Dynamics of Astrophysical Discs — Mathematical Tripos, Part III
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NOTES ON AFD AND MHD

[*These notes are intended for reference but go well beyond what is required for the course.*]

1.1. Astrophysical fluid dynamics

1.1.1. Introduction

Astrophysical fluid dynamics (AFD) is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology itself.

A fluid description is not applicable (i) in solidified regions such as the rocky cores of giant planets, and (ii) in very tenuous regions where the mean free path of the particles is not much less than the characteristic macroscopic length-scale of the system (or indeed in any system if one examines length-scales comparable to or smaller than the mean free path).

There are various flavours of AFD in common use. The basic model involves a compressible, inviscid fluid and is Newtonian (i.e. non-relativistic). The thermal physics of the fluid may be treated in different ways, either by assuming it to be isothermal or adiabatic, or by including radiation processes in varying levels of detail.

Magnetohydrodynamics (MHD) generalizes this theory by including the dynamical effects of magnetic fields. Often the fluid is assumed to be perfectly electrically conducting (*ideal MHD*).

One can also include the dynamical (rather than thermal) effects of radiation, resulting in a theory of *radiation (magneto)hydrodynamics*. Dissipative effects such as viscosity and resistivity can be included. All these theories can also be formulated in a relativistic framework.

AFD typically differs from 'laboratory' or 'engineering' fluid dynamics in the relative importance of certain effects. Effects that may be important in AFD include compressibility, gravitation, magnetic fields, radiation forces and relativistic phenomena. Effects that are usually unimportant include viscosity, surface tension and the presence of solid boundaries.

1.1.2. Basic equations

A *simple fluid* is characterized by a velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ and two independent thermodynamic properties, of which the most useful are the dynamical quantities: the mass density $\rho(\boldsymbol{r},t)$ and the pressure $p(\boldsymbol{r},t)$. Other properties, such as the temperature and the viscosity, can be regarded as functions of ρ and p.

In the *Eulerian viewpoint* we consider how the properties of the fluid vary in time at a point that is fixed in space, i.e. attached to the (usually inertial) coordinate system. In the *Lagrangian viewpoint* we consider how the properties of the fluid vary at a point that moves with the fluid. The *Eulerian time-derivative* of the density, for example, is

$$\frac{\partial \rho}{\partial t},$$

while the *Lagrangian time-derivative* is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho.$$

Equation of mass conservation

The *equation of mass conservation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

has the typical form of a conservation law. Note that $\rho \boldsymbol{u}$ is the *mass flux density.*

This equation can also be written in the form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}.$$

In an *incompressible fluid*, fluid elements preserve their density, and so

$$\nabla \cdot \boldsymbol{u} = 0.$$

Although no fluid is strictly incompressible, this can sometimes be a good approximation for subsonic motions.

Equation of motion

The *equation of motion* is

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T},$$

and derives from Newton's second law. Here Φ is the gravitational potential and $\mathbf{T}$ is the stress tensor, a symmetric tensor field of second rank. Many types of force, such as viscous, turbulent or magnetic forces, can be represented in the form $\nabla \cdot \mathbf{T}$.

Poisson's equation

The gravitational potential is generated by the mass distribution according to *Poisson's equation*,

$$\nabla^2\Phi = 4\pi G\rho,$$

where G is Newton's constant. The solution

$$\Phi(\boldsymbol{r},t) = -G\int \frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}'-\boldsymbol{r}|}\,\mathrm{d}^3\boldsymbol{r}'$$

will, in general, involve contributions from both inside and outside the fluid body being considered. Some astrophysical discs are *non-self-gravitating*, meaning that the internal contribution can be neglected relative to that of external masses, most importantly the central object. In this case Φ is known in advance and Poisson's equation is not coupled to the other equations.

Thermal energy equation

The *thermal energy equation* is

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \mathscr{H} - \mathscr{C},$$

where T is the temperature, s the specific entropy (entropy per unit mass), $\mathscr{H}$ the heating rate per unit volume and $\mathscr{C}$ the cooling rate per unit volume. Note that $\mathscr{H}$ and $\mathscr{C}$ represent non-adiabatic processes.

In the case of a (Navier–Stokes) viscous stress,

$$\mathbf{T} = 2\mu\mathbf{S} + \mu_\mathrm{b}(\nabla\cdot\boldsymbol{u})\mathbf{I},$$

where μ is the (shear) viscosity, μ_b the bulk viscosity and

$$\mathbf{S} = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\mathrm{T}\right] - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\mathbf{I}$$

is the traceless shear tensor. ($\mathbf{I}$ is the unit tensor. Recall that the *dynamic viscosity* μ and the *kinematic viscosity* ν are related by $\mu = \rho\nu$.) The viscous heating rate is

$$\mathscr{H} = \mathbf{T}:\nabla\boldsymbol{u} = 2\mu\mathbf{S}^2 + \mu_\mathrm{b}(\nabla\cdot\boldsymbol{u})^2.$$

Although molecular viscosity is rarely important in astrophysics, it is commonly used to parametrize other processes, especially turbulence, that to some extent mimic it.

The radiative cooling rate can be written

$$\mathscr{C} = \nabla\cdot\boldsymbol{F},$$

where $\boldsymbol{F}$ is the *radiative energy flux density*. In optically thick media the radiation is locally close to a black-body distribution and may be treated in the *diffusion approximation*. The radiative flux is then directed down the temperature gradient,

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho}\nabla T,$$

where σ is the Stefan–Boltzmann constant and κ the *Rosseland mean opacity*.

To relate the thermal quantities T and s to the dynamical variables ρ and p, an *equation of state* is required, together with standard thermodynamic identities. (The local thermodynamic state of a simple fluid is specified by any two of these quantities. This ignores the possible complications of variable chemical composition.) Most important in astrophysics is the case of an *ideal gas together with black-body radiation*, for which

$$p = p_{\mathrm{g}} + p_{\mathrm{r}} = \frac{k_{\mathrm{B}}\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}} + \frac{4\sigma T^4}{3c},$$

where k_{B} is Boltzmann's constant, m_{p} the mass of the proton and c the speed of light. μ_{m} is the mean molecular weight (2.0 for molecular hydrogen, 1.0 for atomic hydrogen, 0.5 for fully ionized hydrogen and about 0.6 for ionized matter of cosmic abundances). Radiation pressure is often negligible but can be important in the centres of stars or in the innermost part of luminous discs around neutron stars and black holes.

In AFD it is usual to define three *adiabatic exponents* γ_1, γ_2 and γ_3 by

$$\gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s,$$

$$\frac{\gamma_2}{\gamma_2 - 1} = \left(\frac{\partial \ln p}{\partial \ln T}\right)_s,$$

$$\gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s.$$

The ratio of specific heats $\gamma = c_{\mathrm{p}}/c_{\mathrm{v}}$ and the three gamma coefficients are related by

$$\gamma_1 = \left(\frac{\gamma_2}{\gamma_2 - 1}\right)(\gamma_3 - 1) = \chi_\rho + \chi_T(\gamma_3 - 1) = \chi_\rho\gamma, \tag{1}$$

where

$$\chi_\rho = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T, \qquad \chi_T = \left(\frac{\partial \ln p}{\partial \ln T}\right)_\rho$$

can be found from the equation of state. In the case of an ideal gas (with negligible radiation pressure), $\chi_\rho = \chi_T = 1$ and so $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$.

We then have the important relation

$$\rho T \mathrm{d}s = \left(\frac{1}{\gamma_3 - 1}\right)\left(\mathrm{d}p - \frac{\gamma_1 p}{\rho}\mathrm{d}\rho\right), \tag{2}$$

which allows the thermal energy equation to be rewritten in terms of the dynamical variables ρ and p, i.e.

$$\left(\frac{1}{\gamma_3-1}\right)\left(\frac{\mathrm{D}p}{\mathrm{D}t}-\frac{\gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)=\mathbf{T}:\nabla\boldsymbol{u}-\nabla\cdot\boldsymbol{F}.$$

It is often assumed that $\gamma_1=\gamma_2=\gamma_3=\gamma$.

Exercise: Derive equations (1) and (2). For the latter you will require the Maxwell relation that comes from the expression $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}(\rho^{-1})$ for the differential of the specific internal energy.

1.2. Elementary derivation of the MHD equations

Magnetohydrodynamics (or MHD) is the dynamics of an electrically conducting fluid containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

1.2.1. The induction equation

We restrict ourselves to a non-relativistic theory in which fluid motions are slow compared to the speed of light. The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are governed by Maxwell's equations without the displacement current,

$$\frac{\partial\boldsymbol{B}}{\partial t}=-\nabla\times\boldsymbol{E},$$
$$\nabla\cdot\boldsymbol{B}=0,$$
$$\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J},$$

where μ_0 is the permeability of free space and $\boldsymbol{J}$ is the electric current density. The fourth Maxwell equation, involving $\nabla\cdot\boldsymbol{E}$, is not required in a non-relativistic theory.

Exercise: Show that these equations are invariant under the *Galilean transformation* to a frame of reference moving with uniform relative velocity $\boldsymbol{v}$,

$$\boldsymbol{r}'=\boldsymbol{r}-\boldsymbol{v}t,$$
$$t'=t,$$
$$\boldsymbol{E}'=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B},$$
$$\boldsymbol{B}'=\boldsymbol{B},$$
$$\boldsymbol{J}'=\boldsymbol{J},$$

as required by a 'non-relativistic' theory. (In fact, this is simply 'Galilean relativity'.)

Ohm's law for a static medium with electrical conductivity σ is

$$\boldsymbol{J}=\sigma\boldsymbol{E}.$$

For an electrically conducting fluid moving with velocity $\boldsymbol{u}$ this becomes

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}),$$

or

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \frac{\boldsymbol{J}}{\sigma}.$$

From Maxwell's equations we then obtain the *induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

where

$$\eta = \frac{1}{\mu_0 \sigma}$$

is the *magnetic diffusivity*. (The *resistivity* is $1/\sigma$.) Note that this is an evolutionary equation for $\boldsymbol{B}$ alone, and $\boldsymbol{E}$ and $\boldsymbol{J}$ have been eliminated. The divergence of the induction equation is

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0,$$

so the solenoidal character of $\boldsymbol{B}$ is preserved.

1.2.2. The Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a *Lorentz force*

$$\boldsymbol{F}_\mathrm{m} = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. (The electrostatic force is negligible in the non-relativistic theory.) In Cartesian components,

$$\begin{aligned}(\mu_0 \boldsymbol{F}_\mathrm{m})_i &= \epsilon_{ijk}(\epsilon_{jlm}\partial_l B_m)B_k \\ &= (\partial_k B_i - \partial_i B_k)B_k \\ &= B_k \partial_k B_i - \partial_i(\tfrac{1}{2}B^2).\end{aligned}$$

Thus

$$\boldsymbol{F}_\mathrm{m} = \frac{1}{\mu_0}\boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla\left(\frac{B^2}{2\mu_0}\right).$$

The first term is a *curvature force* due to a tension in the field lines. The second term is the gradient of an isotropic *magnetic pressure*

$$p_\mathrm{m} = \frac{B^2}{2\mu_0},$$

which is also equal to the energy density of the magnetic field.

Alternatively, one can write

$$\boldsymbol{F}_{\mathrm{m}} = \nabla \cdot \mathbf{M},$$

where

$$\mathbf{M} = \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0} - \frac{B^2}{2\mu_0}\mathbf{I}$$

is the *Maxwell stress tensor*. If the magnetic field is locally aligned with the z-axis, then

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +\frac{B^2}{\mu_0} \end{bmatrix} + \begin{bmatrix} -\frac{B^2}{2\mu_0} & 0 & 0 \\ 0 & -\frac{B^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B^2}{2\mu_0} \end{bmatrix}.$$

The first term represents a *magnetic tension* $T_{\mathrm{m}} = B^2/\mu_0$ per unit area in the field lines. This gives rise to *Alfvén waves*, which travel parallel to the field with characteristic speed

$$v_{\mathrm{A}} = \left(\frac{T_{\mathrm{m}}}{\rho}\right)^{1/2} = \frac{B}{(\mu_0\rho)^{1/2}},$$

the *Alfvén speed*. This is often considered as a vector Alfvén velocity,

$$\boldsymbol{v}_{\mathrm{A}} = \frac{\boldsymbol{B}}{(\mu_0\rho)^{1/2}}.$$

The magnetic pressure also affects the propagation of sound waves, which become *magnetoacoustic waves*.

The combination

$$\Pi = p + \frac{B^2}{2\mu_0}$$

is often referred to as the *total pressure*. The ratio

$$\beta = \frac{p}{B^2/2\mu_0}$$

is known as the *plasma beta*.

1.2.3. Joule heating

Like viscosity, resistivity is a source of irreversibility and dissipation. (Note, however, that while viscosity is the microscopic transport coefficient for momentum, resistivity is *inversely* proportional to the microscopic transport coefficient for electrical current.) In the presence of resistivity, magnetic energy is dissipated at a rate

$$\mathscr{H}_{\mathrm{Joule}} = \frac{J^2}{\sigma} = \mu_0\eta J^2$$

per unit volume and converted into heat.

1.2.4. Summary of the MHD equations

A full set of MHD equations, including compressibility and dissipative effects, might read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T} + \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B},$$

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \mathbf{T} : \nabla \boldsymbol{u} + \mu_0 \eta J^2 - \nabla \cdot \boldsymbol{F},$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

together with constitutive relations determining the viscosity, magnetic diffusivity, opacity, equation of state, etc. Most of these equations can be written in at least one other way that may be useful in different circumstances.

These equations display the essential *nonlinearity* of MHD. When the velocity field is prescribed, an artifice known as the *kinematic approximation*, the induction equation is a relatively straightforward linear evolutionary equation for the magnetic field. However, a sufficiently strong magnetic field will modify the velocity field through its dynamical effect, the Lorentz force. This nonlinear coupling leads to a rich variety of behaviour. (Of course, the purely hydrodynamic nonlinearity of the $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ term is still present. Another, less important nonlinear effect of the magnetic field occurs through Joule heating.)

1.3. Kinematics of the magnetic field

1.3.1. Ideal MHD

For a perfect electrical conductor, $\sigma \to \infty$ and so $\eta \to 0$. This limit is known as *ideal MHD*. The induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}).$$

This equation has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material lines. To show this, write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - (\nabla \cdot \boldsymbol{u}) \boldsymbol{B},$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u},$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right)\cdot\nabla\boldsymbol{u}.$$

This is exactly the same equation satisfied by an infinitesimal *material line element* $\delta\boldsymbol{r}$ as it is stretched by the velocity gradient:

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta\boldsymbol{r} = \delta\boldsymbol{u} = \delta\boldsymbol{r}\cdot\nabla\boldsymbol{u}.$$

Precisely the same equation,

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}),$$

is satisfied by the vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ in barotropic (homentropic) ideal fluid dynamics in the absence of a magnetic field. However, the fact that $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by the curl operation means that the analogy between vorticity dynamics and MHD is not perfect.

Another way to demonstrate the result of flux freezing is to represent the magnetic field using *Euler potentials* α and β,

$$\boldsymbol{B} = \nabla\alpha\times\nabla\beta.$$

This is sometimes called a *Clebsch representation*. By using two scalar potentials we are able to represent a three-dimensional vector field satisfying the constraint $\nabla\cdot\boldsymbol{B} = 0$. A vector potential of the form $\boldsymbol{A} = \alpha\nabla\beta + \nabla\gamma$ generates this magnetic field. The magnetic field lines are the intersections of the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$.

After some algebra it can be shown that

$$\frac{\partial\boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \nabla\left(\frac{\mathrm{D}\alpha}{\mathrm{D}t}\right)\times\nabla\beta + \nabla\alpha\times\nabla\left(\frac{\mathrm{D}\beta}{\mathrm{D}t}\right). \tag{3}$$

The ideal induction equation is therefore satisfied if the Euler potentials are conserved following the fluid flow, i.e. if the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$ are material surfaces. In this case the magnetic field lines can also be identified with material lines.

Exercise: Derive equation (3).

1.3.2. Non-ideal MHD

When $\eta > 0$ the resistivity of the fluid causes *diffusion* of the magnetic field and *dissipation* of magnetic energy. In the case of a uniform magnetic diffusivity, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}.$$

Magnetic field lines are no longer frozen in to the fluid, but are able to slip through it. If L and U represent characteristic scales of length and velocity for the flow, the characteristic time-scales for advection and diffusion of the field are

$$T_{\text{advection}} = \frac{L}{U},$$

$$T_{\text{diffusion}} = \frac{L^2}{\eta}.$$

The relative importance of advection is measured by the *magnetic Reynolds number*

$$\text{Rm} = \frac{T_{\text{diffusion}}}{T_{\text{advection}}} = \frac{LU}{\eta}.$$

When $\text{Rm} \gg 1$, as is typical in astrophysics, ideal MHD is a good approximation. However, a highly conducting fluid can violate the constraints of ideal MHD by developing very small-scale structures for which Rm is not large. This happens in reconnection and in turbulence.

The magnetic diffusivity has the same dimensions as the kinematic viscosity ν. Their ratio is (one definition of) the *magnetic Prandtl number*

$$\text{Pm} = \frac{\nu}{\eta}.$$

1.4. Energetics and conservation laws

1.4.1. Synthesis of the total energy equation, including dissipation

Kinetic energy:

$$\rho \frac{\text{D}}{\text{D}t}(\tfrac{1}{2}u^2) = -\rho \boldsymbol{u} \cdot \nabla \Phi - \boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\nabla \cdot \mathbf{T}) + \frac{1}{\mu_0} \boldsymbol{u} \cdot [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}] \,.$$

Gravitational energy (assuming Φ to be independent of t):

$$\rho \frac{\text{D}\Phi}{\text{D}t} = \rho \boldsymbol{u} \cdot \nabla \Phi.$$

Thermal energy (noting $\text{d}e = T\,\text{d}s - p\,\text{d}(\rho^{-1}) \Rightarrow \rho\,\text{d}e = p\,\text{d}\ln\rho + \rho T\,\text{d}s$):

$$\rho \frac{\text{D}e}{\text{D}t} = -p \nabla \cdot \boldsymbol{u} + \mathbf{T} : \nabla \boldsymbol{u} + \mu_0 \eta J^2 - \nabla \cdot \boldsymbol{F}.$$

Sum of these three:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}u^2+\Phi+e)=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot(-\boldsymbol{u}\times\boldsymbol{B}+\eta\nabla\times\boldsymbol{B})-\nabla\cdot(p\boldsymbol{u}-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F})$$

or (using specific enthalpy $w=e+(p/\rho)$)

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F}\right]=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}.$$

Magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right)=-\frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\times\boldsymbol{E}.$$

Total energy:

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)+\frac{B^2}{2\mu_0}\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F}\right]=0.$$

Note that $(\boldsymbol{E}\times\boldsymbol{B})/\mu_0$ is the electromagnetic *Poynting flux*, while $-\mathbf{T}\cdot\boldsymbol{u}$ is the viscous energy flux.

1.4.2. Other conservation laws in ideal MHD

[*Note: The term 'ideal MHD' is usually understood to mean that all dissipative and non-adiabatic effects, including viscosity, resistivity, viscous and Joule heating, and radiative cooling, are neglected.*]

Mass:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0.$$

z-component (e.g.) of momentum (not conserved in an external gravitational field):

$$\frac{\partial}{\partial t}(\rho u_z)+\nabla\cdot\left(\rho u_z\boldsymbol{u}-\frac{B_z\boldsymbol{B}}{\mu_0}+\Pi\,\boldsymbol{e}_z\right)=-\rho\frac{\partial\Phi}{\partial z}.$$

z-component (e.g.) of angular momentum (conserved only in an axisymmetric gravitational field):

$$\frac{\partial}{\partial t}(\rho r u_\phi)+\nabla\cdot\left[\rho r u_\phi\boldsymbol{u}-\frac{rB_\phi\boldsymbol{B}}{\mu_0}+r\Pi\,\boldsymbol{e}_\phi\right]=-\rho\frac{\partial\Phi}{\partial\phi}.$$

In ideal fluid dynamics there are also certain invariants with a geometrical or topological interpretation. In barotropic (homentropic) flow, for example, vorticity (or, equivalently, circulation) is conserved, while, in non-barotropic flow, potential vorticity is conserved. The Lorentz force breaks these conservation laws because the curl of the Lorentz force per unit mass does not vanish in general. However, some new topological invariants associated with the magnetic field appear.

The *magnetic helicity* in a volume V is

$$H_\mathrm{m} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V,$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Now

$$\begin{aligned}\frac{\partial \boldsymbol{A}}{\partial t} &= -\boldsymbol{E} - \nabla \Phi_\mathrm{e} \\ &= \boldsymbol{u} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B} - \nabla \Phi_\mathrm{e},\end{aligned}$$

where Φ_e is the electrostatic potential. This can be thought of as the 'uncurl' of the induction equation. In ideal MHD, therefore, magnetic helicity is conserved:

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) + \nabla \cdot [\Phi_\mathrm{e} \boldsymbol{B} + \boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B})] = 0.$$

However, care is needed because H_m is not uniquely defined unless $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on the surface S of V. Under a gauge transformation $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla \chi$, $\Phi_\mathrm{e} \mapsto \Phi_\mathrm{e} - \partial \chi / \partial t$, H_m changes by an amount

$$\int_V \boldsymbol{B} \cdot \nabla \chi \, \mathrm{d}V = \int_V \nabla \cdot (\chi \boldsymbol{B}) \, \mathrm{d}V = \int_S \chi \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

The *cross-helicity* in a volume V is

$$H_\mathrm{c} = \int_V \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V.$$

It is helpful here to write the equation of motion in ideal MHD in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla(\tfrac{1}{2}u^2 + \Phi + w) + T \nabla s + \frac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B},$$

using the relation $\mathrm{d}w = \rho^{-1} \, \mathrm{d}p + T \, \mathrm{d}s$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u} \cdot \boldsymbol{B}) + \nabla \cdot \left[\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{B}) + (\tfrac{1}{2}u^2 + w + \Phi)\boldsymbol{B}\right] = T \boldsymbol{B} \cdot \nabla s,$$

and so cross-helicity is conserved in ideal MHD in barotropic (homentropic) flow.