}{\mu_0 \rho} \langle |\nabla \times B|^2 \rangle. \]

Let \( L = \max(L_x, L_y, L_z) \), and define the magnetic Reynolds number

\[ R_m = \frac{L^2 |A|}{\eta}. \]

Show that, if \( \nu = \eta \) and \( \langle v \rangle = \langle B \rangle = 0 \), turbulence cannot be sustained in the box if \( R_m < 4 \pi^2 \).

Please send comments and corrections to gio10@cam.ac.uk