Astrophysical Fluid Dynamics — Mathematical Tripos, Part III
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**Provisional synopsis**

- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation. Microphysical basis and validity of a fluid description.
- Physical interpretation of MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media.
- Nonlinear waves, shocks and other discontinuities.
- Spherical blast waves: supernovae.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Waves and instabilities in stratified rotating astrophysical bodies.

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# 1 Ideal gas dynamics and MHD

## 1.1 Review of ideal gas dynamics

### 1.1.1 Fluid variables

A fluid is characterized by a *velocity field* $\boldsymbol{u}(\boldsymbol{x},t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the *pressure* $p(\boldsymbol{x},t)$ and the *mass density* $\rho(\boldsymbol{x},t)$. Other properties, e.g. temperature $T$, can be regarded as functions of $p$ and $\rho$. The *specific volume* (volume per unit mass) is $v = 1/\rho$.

We neglect the possible complications of variable chemical composition associated with ionization and recombination, or chemical or nuclear reactions.

### 1.1.2 Eulerian and Lagrangian viewpoints

In the *Eulerian viewpoint* we consider how fluid properties vary in time at a point that is fixed in space, i.e. attached to the (usually inertial) coordinate system. The *Eulerian time-derivative* is simply

$$\frac{\partial}{\partial t}.$$

In the *Lagrangian viewpoint* we consider how fluid properties vary in time at a point that moves with the fluid. The *Lagrangian time-derivative* is then

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}.$$

### 1.1.3 Material points and structures

A *material point* is an idealized *fluid element*, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x},t)$ of the fluid. (Note that the true particles of which the fluid is composed have an additional random thermal motion.)

*Material curves, surfaces and volumes* are geometrical structures composed of fluid elements and moving (and distorting) with the fluid flow.

An infinitesimal material line element $\delta\boldsymbol{x}$ evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\boldsymbol{\nabla}\boldsymbol{u}.$$

It changes its length and/or orientation in the presence of a velocity gradient.

Infinitesimal material surface and volume elements can be defined from material line elements according to

$$\delta\boldsymbol{S} = \delta\boldsymbol{x}^{(1)}\times\delta\boldsymbol{x}^{(2)},$$

$$\delta V = \delta\boldsymbol{x}^{(1)}\cdot\delta\boldsymbol{x}^{(2)}\times\delta\boldsymbol{x}^{(3)}.$$

They therefore evolve according to (**exercise**)

$$\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} = (\boldsymbol{\nabla}\cdot\boldsymbol{u})\delta\boldsymbol{S} - (\boldsymbol{\nabla}\boldsymbol{u})\cdot\delta\boldsymbol{S},$$

$$\frac{\mathrm{D}\delta V}{\mathrm{D}t} = (\boldsymbol{\nabla}\cdot\boldsymbol{u})\delta V.$$

(See, e.g., Batchelor, *An Introduction to Fluid Dynamics*, chapter 3.) In Cartesian coordinates and suffix notation the equation for $\delta\boldsymbol{S}$ reads

$$\frac{\mathrm{D}\delta S_i}{\mathrm{D}t} = \frac{\partial u_j}{\partial x_j}\,\delta S_i - \frac{\partial u_j}{\partial x_i}\,\delta S_j$$

### 1.1.4 Equation of mass conservation

The equation of mass conservation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0,$$

has the typical form of a conservation law: $\rho$ is the mass density and $\rho\boldsymbol{u}$ is the mass flux density. An alternative form is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}.$$



If $\delta m = \rho\,\delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0.$$

### 1.1.5 Equation of motion

The equation of motion,

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}p,$$

derives from Newton's second law with gravitational and pressure forces. $\Phi(\boldsymbol{x},t)$ is the gravitational potential. Viscous forces are neglected in ideal gas dynamics.

### 1.1.6 Poisson's equation

The gravitational potential is related to the mass density by Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho,$$

where $G$ is Newton's constant. The solution

$$\begin{aligned}\Phi(\boldsymbol{x},t) &= -G\int_V \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}' - G\int_{\hat{V}} \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}'\\ &= \Phi_{\text{int}} + \Phi_{\text{ext}}\end{aligned}$$

generally involves contributions from both inside and outside the fluid region $V$ under consideration.

*Non-self-gravitating* means that (variations in) $\Phi_{\text{int}}$ can be neglected. Then $\Phi(\boldsymbol{x},t)$ is known in advance and Poisson's equation is not coupled to the other equations.

### 1.1.7 Thermal energy equation

In the absence of non-adiabatic heating (e.g. by viscous dissipation or nuclear reactions) and cooling (e.g. by radiation or conduction),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0,$$

where $s$ is the *specific entropy* (entropy per unit mass). Fluid elements undergo reversible thermodynamic changes and preserve their entropy.

This condition is violated in shocks (see later).

The thermal variables $(T, s)$ can be related to the dynamical variables $(p, \rho)$ via an *equation of state* and standard thermodynamic identities. The most important case is that of an *ideal gas together with black-body radiation*,

$$p = p_\mathrm{g} + p_\mathrm{r} = \frac{k\rho T}{\mu m_\mathrm{p}} + \frac{4\sigma T^4}{3c},$$

where $k$ is Boltzmann's constant, $m_\mathrm{p}$ is the mass of the proton and $c$ the speed of light. $\mu$ is the mean molecular weight (the average mass of the particles in units of $m_\mathrm{p}$), equal to 2.0 for molecular hydrogen, 1.0 for atomic hydrogen, 0.5 for fully ionized hydrogen and about 0.6 for ionized matter of typical cosmic abundances. Radiation pressure is usually negligible except in the centres of high-mass stars and in the immediate environments of neutron stars and black holes.

We define the first *adiabatic exponent*

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s,$$

related to the ratio of specific heats $\gamma = c_\mathrm{p}/c_\mathrm{v}$ by (**exercise**)

$$\Gamma_1 = \chi_\rho \gamma,$$

where

$$\chi_\rho = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T$$



can be found from the equation of state. For an ideal gas with negligible radiation pressure, $\chi_\rho = 1$ and so $\Gamma_1 = \gamma$.

We often rewrite the thermal energy equation as

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t},$$

and generally write $\gamma$ for $\Gamma_1$.

### 1.1.8 Simplified models

A *polytropic gas* is an ideal gas with constant $c_\mathrm{v}$, $c_\mathrm{p}$, $\gamma$ and $\mu$. The *polytropic index* $n$ (not generally an integer) is defined by $\gamma = 1 + 1/n$. Equipartition of energy for a classical gas with $N$ degrees of freedom per particle gives $\gamma = 1 + 2/N$. For a classical monatomic gas with $N = 3$ translational degrees of freedom, $\gamma = 5/3$ and $n = 3/2$. In reality $\Gamma_1$ is variable when the gas undergoes ionization or when the gas and radiation pressure are comparable. The specific *internal energy* of a polytropic gas is

$$e = \frac{p}{(\gamma - 1)\rho} \qquad \left[= \frac{N}{\mu m_\mathrm{p}}\tfrac{1}{2}kT\right].$$

A *barotropic fluid* is an idealized situation in which the relation $p(\rho)$ is known in advance. We can then dispense with the thermal energy equation. e.g. if the gas is strictly isothermal and ideal, then $p = c_\mathrm{s}^2\rho$ with $c_\mathrm{s}$ = constant being the isothermal sound speed. Alternatively, if the gas is strictly isentropic and polytropic, then $p = K\rho^\gamma$ with $K$ = constant.

An *incompressible fluid* is an idealized situation in which $\mathrm{D}\rho/\mathrm{D}t = 0$, implying $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. This can be achieved formally by taking the limit $\gamma \to \infty$. The approximation of incompressibility eliminates acoustic phenomena from the dynamics.

The ideal gas law itself is not valid at very high densities or where quantum degeneracy is important.

## 1.2 Elementary derivation of the MHD equations

*Magnetohydrodynamics* (MHD) is the dynamics of an electrically conducting fluid (ionized plasma or liquid metal) containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

### 1.2.1 Induction equation

We consider a non-relativistic theory in which the fluid motions are slow compared to the speed of light. The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are governed by Maxwell's equations without the displacement current,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J},$$

where $\mu_0$ is the permeability of free space and $\boldsymbol{J}$ is the electric current density. The fourth Maxwell equation, involving $\boldsymbol{\nabla} \cdot \boldsymbol{E}$, is not required in a non-relativistic theory. These are sometimes called the *pre-Maxwell equations*.

**Exercise**: Show that these equations are invariant under the *Galilean transformation* to a frame of reference moving with uniform relative velocity $\boldsymbol{v}$,

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t,$$

$$t' = t,$$

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B},$$

$$\boldsymbol{B}' = \boldsymbol{B},$$

$$\boldsymbol{J}' = \boldsymbol{J},$$

as required by a 'non-relativistic' theory. (In fact, this is simply Galilean, rather than Einsteinian, relativity.)



In the *ideal MHD approximation* we regard the fluid as a perfect electrical conductor. The electric field in the rest frame of the fluid vanishes, implying that

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$$

in a frame in which the fluid velocity is $\boldsymbol{u}(\boldsymbol{x}, t)$.

This condition can be regarded as the limit of a constitutive relation such as Ohm's law, in which the effects of resistivity (i.e. finite conductivity) are neglected.

From Maxwell's equations, we then obtain the ideal *induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}).$$

This is an evolutionary equation for $\boldsymbol{B}$ alone, and $\boldsymbol{E}$ and $\boldsymbol{J}$ have been eliminated. The divergence of the induction equation

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) = 0$$

ensures that the solenoidal character of $\boldsymbol{B}$ is preserved.

### 1.2.2 The Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a bulk *Lorentz force*

$$\boldsymbol{F}_{\mathrm{m}} = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. This can be understood as the sum of the Lorentz forces on individual particles,

$$\sum q\boldsymbol{v} \times \boldsymbol{B} = \left(\sum q\boldsymbol{v}\right) \times \boldsymbol{B}.$$

(The electrostatic force can be shown to be negligible in the non-relativistic theory.)

In Cartesian coordinates

$$\begin{aligned}(\mu_0\boldsymbol{F}_\mathrm{m})_i &= \epsilon_{ijk}\left(\epsilon_{jlm}\frac{\partial B_m}{\partial x_l}\right)B_k\\ &= \left(\frac{\partial B_i}{\partial x_k}-\frac{\partial B_k}{\partial x_i}\right)B_k\\ &= B_k\frac{\partial B_i}{\partial x_k}-\frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right).\end{aligned}$$

Thus

$$\boldsymbol{F}_\mathrm{m}=\frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}-\boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right).$$

The first term can be interpreted as a *curvature force* due to a *magnetic tension* $T_\mathrm{m}=B^2/\mu_0$ per unit area in the field lines. The second term is the gradient of an isotropic *magnetic pressure*

$$p_\mathrm{m}=\frac{B^2}{2\mu_0},$$

which is also equal to the energy density of the magnetic field.

The magnetic tension gives rise to *Alfvén waves* (see later), which travel parallel to the field with characteristic speed

$$v_\mathrm{a}=\left(\frac{T_\mathrm{m}}{\rho}\right)^{1/2}=\frac{B}{(\mu_0\rho)^{1/2}},$$

the *Alfvén speed*. This is often considered as a vector Alfvén velocity,

$$\boldsymbol{v}_\mathrm{a}=\frac{\boldsymbol{B}}{(\mu_0\rho)^{1/2}}.$$

The magnetic pressure also affects the propagation of sound waves, which become *magnetoacoustic waves* (see later).

The combination

$$\Pi=p+\frac{B^2}{2\mu_0}$$



is often referred to as the *total pressure*. The ratio

$$\beta=\frac{p}{B^2/2\mu_0}$$

is known as the *plasma beta*.

### 1.2.3 Summary of the MHD equations

A full set of ideal MHD equations might read

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})=0,\\ &\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\rho\boldsymbol{\nabla}\Phi-\boldsymbol{\nabla}p+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B},\\ &\frac{\mathrm{D}s}{\mathrm{D}t}=0,\\ &\frac{\partial\boldsymbol{B}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}),\\ &\boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\end{aligned}$$

together with the equation of state, Poisson's equation, etc., as required. Most of these equations can be written in at least one other way that may be useful in different circumstances.

These equations display the essential *nonlinearity* of MHD. When the velocity field is prescribed, an artifice known as the *kinematic approximation*, the induction equation is a relatively straightforward linear evolutionary equation for the magnetic field. However, a sufficiently strong magnetic field will modify the velocity field through its dynamical effect, the Lorentz force. This nonlinear coupling leads to a rich variety of behaviour. (Of course, the purely hydrodynamic nonlinearity of the $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ term, which is responsible for much of the complexity of fluid dynamics, is still present.)

## 1.3 Microphysical basis

It is useful to understand the way in which the fluid dynamical equations are derived from microphysical considerations. The simplest model involves identical neutral particles with no internal degrees of freedom.

### 1.3.1 The Boltzmann equation

Between collisions, particles follow Hamiltonian trajectories in their six-dimensional $(\boldsymbol{x}, \boldsymbol{v})$ phase space:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = a_i = -\frac{\partial \Phi}{\partial x_i}.$$

The *distribution function* $f(\boldsymbol{x}, \boldsymbol{v}, t)$ specifies the number density of particles in phase space. The velocity moments of $f$ define the number density $n(\boldsymbol{x}, t)$ in real space, the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the velocity dispersion $c(\boldsymbol{x}, t)$ according to

$$\int f \, \mathrm{d}^3\boldsymbol{v} = n,$$

$$\int \boldsymbol{v} f \, \mathrm{d}^3\boldsymbol{v} = n\boldsymbol{u},$$

$$\int |\boldsymbol{v} - \boldsymbol{u}|^2 f \, \mathrm{d}^3\boldsymbol{v} = 3nc^2.$$

Equivalently,

$$\int v^2 f \, \mathrm{d}^3\boldsymbol{v} = n(u^2 + 3c^2).$$

The relation between velocity dispersion and temperature is $kT = mc^2$.

In the absence of collisions, $f$ is conserved following the Hamiltonian flow in phase space. This is because particles are conserved and the



flow in phase space is incompressible (Liouville's theorem). $f$ evolves according to *Boltzmann's equation*

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}.$$

The collision term on the right-hand side is a complicated integral operator but has three simple properties corresponding to the conservation of mass, momentum and energy in collisions:

$$\int m \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v} = 0,$$

$$\int m\boldsymbol{v} \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v} = \boldsymbol{0},$$

$$\int \tfrac{1}{2} m v^2 \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v} = 0.$$

The collision term is strictly local in $\boldsymbol{x}$ (not even involving derivatives) although it involves integrals over $\boldsymbol{v}$. The *Maxwellian distribution*

$$f_{\mathrm{M}} = (2\pi c^2)^{-3/2} n \exp\left(-\frac{|\boldsymbol{v} - \boldsymbol{u}|^2}{2c^2}\right)$$

is the unique solution of $(\partial f_{\mathrm{M}}/\partial t)_{\mathrm{c}} = 0$ and can have any parameters $n$, $\boldsymbol{u}$ and $c$.

### 1.3.2 Derivation of fluid equations

A crude but illuminating model of the collision operator is the *BGK approximation*

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \approx -\frac{1}{\tau}(f - f_{\mathrm{M}})$$

where $f_{\mathrm{M}}$ is a Maxwellian with the same $n$, $\boldsymbol{u}$ and $c$ as $f$, and $\tau$ is the *relaxation time*. This can be identified with the mean free flight time of particles between collisions. In other words the collisions attempt to restore a Maxwellian distribution on a characteristic time-scale $\tau$. They do this by randomizing the particle velocities in a way consistent with the conservation of momentum and energy.

If the characteristic time-scale of the fluid flow is $T \gg \tau$, then the collision term dominates the Boltzmann equation and $f$ must be very close to $f_{\mathrm{M}}$. This is the *hydrodynamic limit.*

The velocity moments of $f_{\mathrm{M}}$ can be determined from standard Gaussian integrals, in particular (**exercise**)

$$\int f_{\mathrm{M}}\,\mathrm{d}^3\boldsymbol{v} = n,$$

$$\int v_i f_{\mathrm{M}}\,\mathrm{d}^3\boldsymbol{v} = nu_i,$$

$$\int v_i v_j f_{\mathrm{M}}\,\mathrm{d}^3\boldsymbol{v} = n(u_i u_j + c^2\delta_{ij}),$$

$$\int v^2 v_i f_{\mathrm{M}}\,\mathrm{d}^3\boldsymbol{v} = n(u^2 + 5c^2)u_i.$$

We obtain equations for mass, momentum and energy by taking moments of the Boltzmann equation weighted by $(m, mv_i, \frac{1}{2}mv^2)$. In each case the collision term integrates to zero and the $\partial/\partial v_j$ term can be integrated by parts. We replace $f$ with $f_{\mathrm{M}}$ when evaluating the left-hand sides and note that $mn = \rho$:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0,$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left[\rho(u_i u_j + c^2\delta_{ij})\right] - \rho a_i = 0,$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}\rho c^2\right) + \frac{\partial}{\partial x_i}\left[(\tfrac{1}{2}\rho u^2 + \tfrac{5}{2}\rho c^2)u_i\right] - \rho u_i a_i = 0.$$

These are equivalent to the equations of ideal gas dynamics in conservative form (see later) for a monatomic ideal gas ($\gamma = 5/3$). The specific internal energy is $e = \frac{3}{2}c^2 = \frac{3}{2}kT/m$.



### 1.3.3 Validity of a fluid approach

The basic idea here is that deviations from the Maxwellian distribution are small because collisions are frequent compared to the characteristic time-scale of the flow. In higher-order approximations these deviations can be estimated, leading to the equations of *dissipative gas dynamics* including *transport effects* (viscosity and heat conduction).

The fluid approach breaks down if $\tau$ is not $\ll T$, or if the mean free path $\lambda \approx c\tau$ between collisions is not $\ll$ the characteristic length-scale $L$ of the flow. $\lambda$ can be very long (measured in AU or pc) in very tenuous gases such as the interstellar medium, but may still be smaller than the size of the system.

This approach can be generalized to deal with molecules with internal degrees of freedom and also to plasmas or partially ionized gases where there are various species of particle with different charges and masses experiencing electromagnetic forces. The equations of MHD can be derived using a similar method.

Some typical numbers:

Solar-type star: centre $\rho \sim 10^2\,\mathrm{g\,cm^{-3}}$, $T \sim 10^7\,\mathrm{K}$; photosphere $\rho \sim 10^{-7}\,\mathrm{g\,cm^{-3}}$, $T \sim 10^4\,\mathrm{K}$; corona $\rho \sim 10^{-15}\,\mathrm{g\,cm^{-3}}$, $T \sim 10^6\,\mathrm{K}$.

Interstellar medium: molecular clouds $n \sim 10^3\,\mathrm{cm^{-3}}$, $T \sim 10\,\mathrm{K}$; cold medium (neutral) $n \sim 10 - 100\,\mathrm{cm^{-3}}$, $T \sim 10^2\,K$; warm medium (neutral/ionized) $n \sim 0.1 - 1\,\mathrm{cm^{-3}}$, $T \sim 10^4\,K$; hot medium (ionized) $n \sim 10^{-3} - 10^{-2}\,\mathrm{cm^{-3}}$, $T \sim 10^6\,K$.

The Coulomb cross-section for 'collisions' between charged particles (electrons or ions) is $\sigma \approx 1 \times 10^{-4}(T/\mathrm{K})^{-2}\,\mathrm{cm^2}$. The mean free path is $\lambda = 1/(n\sigma)$.

# 2 Physical interpretation of MHD

There are two aspects to MHD: the advection of $\boldsymbol{B}$ by $\boldsymbol{u}$ (induction equation) and the dynamical back-reaction of $\boldsymbol{B}$ on $\boldsymbol{u}$ (Lorentz force).

## 2.1 Kinematics of the magnetic field

The ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})$$

has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material curves. This is sometimes known as *Alfvén's theorem*.

One way to show this is to use the identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{B} + \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{B})$$

to write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{u}),$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right) \cdot \boldsymbol{\nabla} \boldsymbol{u}.$$

This is exactly the same equation satisfied by a material line element $\delta \boldsymbol{x}$. Therefore a magnetic field line (an integral curve of $\boldsymbol{B}/\rho$) is advected and distorted by the fluid in the same way as a material curve.



Precisely the same equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}),$$

is satisfied by the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ in homentropic/barotropic ideal fluid dynamics in the absence of a magnetic field. However, the fact that $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by the curl operation means that the analogy between vorticity dynamics and MHD is not perfect.

Another way to demonstrate the result of flux freezing is to represent the magnetic field using *Euler potentials* $\alpha$ and $\beta$,

$$\boldsymbol{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta.$$

This is sometimes called a *Clebsch representation*. By using two scalar potentials we are able to represent a three-dimensional vector field satisfying the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. A vector potential of the form $\boldsymbol{A} = \alpha \boldsymbol{\nabla}\beta + \boldsymbol{\nabla}\gamma$ generates this magnetic field via $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. The magnetic field lines are the intersections of the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$.

After some algebra it can be shown that

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{\nabla}\left(\frac{\mathrm{D}\alpha}{\mathrm{D}t}\right) \times \boldsymbol{\nabla}\beta + \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\left(\frac{\mathrm{D}\beta}{\mathrm{D}t}\right).$$

The ideal induction equation is therefore satisfied if the Euler potentials are conserved following the fluid flow, i.e. if the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$ are material surfaces. In this case the magnetic field lines can also be identified with material lines.

Yet another viewpoint is that the magnetic flux $\delta\Phi = \boldsymbol{B} \cdot \delta\boldsymbol{S}$ through a material surface element is conserved:

$$\begin{aligned}
\frac{\mathrm{D}\delta\Phi}{\mathrm{D}t} &= \frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} \cdot \delta\boldsymbol{S} + \boldsymbol{B} \cdot \frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} \\
&= \left(B_j \frac{\partial u_i}{\partial x_j} - B_i \frac{\partial u_j}{\partial x_j}\right) \delta S_i + B_i \left(\frac{\partial u_j}{\partial x_j}\, \delta S_i - \frac{\partial u_j}{\partial x_i}\, \delta S_j\right) \\
&= 0.
\end{aligned}$$

By extension, we have conservation of the magnetic flux passing through any material surface.

## 2.2 The Lorentz force

The Lorentz force

$$\boldsymbol{F}_\mathrm{m} = \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right)$$

can also be written as the divergence of the *Maxwell stress tensor*:

$$\boldsymbol{F}_\mathrm{m} = \boldsymbol{\nabla}\cdot\mathbf{M}, \qquad \mathbf{M} = \frac{1}{\mu_0}\left(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\mathbf{1}\right).$$

In Cartesian coordinates

$$(\boldsymbol{F}_\mathrm{m})_i = \frac{\partial M_{ji}}{\partial x_j}, \qquad M_{ij} = \frac{1}{\mu_0}\left(B_iB_j - \frac{B^2}{2}\delta_{ij}\right).$$

If the magnetic field is locally aligned with the $x$-axis, then

$$\mathbf{M} = \begin{bmatrix} T_\mathrm{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} p_\mathrm{m} & 0 & 0 \\ 0 & p_\mathrm{m} & 0 \\ 0 & 0 & p_\mathrm{m} \end{bmatrix},$$

showing the magnetic tension and pressure.

Combining the ideas of magnetic tension and a frozen-in field leads to the picture of field lines as elastic strings embedded in the fluid. Indeed there is a close analogy between MHD and the dynamics of dilute solutions of long-chain polymer molecules.

## 2.3 Differential rotation and torsional Alfvén waves

We first consider the kinematic behaviour of a magnetic field in the presence of a prescribed velocity field involving *differential rotation*. In cylindrical polar coordinates $(R, \phi, z)$, let

$$\boldsymbol{u} = R\Omega(R, z)\,\boldsymbol{e}_\phi.$$



Consider an axisymmetric magnetic field, which we separate into *poloidal* (meridional) and *toroidal* (azimuthal) parts:

$$\boldsymbol{B} = \boldsymbol{B}_\mathrm{p}(R, z, t) + B_\phi(R, z, t)\,\boldsymbol{e}_\phi.$$

The ideal induction equation reduces to (**exercise**)

$$\frac{\partial \boldsymbol{B}_\mathrm{p}}{\partial t} = 0,$$

$$\frac{\partial B_\phi}{\partial t} = R\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\Omega.$$

Differential rotation winds the poloidal field to generate a toroidal field. To obtain an steady state without winding, we require

$$\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\Omega = 0,$$

known as *Ferraro's law of isorotation.*

There is an energetic cost to winding the field, as work is done against magnetic tension. In a dynamical situation a strong magnetic field tends to enforce isorotation along its length.

We now generalize the analysis to allow for axisymmetric *torsional oscillations*:

$$\boldsymbol{u} = R\Omega(R, z, t)\,\boldsymbol{e}_\phi.$$

The azimuthal component of equation of motion is (**exercise**)

$$\rho R\frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0 R}\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}(RB_\phi).$$

This combines with the induction equation to give

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\mu_0\rho R^2}\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}(R^2\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\Omega).$$

This equation describes *torsional Alfvén waves.* e.g. if $\boldsymbol{B}_\mathrm{p} = B_z\,\boldsymbol{e}_z$ is vertical and uniform, then

$$\frac{\partial^2 \Omega^2}{\partial t^2} = v_\mathrm{a}^2\frac{\partial^2 \Omega^2}{\partial z^2}.$$

This is not strictly an exact nonlinear analysis because we have neglected the force balance (and indeed motion) in the meridional plane.

## 2.4 Magnetostatic equilibrium

A *magnetostatic equilibrium* is a static solution ($\boldsymbol{u} = \boldsymbol{0}$) of the equation of motion, i.e. one satisfying

$$\boldsymbol{0} = -\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}p + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B},$$

together with $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$.

In regions of low density, such as the solar corona, the magnetic field may be dynamically dominant over the effects of gravity or gas pressure. Under these circumstances we have (approximately) a *force-free magnetic field* such that

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{0}.$$

Magnetic fields $\boldsymbol{B}$ satisfying this equation are known in a wider mathematical context as *Beltrami fields*. Since $\boldsymbol{\nabla}\times\boldsymbol{B}$ must be parallel to $\boldsymbol{B}$, we may write

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \lambda\boldsymbol{B}, \tag{1}$$

for some scalar field $\lambda(\boldsymbol{x})$. The divergence of this equation is

$$0 = \boldsymbol{B}\cdot\boldsymbol{\nabla}\lambda,$$

so that $\lambda$ is constant on each magnetic field line. In the special case $\lambda =$ constant, known as a *linear force-free magnetic field*, the curl of equation (1) results in the Helmholtz equation

$$-\nabla^2\boldsymbol{B} = \lambda^2\boldsymbol{B},$$

which admits a wide variety of non-trivial solutions.

A subset of force-free magnetic fields consists of *potential* or *current-free* magnetic fields for which

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{0}.$$



In a true vacuum, the magnetic field must be potential. However, only an extremely low density of charge carriers (i.e. electrons) is needed to make the force-free description more relevant.

An example of a force-free field is

$$\boldsymbol{B} = B_\phi(R)\,\boldsymbol{e}_\phi + B_z(R)\,\boldsymbol{e}_z,$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = -\frac{\mathrm{d}B_z}{\mathrm{d}R}\,\boldsymbol{e}_\phi + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}(RB_\phi)\,\boldsymbol{e}_z.$$

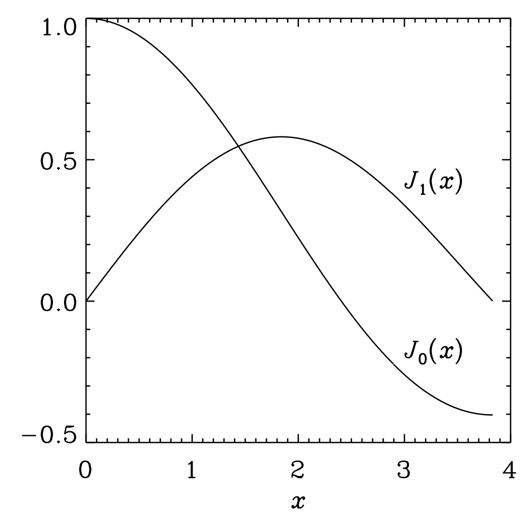
Now $\boldsymbol{\nabla}\times\boldsymbol{B} = \lambda\boldsymbol{B}$ implies

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}B_z}{\mathrm{d}R}\right) + \lambda^2 B_z = 0.$$

The solution regular at $R = 0$ is

$$B_z = B_0 J_0(\lambda R), \qquad B_\phi = B_0 J_1(\lambda R),$$

where $J_n$ is the Bessel function of order $n$. This solution can be matched smoothly to a uniform exterior field at a zero of $J_1$.




The helical nature of this field is typical of force-free fields with $\lambda \neq 0$.

### 2.5 Magnetic buoyancy

A *magnetic flux tube* is an idealized situation in which the field is localized in a tube and vanishes outside. To balance the total pressure at the interface, the gas pressure must be lower inside. Unless the temperatures are different, the density is lower inside. In a gravitational field the tube therefore experiences an upward buoyancy force and tends to rise.



## 3 Conservation laws, symmetries and hyperbolic structure

### 3.1 Synthesis of the total energy equation

Starting from the ideal MHD equations, we construct the total energy equation piece by piece.

Kinetic energy:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}u^2) = -\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi - \boldsymbol{u}\cdot\boldsymbol{\nabla}p + \frac{1}{\mu_0}\boldsymbol{u}\cdot[(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]\,.$$

Gravitational energy (assuming first that the system is non-self-gravitating and $\Phi$ is independent of $t$):

$$\rho\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi.$$

Thermal energy (using the thermodynamic identity $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}v$):

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = \rho T\frac{\mathrm{D}s}{\mathrm{D}t} + p\frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -p\boldsymbol{\nabla}\cdot\boldsymbol{u}.$$

Sum of these three:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}u^2+\Phi+e) = -\boldsymbol{\nabla}\cdot(p\boldsymbol{u}) + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\cdot(-\boldsymbol{u}\times\boldsymbol{B}).$$

Using mass conservation:

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)\right] + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+e)+p\boldsymbol{u}\right] = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\boldsymbol{E}.$$

Magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) = -\frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}.$$

Total energy:

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)+\frac{B^2}{2\mu_0}\right] + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0,$$

where $w = e + p/\rho$ is the *specific enthalpy*. Note that $(\boldsymbol{E}\times\boldsymbol{B})/\mu_0$ is the electromagnetic *Poynting flux*. The total energy is therefore conserved.

To allow for self-gravitation we write $\Phi = \Phi_{\text{int}} + \Phi_{\text{ext}}$. Now

$$\begin{aligned}\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}\Phi_{\text{int}}) &= -\tfrac{1}{2}G\rho\frac{\mathrm{D}}{\mathrm{D}t}\int\frac{\mathrm{d}m'}{|\boldsymbol{x}'-\boldsymbol{x}|}\\ &= \tfrac{1}{2}G\rho\int\frac{(\boldsymbol{u}'-\boldsymbol{u})\cdot(\boldsymbol{x}'-\boldsymbol{x})}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\,\mathrm{d}m'\\ &= \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi_{\text{int}} + \tfrac{1}{2}G\rho\int\frac{(\boldsymbol{u}'+\boldsymbol{u})\cdot(\boldsymbol{x}'-\boldsymbol{x})}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\,\mathrm{d}m'.\end{aligned}$$

The volume integral of the last term vanishes because it is antisymmetric:

$$\tfrac{1}{2}G\iint\frac{(\boldsymbol{u}'+\boldsymbol{u})\cdot(\boldsymbol{x}'-\boldsymbol{x})}{|\boldsymbol{x}'-\boldsymbol{x}|^3}\,\mathrm{d}m'\,\mathrm{d}m = 0.$$

The final conservation equation is therefore non-local:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2 + \tfrac{1}{2}\Phi_{\text{int}} + \Phi_{\text{ext}} + e) + \frac{B^2}{2\mu_0}\right]\\ &\quad + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2 + \Phi + w) + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]\\ &\quad + (\text{term that integrates to zero}) = 0.\end{aligned}$$

## 3.2 Other conservation laws in ideal MHD

In ideal fluid dynamics there are certain invariants with a geometrical or topological interpretation. In homentropic/barotropic flow, for example, vorticity (or, equivalently, circulation) and kinetic helicity are conserved, while, in non-barotropic flow, potential vorticity is conserved. The Lorentz force breaks these conservation laws because the curl of the Lorentz force per unit mass does not vanish in general. However, some new topological invariants associated with the magnetic field appear.



The *magnetic helicity* in a volume $V$ is

$$H_{\mathrm{m}} = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V,$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. Now

$$\frac{\partial\boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \boldsymbol{\nabla}\Phi_{\mathrm{e}} = \boldsymbol{u}\times\boldsymbol{B} - \boldsymbol{\nabla}\Phi_{\mathrm{e}},$$

where $\Phi_{\mathrm{e}}$ is the electrostatic potential. This can be thought of as the 'uncurl' of the induction equation. In ideal MHD, therefore, magnetic helicity is conserved:

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) + \boldsymbol{\nabla}\cdot[\Phi_{\mathrm{e}}\boldsymbol{B} + \boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})] = 0.$$

However, care is needed because $H_{\mathrm{m}}$ is not uniquely defined unless $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on the surface $S$ of $V$. Under a gauge transformation $\boldsymbol{A}\mapsto\boldsymbol{A}+\boldsymbol{\nabla}\chi$, $\Phi_{\mathrm{e}}\mapsto\Phi_{\mathrm{e}} - \partial\chi/\partial t$, $H_{\mathrm{m}}$ changes by an amount

$$\int_V \boldsymbol{B}\cdot\boldsymbol{\nabla}\chi\,\mathrm{d}V = \int_V \boldsymbol{\nabla}\cdot(\chi\boldsymbol{B})\,\mathrm{d}V = \int_S \chi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

Magnetic helicity is a *pseudoscalar* quantity (it changes sign under a reflection of the spatial coordinates). It is related to the lack of reflectional symmetry in the magnetic field. It can also be interpreted topologically in terms of the twistedness and/or knottedness of the magnetic field (see Example 1.5). Since the field is 'frozen in' to the fluid and deformed continuously by it, the topological properties of the field are conserved. The equivalent conserved quantity in ideal gas dynamics (without a magnetic field) is the *kinetic helicity*

$$H_{\mathrm{k}} = \int_V \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\,\mathrm{d}V.$$

The *cross-helicity* in a volume $V$ is

$$H_{\mathrm{c}} = \int_V \boldsymbol{u}\cdot\boldsymbol{B}\,\mathrm{d}V.$$

It is helpful here to write the equation of motion in ideal MHD in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u} + \boldsymbol{\nabla}(\tfrac{1}{2}u^2 + \Phi + w) = T\boldsymbol{\nabla}s + \frac{1}{\mu_0\rho}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \tag{1}$$

using the relation $\mathrm{d}w = T\,\mathrm{d}s + v\,\mathrm{d}p$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{B}) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{B}) + (\tfrac{1}{2}u^2 + w + \Phi)\boldsymbol{B}\right] = T\boldsymbol{B}\cdot\boldsymbol{\nabla}s,$$

and so cross-helicity is conserved in ideal MHD in homentropic/barotropic flow.

*Bernoulli's theorem* follows from the inner product of equation (1) with $\boldsymbol{u}$. In steady flow

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}(\tfrac{1}{2}u^2 + \Phi + w) = 0,$$

but only if $\boldsymbol{u}\cdot\boldsymbol{F}_{\mathrm{m}} = 0$ (e.g. if $\boldsymbol{u}\parallel\boldsymbol{B}$), i.e. if $\boldsymbol{B}$ does no work on the flow.

## 3.3 Symmetries of the equations

- translation of space and time, and rotation of space (if $\Phi_{\mathrm{ext}}$ has those symmetries): related to conservation of momentum, energy and angular momentum
- reversal of time: related to absence of dissipation
- reflection of space (but note that $\boldsymbol{B}$ is a *pseudovector* and does not change sign)
- Galilean invariance (relativity principle)
- reversal of the sign of $\boldsymbol{B}$



## 3.4 Hyperbolic structure

We neglect the gravitational force here, since it involves action at a distance.

The equation of mass conservation, the thermal energy equation, the equation of motion and the induction equation can be written in the combined form

$$\frac{\partial \mathsf{U}}{\partial t} + \mathsf{A}_i\frac{\partial \mathsf{U}}{\partial x_i} = 0,$$

where

$$\mathsf{U} = [\rho, p, \boldsymbol{u}, \boldsymbol{B}]^{\mathrm{T}}$$

is an eight-dimensional state vector and the $\mathsf{A}_i$ are three $8\times 8$ matrices, e.g.

$$\mathsf{A}_x = \begin{bmatrix} u_x & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u_x & \gamma p & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & u_x & 0 & 0 & 0 & \frac{B_y}{\mu_0\rho} & \frac{B_z}{\mu_0\rho} \\ 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_x}{\mu_0\rho} & 0 \\ 0 & 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_x}{\mu_0\rho} \\ 0 & 0 & 0 & 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & B_y & -B_x & 0 & 0 & u_x & 0 \\ 0 & 0 & B_z & 0 & -B_x & 0 & 0 & u_x \end{bmatrix}.$$

The system of equations is said to be *hyperbolic* if the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector $\boldsymbol{n}$ and if the eigenvectors span the eight-dimensional space. The eigenvalues can be identified as wave speeds, and the eigenvectors as wave modes. $\boldsymbol{n}$ is the local normal to the wavefronts.

Taking $\boldsymbol{n} = \boldsymbol{e}_x$ WLOG, we find

$$\det(\mathsf{A}_x - v\mathsf{1}) = (v-u_x)^2\left[(v-u_x)^2 - v_{\mathrm{a}x}^2\right] \times\left[(v-u_x)^4 - (v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)(v-u_x)^2 + v_{\mathrm{s}}^2 v_{\mathrm{a}x}^2\right],$$

where

$$v_{\mathrm{s}} = \left(\frac{\gamma p}{\rho}\right)^{1/2}$$

is the *adiabatic sound speed*. The wave speeds $v$ are always real and the system is indeed hyperbolic. The various wave modes will be examined later.

For ideal gas dynamics without a magnetic field, the state vector is five-dimensional and the wave speeds for $\boldsymbol{n} = \boldsymbol{e}_x$ are $u_x$ and $u_x \pm v_{\mathrm{s}}$.

In this representation, there are two modes that propagate at the fluid velocity, so they do not propagate relative to the fluid. One is the *entropy mode*, which is physical but involves only a density perturbation. The other is the '$\boldsymbol{\nabla} \cdot \boldsymbol{B}$' mode, which is unphysical and involves a perturbation of $\boldsymbol{\nabla} \cdot \boldsymbol{B}$. This is eliminated by imposing the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$.

## 3.5 Stress tensor and virial theorem

The equation of motion can be written in the form

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{\nabla} \cdot \mathbf{T},$$

where

$$\mathbf{T} = -p\,\mathbf{1} - \frac{1}{4\pi G}\left(\boldsymbol{g}\boldsymbol{g} - \frac{g^2}{2}\mathbf{1}\right) + \frac{1}{\mu_0}\left(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\mathbf{1}\right)$$

is a symmetric stress tensor and $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$. The idea of gravitational stress only works if the system is self-gravitating and $\Phi$ and $\rho$ are related through Poisson's equation:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{g}\boldsymbol{g} - \tfrac{1}{2}g^2\mathbf{1}) = (\boldsymbol{\nabla} \cdot \boldsymbol{g})\boldsymbol{g} = -4\pi G\rho\boldsymbol{g}$$

Consider

$$\rho \frac{\mathrm{D}^2}{\mathrm{D}t^2}(x_i x_j) = \rho \frac{\mathrm{D}}{\mathrm{D}t}(u_i x_j + x_i u_j) = 2\rho u_i u_j + x_j \frac{\partial T_{ki}}{\partial x_k} + x_i \frac{\partial T_{kj}}{\partial x_k}$$



Integrate over a material volume $V$ bounded by a surface $S$ (mass element $\mathrm{d}m = \rho\,\mathrm{d}V$):

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V x_i x_j\,\mathrm{d}m &= \int_V \left(2\rho u_i u_j + x_j \frac{\partial T_{ki}}{\partial x_k} + x_i \frac{\partial T_{kj}}{\partial x_k}\right)\mathrm{d}V \\ &= \int_V (2\rho u_i u_j - T_{ji} - T_{ij})\,\mathrm{d}V + \int_S (x_j T_{ki} + x_i T_{kj}) n_k\,\mathrm{d}S\end{aligned}$$

If the surface term vanishes (e.g. if $T_{ij}$ falls off faster than $r^{-3}$ and we let $V$ occupy the whole of space) and $T_{ij}$ is symmetric, we obtain the *tensor virial theorem*

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = 2K_{ij} - \mathcal{T}_{ij},$$

where

$$I_{ij} = \int x_i x_j\,\mathrm{d}m,$$

$$K_{ij} = \int \tfrac{1}{2} u_i u_j\,\mathrm{d}m,$$

$$\mathcal{T}_{ij} = \int T_{ij}\,\mathrm{d}V.$$

The *scalar virial theorem* is the trace of this expression:

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K - \mathcal{T}.$$

$K$ is the total kinetic energy. Now

$$\begin{aligned}-\mathcal{T} &= \int \left(3p - \frac{g^2}{8\pi G} + \frac{B^2}{2\mu_0}\right)\mathrm{d}V \\ &= 3(\gamma - 1)U + W + M,\end{aligned}$$

for a polytropic gas with no external gravitational field, where $U$, $W$ and $M$ are the total internal, gravitational and magnetic energies. The gravitational integral is

$$-\int_V \frac{g^2}{8\pi G}\,\mathrm{d}V = -\int_V \frac{|\boldsymbol{\nabla}\Phi|^2}{8\pi G}\,\mathrm{d}V = \int_V \frac{\Phi\nabla^2\Phi}{8\pi G}\,\mathrm{d}V = \frac{1}{2}\int_V \rho\Phi\,\mathrm{d}V.$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + 3(\gamma - 1)U + W + M.$$

On the RHS, only $W$ is negative. For the system to be bound (i.e. not fly apart) the kinetic, internal and magnetic energies are limited by

$$2K + 3(\gamma - 1)U + M \leqslant |W|.$$

The tensor virial theorem provides more specific information relating to the energies associated with individual directions.



# 4 Linear waves in homogeneous media

In ideal MHD the density, pressure and magnetic field evolve according to

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla}\rho - \rho \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

$$\frac{\partial p}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla}p - \gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}).$$

Consider a magnetostatic equilibrium in which the density, pressure and magnetic field are $\rho_0(\boldsymbol{x})$, $p_0(\boldsymbol{x})$ and $\boldsymbol{B}_0(\boldsymbol{x})$. Now consider small perturbations from equilibrium, such that $\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}) + \delta\rho(\boldsymbol{x}, t)$ with $|\delta\rho| \ll \rho_0$, etc. The linearized equations are

$$\frac{\partial \delta\rho}{\partial t} = -\delta\boldsymbol{u} \cdot \boldsymbol{\nabla}\rho_0 - \rho_0 \boldsymbol{\nabla} \cdot \delta\boldsymbol{u},$$

$$\frac{\partial \delta p}{\partial t} = -\delta\boldsymbol{u} \cdot \boldsymbol{\nabla}p_0 - \gamma p_0 \boldsymbol{\nabla} \cdot \delta\boldsymbol{u},$$

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta\boldsymbol{u} \times \boldsymbol{B}_0).$$

By introducing the *displacement* $\boldsymbol{\xi}(\boldsymbol{x}, t)$ such that $\delta\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$, we can integrate these equations to obtain

$$\delta\rho = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\rho - \rho \boldsymbol{\nabla} \cdot \boldsymbol{\xi},$$

$$\delta p = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla}p - \gamma p \boldsymbol{\nabla} \cdot \boldsymbol{\xi},$$

$$\begin{aligned}\delta\boldsymbol{B} &= \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) \\ &= \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\boldsymbol{B} - (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})\boldsymbol{B}.\end{aligned}$$

We have now dropped the subscript '0' without danger of confusion. We have also eliminated the entropy mode, which would consist in this case of a time-independent perturbation of the density distribution.

The linearized equation of motion is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\delta\rho\boldsymbol{\nabla}\Phi - \rho\boldsymbol{\nabla}\delta\Phi - \boldsymbol{\nabla}\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\boldsymbol{B}),$$

where the perturbation of total pressure is

$$\begin{aligned}\delta\Pi &= \delta p + \frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{\mu_0}\\ &= -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}).\end{aligned}$$

The gravitational potential perturbation satisfies the linearized Poisson equation

$$\nabla^2\delta\Phi = 4\pi G\delta\rho.$$

We consider a basic state of uniform density, pressure and magnetic field, in the absence of gravity. Such a system is homogeneous but anisotropic, because the uniform field distinguishes a particular direction. The problem simplifies to

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla}\delta\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\left[\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\boldsymbol{B}\right],$$

with

$$\delta\Pi = -\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}).$$

Owing to the symmetries of the basic state, plane-wave solutions exist of the form

$$\boldsymbol{\xi}(\boldsymbol{x},t) = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\right],$$

where $\omega$ and $\boldsymbol{k}$ are the frequency and wavevector, and $\tilde{\boldsymbol{\xi}}$ is a constant vector representing the amplitude of the wave. Then (omitting the tilde and changing the sign)

$$\begin{aligned}\rho\omega^2\boldsymbol{\xi} = &\left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]\boldsymbol{k}\\ &+ \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\left[(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{\xi} - (\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{B}\right].\end{aligned} \tag{1}$$

For transverse displacements that are orthogonal to both the wavevector and the field, i.e. $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$, this simplifies to

$$\rho\omega^2\boldsymbol{\xi} = \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2\boldsymbol{\xi}.$$

Such solutions are called *Alfvén waves*. Their *dispersion relation* is

$$\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{v}_\mathrm{a})^2.$$

Given the dispersion relation $\omega(\boldsymbol{k})$ of any wave mode, the *phase and group velocities* of the wave can be identified as

$$\boldsymbol{v}_\mathrm{p} = \frac{\omega}{k}\hat{\boldsymbol{k}},$$

$$\boldsymbol{v}_\mathrm{g} = \frac{\partial\omega}{\partial\boldsymbol{k}} = \boldsymbol{\nabla}_{\boldsymbol{k}}\omega,$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. The phase velocity is that with which the phase of the wave travels; the group velocity is that which the energy of the wave (or the centre of a wavepacket) travels.

For Alfvén waves, therefore,

$$\boldsymbol{v}_\mathrm{p} = \pm v_\mathrm{a}\cos\theta\,\hat{\boldsymbol{k}},$$

$$\boldsymbol{v}_\mathrm{g} = \pm\boldsymbol{v}_\mathrm{a},$$

where $\theta$ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$.

To find the other solutions, we take the inner product of equation (1) with $\boldsymbol{k}$ and then with $\boldsymbol{B}$ to obtain first

$$\rho\omega^2\boldsymbol{k}\cdot\boldsymbol{\xi} = \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]k^2$$

and then

$$\rho\omega^2 \boldsymbol{B}\cdot\boldsymbol{\xi} = \gamma p(\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{k}\cdot\boldsymbol{B}.$$

These can be written in the form

$$\begin{bmatrix} \rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2 & \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})k^2 \\ -\gamma p(\boldsymbol{k}\cdot\boldsymbol{B}) & \rho\omega^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{k}\cdot\boldsymbol{\xi} \\ \boldsymbol{B}\cdot\boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The 'trivial solution' $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ corresponds to the Alfvén wave that we have already identified. The other solutions satisfy

$$\rho\omega^2 \left[\rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2\right] + \gamma p k^2 \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2 = 0,$$

which simplifies to

$$v_{\mathrm{p}}^4 - (v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)v_{\mathrm{p}}^2 + v_{\mathrm{s}}^2 v_{\mathrm{a}}^2 \cos^2\theta = 0.$$

The two solutions

$$v_{\mathrm{p}}^2 = \tfrac{1}{2}(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2) \pm \left[\tfrac{1}{4}(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)^2 - v_{\mathrm{s}}^2 v_{\mathrm{a}}^2 \cos^2\theta\right]^{1/2}$$

are called *fast and slow magnetoacoustic waves*, respectively.

In the special case $\theta = 0$ ($\boldsymbol{k}\|\boldsymbol{B}$), we have

$$v_{\mathrm{p}}^2 = v_{\mathrm{s}}^2 \quad \text{or} \quad v_{\mathrm{a}}^2,$$

together with $v_{\mathrm{p}}^2 = v_{\mathrm{a}}^2$ for the Alfvén wave. Note that the fast wave could be either $v_{\mathrm{p}}^2 = v_{\mathrm{s}}^2$ or $v_{\mathrm{p}}^2 = v_{\mathrm{a}}^2$, whichever is greater.

In the special case $\theta = \pi/2$ ($\boldsymbol{k} \perp \boldsymbol{B}$), we have

$$v_{\mathrm{p}}^2 = v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2 \quad \text{or} \quad 0,$$

together with $v_{\mathrm{p}}^2 = 0$ for the Alfvén wave.

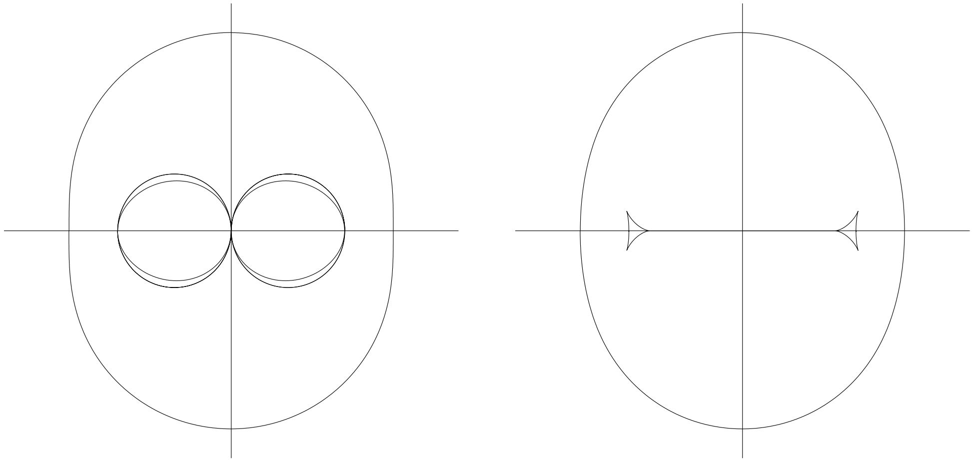
The effects of the magnetic field on wave propagation can be understood as resulting from the two aspects of the Lorentz force. The magnetic



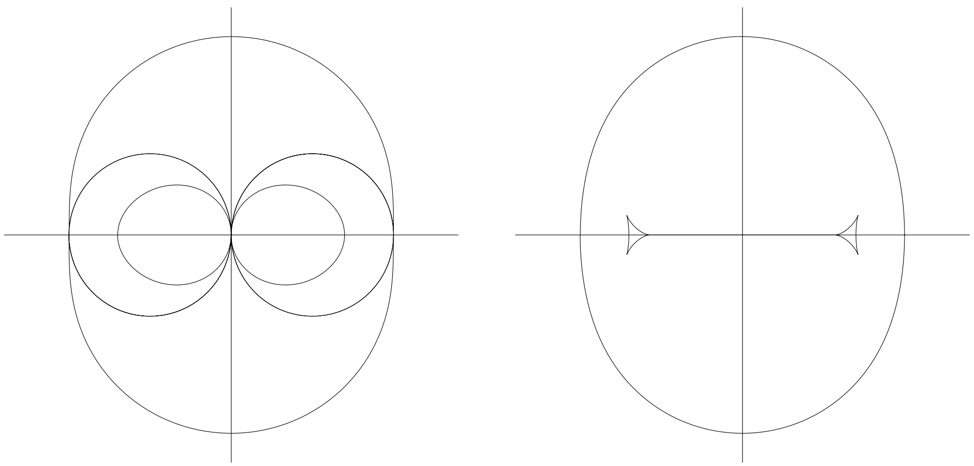
tension gives rise to Alfvén waves, which are similar to waves on an elastic string, and are trivial in the absence of the magnetic field. In addition, the magnetic pressure affects the response of the fluid to compression, and therefore modifies the propagation of acoustic waves.

The phase and group velocity vectors for the full range of $\theta$ are usually exhibited in *Friedrichs diagrams*. We can interpret:

- the fast wave as a quasi-isotropic acoustic-type wave in which both gas and magnetic pressure contribute
- the slow wave as an acoustic-type wave that is strongly guided by the magnetic field
- the Alfvén waves as analogous to a wave on an elastic string, propagating by means of magnetic tension and perfectly guided by the magnetic field



Phase and group velocity diagrams for the case $v_{\mathrm{a}} = 0.7 v_{\mathrm{s}}$.



Phase and group velocity diagrams for the case $v_{\mathrm{s}} = 0.7 v_{\mathrm{a}}$.



# 5 Nonlinear waves, shocks and other discontinuities

## 5.1 One-dimensional gas dynamics

### 5.1.1 Riemann's analysis

The equations of mass conservation and motion in one dimension are

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x},$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

We assume the gas is homentropic ($s = \text{constant}$) and polytropic. Then $p \propto \rho^\gamma$ and $v_{\mathrm{s}}^2 = \gamma p/\rho \propto \rho^{\gamma-1}$. It is convenient to use $v_{\mathrm{s}}$ as a variable in place of $\rho$ or $p$:

$$\mathrm{d}p = v_{\mathrm{s}}^2\,\mathrm{d}\rho, \qquad \mathrm{d}\rho = \frac{\rho}{v_{\mathrm{s}}}\left(\frac{2\,\mathrm{d}v_{\mathrm{s}}}{\gamma-1}\right).$$

Then

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v_{\mathrm{s}}\frac{\partial}{\partial x}\left(\frac{2v_{\mathrm{s}}}{\gamma-1}\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{2v_{\mathrm{s}}}{\gamma-1}\right) + u\frac{\partial}{\partial x}\left(\frac{2v_{\mathrm{s}}}{\gamma-1}\right) + v_{\mathrm{s}}\frac{\partial u}{\partial x} = 0.$$

We add and subtract to obtain

$$\left[\frac{\partial}{\partial t} + (u+v_{\mathrm{s}})\frac{\partial}{\partial x}\right]\left(u + \frac{2v_{\mathrm{s}}}{\gamma-1}\right) = 0,$$

$$\left[\frac{\partial}{\partial t} + (u-v_{\mathrm{s}})\frac{\partial}{\partial x}\right]\left(u - \frac{2v_{\mathrm{s}}}{\gamma-1}\right) = 0.$$

Define the two *Riemann invariants*

$$R_\pm = u \pm \frac{2v_{\mathrm{s}}}{\gamma-1}.$$

Then we deduce that $R_\pm$ = constant along a *characteristic (curve)* of gradient $\mathrm{d}x/\mathrm{d}t = u \pm v_\mathrm{s}$ in the $(x,t)$ plane. The + and − characteristics form an interlocking web covering the space-time diagram.

Note that both Riemann invariants are needed to reconstruct the solution ($u$ and $v_\mathrm{s}$). Half of the information is propagated along one set of characteristics and half along the other.

In general the characteristics are not known in advance but must be determined along with the solution. The + and − characteristics propagate at the speed of sound to the right and left, respectively, *with respect to the motion of the fluid*.

This concept generalizes to nonlinear waves the solution of the classical wave equation for acoustic waves on a uniform background, of the form $f(x - v_\mathrm{s}t) + g(x + v_\mathrm{s}t)$.

### 5.1.2 Method of characteristics

A numerical method of solution can be based on the following idea.

- start with the initial data ($u$ and $v_\mathrm{s}$) for all relevant $x$ at $t = 0$
- determine the characteristic slopes at $t = 0$
- propagate the $R_\pm$ information for a small increment of time, neglecting the variation of the characteristic slopes
- combine the $R_\pm$ information to find $u$ and $v_\mathrm{s}$ at each $x$ at the new value of $t$
- re-evaluate the slopes and repeat

The *domain of dependence* of a point $P$ in the space-time diagram is that region of the diagram bounded by the ± characteristics through $P$ and located in the past of $P$. The solution at $P$ cannot depend on anything that occurs outside the domain of dependence. Similarly, the *domain of influence* of $P$ is the region in the future of $P$ bounded by the characteristics through $P$.



### 5.1.3 A simple wave

Suppose that $R_-$ is uniform (the same constant on every characteristic emanating from an undisturbed region to the right). Its value everywhere is that of the undisturbed region:

$$u - \frac{2v_\mathrm{s}}{\gamma - 1} = -\frac{2v_{\mathrm{s}0}}{\gamma - 1}$$

Then both $R_+$ and $R_-$ must be constant on the + characteristics, so both $u$ and $v_\mathrm{s}$ are constant on them, and the + characteristics have constant slope $v = u + v_\mathrm{s}$ (they are straight lines).

The statement that the wave speed $v$ = constant on the straight lines $\mathrm{d}x/\mathrm{d}t = v$ is expressed by the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0.$$

This is known as the *inviscid Burgers equation* or the *nonlinear advection equation*.

It is easily solved by the method of characteristics. The initial data define $v_0(x) = v(x,0)$. The characteristics are straight lines. In regions where $\mathrm{d}v_0/\mathrm{d}x > 0$ the characteristics diverge in the future. In regions where $\mathrm{d}v_0/\mathrm{d}x < 0$ the characteristics converge and will form a *shock* at some point. Contradictory information arrives at the same point in the space-time diagram, leading to a breakdown of the solution.

Another viewpoint is that of *wave steepening*. The graph of $v$ versus $x$ evolves in time by moving each point at its wave speed $v$. The crest of the wave moves fastest and eventually overtakes the trough to the right of it. The profile would become multiple-valued, but this is physically meaningless and the wave breaks, forming a discontinuity.

Indeed, the formal solution of the inviscid Burgers equation is

$$v(x,t) = v_0(x_0) \qquad \text{with} \quad x = x_0 + v_0(x_0)t.$$

Then $\partial v/\partial x = v_0'/(1 + v_0't)$ diverges first at the breaking time $t = 1/\max(-v_0')$.

The crest of a sound wave move faster than the trough for two reasons. It is partly because the crest is denser and hotter, so the sound speed is higher (unless the gas is isothermal). But it is also because of the self-advection of the wave (the wave speed is $u+v_\mathrm{s}$). The breaking time depends on the amplitude and wavelength of the wave.

## 5.2 General analysis of simple nonlinear waves

Recall the hyperbolic structure of the ideal MHD equations:

$$\frac{\partial \mathsf{U}}{\partial t}+\mathsf{A}_i\frac{\partial \mathsf{U}}{\partial x_i}=0,$$

$$\mathsf{U}=[\rho,p,\boldsymbol{U},\boldsymbol{B}]^\mathrm{T}.$$

The system is hyperbolic because the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector $n_i$. The eigenvalues are identified as the wave speeds, and the corresponding eigenvectors as wave modes.

In a simple wave propagating in the $x$-direction, all physical quantities are functions of a single variable, the one-dimensional phase $\varphi(x,t)$. Thus $\mathsf{U}=\mathsf{U}(\varphi)$ and

$$\frac{\mathrm{d}\mathsf{U}}{\mathrm{d}\varphi}\frac{\partial\varphi}{\partial t}+\mathsf{A}_x\frac{\mathrm{d}\mathsf{U}}{\mathrm{d}\varphi}\frac{\partial\varphi}{\partial x}=0.$$

This works if $\mathrm{d}\mathsf{U}/\mathrm{d}\varphi$ is an eigenvector of the hyperbolic system and then

$$\frac{\partial\varphi}{\partial t}+v\frac{\partial\varphi}{\partial x}=0$$

where $v$ is the corresponding wavespeed. But since $v=v(\varphi)$ we again find

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=0,$$

the inviscid Burgers equation.



Wave steepening is therefore generic for simple waves. However, waves do not always steepen in practice. For example, linear dispersion arising from Coriolis or buoyancy forces (see later) can counteract nonlinear wave steepening. Waves propagating on a non-uniform background are not simple waves.

## 5.3 Shocks and other discontinuities

### 5.3.1 Jump conditions

Discontinuities are resolved in reality by additional physical effects (viscosity, thermal conduction and resistivity, i.e. diffusive effects) that become more important on smaller length-scales.

Properly, we should solve an enhanced set of equations to resolve the internal structure of a shock. This would then be matched on to the external solution where diffusion is neglected. But the matching conditions can in fact be determined from general principles without resolving the internal structure.

We consider a shock front at rest at $x=0$ (making a Galilean transformation if necessary). We look for a stationary solution in which gas flows from left ($\rho_1$, etc.) to right ($\rho_2$, etc.). On the left is upstream, pre-shocked material.

Consider any equation in conservative form

$$\frac{\partial Q}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{F}=0.$$

For a stationary solution in one dimension,

$$\frac{\mathrm{d}F_x}{\mathrm{d}x}=0,$$

which implies that the flux density $F_x$ has the same value on each side of the shock. We write the matching condition as

$$[F_x]_1^2=F_{x2}-F_{x1}=0.$$

Including additional physics means that additional diffusive fluxes (not of mass but of momentum, energy, magnetic flux, etc.) are present. But these fluxes are negligible outside the shock, so they do not affect the jump conditions. This approach is permissible as long as the new physics doesn't introduce any source terms in the equations. So the total energy is a properly conserved quantity but *not the entropy* (see later).

From mass conservation:

$$[\rho u_x]_1^2 = 0.$$

From momentum conservation:

$$\left[\rho u_x^2 + \Pi - \frac{B_x^2}{\mu_0}\right]_1^2 = 0,$$

$$\left[\rho u_x u_y - \frac{B_x B_y}{\mu_0}\right]_1^2 = 0,$$

$$\left[\rho u_x u_z - \frac{B_x B_z}{\mu_0}\right]_1^2 = 0,$$

From $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$:

$$[B_x]_1^2 = 0.$$

From $\partial\boldsymbol{B}/\partial t + \boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$:

$$[u_x B_y - u_y B_x]_1^2 = -[E_z]_1^2 = 0,$$

$$[u_x B_z - u_z B_x]_1^2 = [E_y]_1^2 = 0.$$

(These are the standard electromagnetic conditions at an interface: the normal component of $\boldsymbol{B}$ and the parallel components of $\boldsymbol{E}$ are continuous.) From total energy conservation:

$$\left[\rho u_x(\tfrac{1}{2}u^2 + \Phi + w) + \frac{1}{\mu_0}\left(B^2 u_x - (\boldsymbol{u}\cdot\boldsymbol{B})B_x\right)\right]_1^2 = 0.$$



Note that the conservative form of the momentum equation is

$$\frac{\partial}{\partial t}(\rho u_i) + \boldsymbol{\nabla}\cdot\left(\rho u_i\boldsymbol{u} + \Pi\,\boldsymbol{e}_i - \frac{B_i\boldsymbol{B}}{\mu_0}\right) = 0.$$

Including gravity makes no difference to the shock relations because $\Phi$ is always continuous (it satisfies $\nabla^2\Phi = 4\pi G\rho$).

Although the entropy in ideal MHD satisfies an equation of conservative form,

$$\frac{\partial}{\partial t}(\rho s) + \boldsymbol{\nabla}\cdot(\rho s\boldsymbol{u}) = 0,$$

the dissipation of energy within the shock provides a source term for entropy. Therefore the entropy flux is not continuous across the shock.

### 5.3.2 Non-magnetic shocks

Consider a *normal shock* ($u_y = u_z = 0$) with no magnetic field. We obtain the *Rankine–Hugionot relations*

$$[\rho u_x]_1^2 = 0,$$

$$[\rho u_x^2 + p]_1^2 = 0,$$

$$[\rho u_x(\tfrac{1}{2}u_x^2 + w)]_1^2 = 0.$$

For a polytropic gas,

$$w = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho}$$

and these equations can be solved algebraically (see example 2.2). Introduce the upstream Mach number (the *shock Mach number*)

$$\mathcal{M}_1 = \frac{u_{x1}}{v_{s1}}.$$

Then we find

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2},$$

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{(\gamma+1)},$$

$$\mathcal{M}_2^2 = \frac{2+(\gamma-1)\mathcal{M}_1^2}{2\gamma\mathcal{M}_1^2-(\gamma-1)}.$$

Case (i): $\mathcal{M}_1^2 > 1$, $\mathcal{M}_2^2 < 1$: supersonic upstream, subsonic downstream. This is a *compression shock* ($\rho_2 > \rho_1$, $p_2 > p_1$).

Case (ii): $\mathcal{M}_1^2 < 1$, $\mathcal{M}_2^2 > 1$: supersonic upstream, subsonic downstream. This is a *rarefaction shock* ($\rho_2 < \rho_1$, $p_2 < p_1$).

Trivial case: $\mathcal{M}_1^2 = 1$, $\mathcal{M}_2^2 = 1$ (no shock).

It is shown in example 2.2 that the entropy change in passing through the shock is positive for compression shocks and negative for rarefaction shocks. Therefore *only compression shocks are physically realizable.* Rarefaction shocks are excluded by the second law of thermodynamics. All shocks involve dissipation and irreversibility.

In the *strong shock* limit $\mathcal{M}_1 \gg 1$, common in astrophysical applications, we have

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} \to \frac{\gamma+1}{\gamma-1},$$

$$\frac{p_2}{p_1} \gg 1,$$

$$\mathcal{M}_2^2 \to \frac{\gamma-1}{2\gamma}.$$

Note that the compression ratio $\rho_2/\rho_1$ is finite (and equal to 4 when $\gamma = 5/3$). In the rest frame of the undisturbed (upstream) gas the *shock speed* is $u_{\text{shock}} = -u_{x1}$. The downstream density, velocity (in



that frame) and pressure in the limit of a strong shock are (to be used later)

$$\rho_2 = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1,$$

$$u_{x2} - u_{x1} = \left(\frac{2}{\gamma+1}\right)u_{\text{shock}},$$

$$p_2 = \left(\frac{2}{\gamma+1}\right)\rho_1 u_{\text{shock}}^2.$$

This last equation can be thought of as determining the thermal energy that is generated out of kinetic energy by the passage of a strong shock.

### 5.3.3 Oblique shocks

When $u_y$ or $u_z$ are non-zero, we have the additional relations

$$[\rho u_x u_y]_1^2 = [\rho u_x u_z]_1^2 = 0.$$

Since $\rho u_x$ is continuous across the shock (and non-zero), we deduce that $[u_y]_1^2 = [u_z]_1^2 = 0$. Momentum and energy conservation apply as before, and we recover the Rankine–Hugoniot relations.

### 5.3.4 Tangential discontinuities

There is a separate case with no flow through the discontinuity ($u_x = 0$). This is usually called an interface, not a shock. We can deduce only that $[p]_1^2 = 0$. Arbitrary discontinuities are allowed in $\rho$, $u_y$ and $u_z$. If $[u_y]_1^2 = [u_z]_1^2 = 0$ we have a *contact discontinuity* (only $\rho$ and $s$ may change across the shock), otherwise a *vortex sheet* (the vorticity is proportional to $\delta(x)$).

### 5.3.5 MHD shocks and discontinuities

When a magnetic field is included, the jump conditions allow a wider variety of solutions. There are different types of discontinuity associated with the three MHD waves (Alfvén, slow and fast). Since the parallel components of $\boldsymbol{B}$ need not be continuous, it is possible for them to 'switch on' or 'switch off' on passage through a shock.

A *current sheet* is a tangential discontinuity in the parallel magnetic field. A classic case would be where $B_y$, say, changes sign across the interface. The current density is proportional to $\delta(x)$.



# 6 Spherical blast waves: supernovae

## 6.1 Introduction

In a supernova, an energy of order $10^{51}$ erg is released into the interstellar medium. An expanding spherical blast wave is formed as the explosion sweeps up the surrounding gas. Several good examples of these supernova remnants are observed in the Galaxy.

The effect is similar to a bomb. When photographs (complete with length and time scales) were released of the first atomic bomb test in New Mexico in 1945, both L. I. Sedov in the Soviet Union and G. I. Taylor in the UK were able to work out the energy of the bomb (about 20 kilotons), which was supposed to be a secret.

We suppose that an energy $E$ is released at $t = 0$, $r = 0$ and that the explosion is spherically symmetric. The external medium has density $\rho_0$ and pressure $p_0$. In the Sedov–Taylor phase of the explosion, the pressure $p \gg p_0$. Then a strong shock is formed and the external pressure $p_0$ can be neglected (formally set to zero). Gravity is also negligible in the dynamics.

## 6.2 Governing equations

For a spherically symmetric flow of a polytropic gas,

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\rho = -\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u),$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)u = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln(p\rho^{-\gamma}) = 0.$$

These imply the total energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1}\right)u\right] = 0.$$

The shock is at $r = R(t)$, and the shock speed is $\dot{R}$. The equations are solved in $0 < r < R$ with the strong shock conditions at $r = R$:

$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0,$$

$$u = \frac{2\dot{R}}{\gamma+1},$$

$$p = \frac{2\rho_0\dot{R}^2}{\gamma+1}.$$

The total energy of the explosion is

$$E = \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) 4\pi r^2\,\mathrm{d}r.$$

(The thermal energy of the external medium is negligible.)

## 6.3 Dimensional analysis

The dimensional parameters of the problem on which the solution might depend are $E$ and $\rho_0$. Their dimensions are

$$[E] = ML^2T^{-2}, \qquad [\rho_0] = ML^{-3}$$

Together, they do not define a characteristic length-scale, so the explosion is 'scale-free' or 'self-similar'. If the dimensional analysis includes the time $t$ since the explosion, however, we find a time-dependent characteristic length-scale. The radius of the shock must be

$$R = \alpha\left(\frac{Et^2}{\rho_0}\right)^{1/5},$$

where $\alpha$ is a dimensionless constant to be determined.



## 6.4 Similarity solution

The self-similarity of the explosion is expressed using the dimensionless *similarity variable* $\xi = r/R(t)$. The solution has the form

$$\rho = \rho_0\,\tilde{\rho}(\xi),$$

$$u = \dot{R}\,\tilde{u}(\xi),$$

$$p = \rho_0\dot{R}^2\,\tilde{p}(\xi),$$

where $\tilde{\rho}(\xi)$, $\tilde{u}(\xi)$ and $\tilde{p}(\xi)$ are dimensionless functions to be determined.

## 6.5 Dimensionless equations

We substitute these forms into the equation and cancel the dimensional factors:

$$(\tilde{u}-\xi)\tilde{\rho}' = -\frac{\tilde{\rho}}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2\tilde{u}),$$

$$(\tilde{u}-\xi)\tilde{u}' - \frac{3}{2}\tilde{u} = -\frac{\tilde{p}'}{\tilde{\rho}},$$

$$(\tilde{u}-\xi)\left(\frac{\tilde{p}'}{\tilde{p}} - \frac{\gamma\tilde{\rho}'}{\tilde{\rho}}\right) - 3 = 0.$$

The shock conditions at $\xi = 1$ are:

$$\tilde{\rho} = \frac{\gamma+1}{\gamma-1},$$

$$\tilde{u} = \frac{2}{\gamma+1},$$

$$\tilde{p} = \frac{2}{\gamma+1}.$$

The total energy integral provides a normalization condition,

$$1 = \frac{16\pi}{25}\alpha^5\int_0^1\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right)\xi^2\,\mathrm{d}\xi,$$

which will determine the value of $\alpha$.

## 6.6 First integral

Consider the total energy equation in conservative form (multiply through by $r^2$):

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial r} = 0.$$

The dimensions are

$$[Q] = \left[\frac{E}{r}\right], \qquad [F] = \left[\frac{E}{t}\right].$$

In dimensionless form,

$$Q = \rho_0 R^2 \dot{R}^2 \,\tilde{Q}(\xi), \qquad F = \rho_0 R^2 \dot{R}^3 \,\tilde{F}(\xi).$$

Substitute into the conservation equation to find

$$-\xi\tilde{Q}' - \tilde{Q} + \tilde{F}' = 0.$$

We deduce the first integral

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\tilde{F} - \xi\tilde{Q}) = 0.$$

Now

$$Q = r^2\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}\right),$$

$$F = r^2\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1}\right)u,$$

and so

$$\tilde{Q} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma - 1}\right),$$

$$\tilde{F} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma\tilde{p}}{\gamma - 1}\right)\tilde{u}.$$



Thus

$$\tilde{F} - \xi\tilde{Q} = \text{constant} = 0$$

(for a solution finite at $\xi = 0$).

Solve for $\tilde{p}$:

$$\tilde{p} = \frac{(\gamma - 1)\tilde{\rho}\tilde{u}^2(\xi - \tilde{u})}{2(\gamma\tilde{u} - \xi)}.$$

Note that this is compatible with the shock boundary conditions. Having found a first integral, we can now dispense with (e.g.) the thermal energy equation.

Let $\tilde{u} = v\xi$. We now have

$$(v-1)\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}\ln\xi} = -\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} - 3v,$$

$$(v-1)\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} + \frac{1}{\tilde{\rho}\xi^2}\frac{\mathrm{d}}{\mathrm{d}\ln\xi}\left[\frac{(\gamma - 1)\tilde{\rho}\xi^2 v^2(1-v)}{2(\gamma v - 1)}\right] = \frac{3}{2}v.$$

Eliminate $\tilde{\rho}$:

$$\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} = \frac{v(\gamma v - 1)[5 - (3\gamma - 1)v]}{\gamma(\gamma + 1)v^2 - 2(\gamma + 1)v + 2}.$$

Invert and split into partial fractions:

$$\frac{\mathrm{d}\ln\xi}{\mathrm{d}v} = -\frac{2}{5v} + \frac{\gamma(\gamma - 1)}{(2\gamma + 1)(\gamma v - 1)} + \frac{13\gamma^2 - 7\gamma + 12}{5(2\gamma + 1)[5 - (3\gamma - 1)v]}$$

The solution is

$$\xi \propto v^{-2/5}(\gamma v - 1)^{(\gamma - 1)/(2\gamma + 1)} \times [5 - (3\gamma - 1)v]^{-(13\gamma^2 - 7\gamma + 12)/5(2\gamma + 1)(3\gamma - 1)}.$$

Now

$$\begin{aligned}\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}v} &= -\frac{1}{v-1} - \frac{3v}{v-1}\frac{\mathrm{d}\ln\xi}{\mathrm{d}v} \\ &= \frac{2}{(2-\gamma)(1-v)} + \frac{3\gamma}{(2\gamma + 1)(\gamma v - 1)} \\ &\quad - \frac{13\gamma^2 - 7\gamma + 12}{(2-\gamma)(2\gamma + 1)[5 - (3\gamma - 1)v]}.\end{aligned}$$

The solution is

$$\tilde{\rho} \propto (1-v)^{-2/(2-\gamma)}(\gamma v-1)^{3/(2\gamma+1)} \times [5-(3\gamma-1)v]^{(13\gamma^2-7\gamma+12)/(2-\gamma)(2\gamma+1)(3\gamma-1)}.$$

e.g. for $\gamma = 5/3$:

$$\xi \propto v^{-2/5}\left(\frac{5v}{3}-1\right)^{2/13}(5-4v)^{-82/195},$$

$$\tilde{\rho} \propto (1-v)^{-6}\left(\frac{5v}{3}-1\right)^{9/13}(5-4v)^{82/13}.$$

To satisfy $v = 2/(\gamma+1) = 3/4$ and $\tilde{\rho} = (\gamma+1)/(\gamma-1) = 4$ at $\xi = 1$:

$$\xi = \left(\frac{4v}{3}\right)^{-2/5}\left(\frac{20v}{3}-4\right)^{2/13}\left(\frac{5}{2}-2v\right)^{-82/195},$$

$$\tilde{\rho} = 4\,(4-4v)^{-6}\left(\frac{20v}{3}-4\right)^{9/13}\left(\frac{5}{2}-2v\right)^{82/13}.$$

Then, from the first integral,

$$\tilde{p} = \frac{3}{4}\left(\frac{4v}{3}\right)^{6/5}(4-4v)^{-5}\left(\frac{5}{2}-2v\right)^{82/15}.$$

$\xi$ ranges from 0 to 1, and $v$ from 3/5 to 3/4.

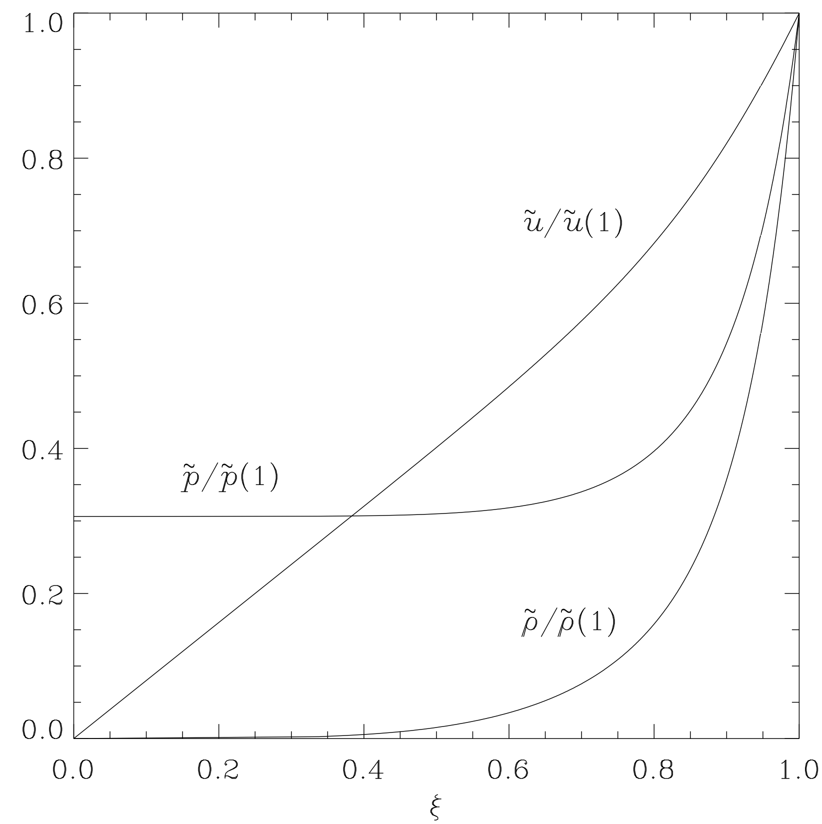
The normalization integral (numerically) yields $\alpha \approx 1.152$.

## 6.7 Application

Supernova: $E \sim 10^{51}\,\mathrm{erg}$. Estimate $\rho_0 \sim 2\times10^{-24}\,\mathrm{g\,cm^{-3}}$. Then $R \approx 5.1\,\mathrm{pc}$ and $\dot{R} \approx 2000\,\mathrm{km\,s^{-1}}$ at $t = 1000\,\mathrm{yr}$.

1945 New Mexico explosion: $E \approx 7.1\times10^{21}\,\mathrm{erg}$, $\rho_0 \approx 1.2\times10^{-3}\,\mathrm{g\,cm^{-3}}$. Then $R \approx 140\,\mathrm{m}$ and $\dot{R} \approx 5.7\,\mathrm{km\,s^{-1}}$ at $t = 0.01\,\mathrm{s}$.






Sedov solution for $\gamma = 5/3$.

The similarity method is useful in a very wide range of nonlinear problems. In this case it reduced partial differential equations to integrable ordinary differential equations.

# 7 Spherically symmetric steady flows: stellar winds and accretion

## 7.1 Basic equations

We consider spherically symmetric steady flow that is purely radial, either towards or away from a body of mass $M$. We neglect the effects of rotation and magnetic fields. The gas is polytropic and non-self-gravitating, so $\Phi = -GM/r$.

Mass conservation for such a flow implies

$$r^2 \rho u = -\frac{\dot{M}}{4\pi} = \text{constant}$$

If $u > 0$ (a stellar wind), $-\dot{M}$ is the mass loss rate. If $u < 0$ (an accretion flow), $\dot{M}$ is the mass accretion rate. We ignore the secular change in the mass $M$.

The thermal energy equation implies homentropic flow:

$$p = K\rho^\gamma, \qquad K = \text{constant}$$

The equation of motion has only one component (where $u = u_r$):

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} - \rho \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Alternatively, we can use the integral form (Bernoulli's equation):

$$\tfrac{1}{2}u^2 + w + \Phi = B = \text{constant}, \qquad w = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho}$$

In highly subsonic flow the first term is negligible and the gas is quasi-hydrostatic.

In highly supersonic flow the second term is negligible and the flow is quasi-ballistic (freely falling).

Our aim is to solve for $u(r)$, and to determine $\dot{M}$ if appropriate.



## 7.2 First treatment

We first use the differential form of the equation of motion.

Rewrite the pressure term using the other two equations:

$$-\frac{\mathrm{d}p}{\mathrm{d}r} = -\gamma p \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = \rho v_\mathrm{s}^2 \left(\frac{2}{r} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

Thus

$$(u^2 - v_\mathrm{s}^2)\frac{\mathrm{d}u}{\mathrm{d}r} = u\left(\frac{2v_\mathrm{s}^2}{r} - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

There is a critical point (sonic point) at $r = r_\mathrm{s}$ where $|u| = v_\mathrm{s}$. For the flow to pass smoothly from subsonic to supersonic, the RHS must vanish at the sonic point:

$$\frac{2v_\mathrm{ss}^2}{r_\mathrm{s}} - \frac{GM}{r_\mathrm{s}^2} = 0$$

Evaluate Bernoulli's equation at sonic point:

$$\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right)v_\mathrm{ss}^2 - \frac{GM}{r_\mathrm{s}} = B$$

We deduce that

$$v_\mathrm{ss}^2 = \frac{2(\gamma - 1)}{(5 - 3\gamma)}B, \qquad r_\mathrm{s} = \frac{(5 - 3\gamma)}{4(\gamma - 1)}\frac{GM}{B}$$

There is a unique transonic solution, which exists only for $1 < \gamma < 5/3$ (the case $\gamma = 1$ can be treated separately or by taking a limit).

Now evaluate $\dot{M}$ at the sonic point:

$$|\dot{M}| = 4\pi r_\mathrm{s}^2 \rho_\mathrm{s} v_\mathrm{ss}$$

## 7.3 Second treatment

We now use Bernoulli's equation instead of the equation of motion.

Introduce the local Mach number $\mathcal{M} = |u|/v_{\rm s}$. Then

$$r^2 \rho v_{\rm s} \mathcal{M} = \frac{|\dot{M}|}{4\pi}$$

$$v_{\rm s}^2 = \gamma K \rho^{\gamma-1}$$

Eliminate $\rho$:

$$v_{\rm s}^{\gamma+1} = \gamma K \left( \frac{|\dot{M}|}{4\pi r^2 \mathcal{M}} \right)^{\gamma-1}$$

Bernoulli's equation becomes

$$\tfrac{1}{2} v_{\rm s}^2 \mathcal{M}^2 + \frac{v_{\rm s}^2}{\gamma - 1} = B + \frac{GM}{r}$$

Substitute for $v_{\rm s}$:

$$(\gamma K)^{2/(\gamma+1)} \left( \frac{|\dot{M}|}{4\pi} \right)^{2(\gamma-1)/(\gamma+1)} \left[ \frac{\mathcal{M}^{4/(\gamma+1)}}{2} + \frac{\mathcal{M}^{-2(\gamma-1)/(\gamma+1)}}{\gamma - 1} \right]$$
$$= B r^{4(\gamma-1)/(\gamma+1)} + GM r^{-(5-3\gamma)/(\gamma+1)}$$

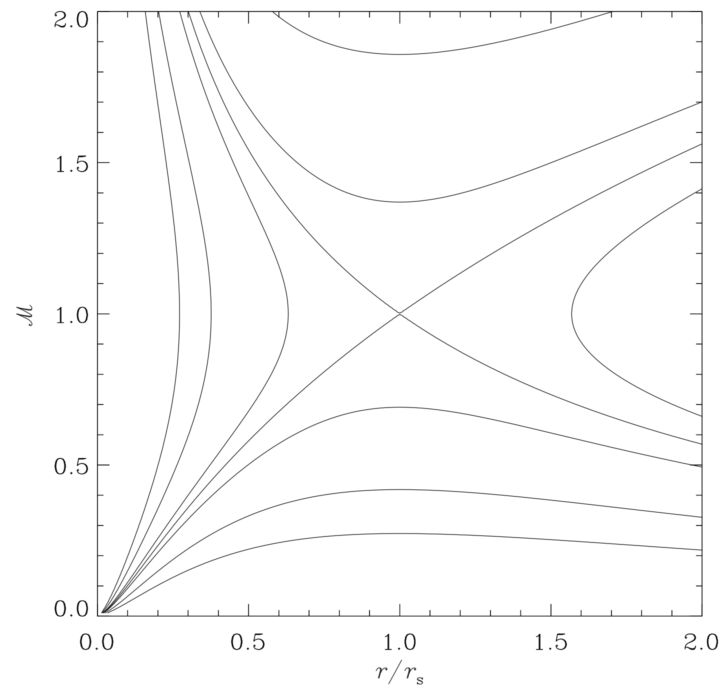
Think of this as $f(\mathcal{M}) = g(r)$. Assume that $1 < \gamma < 5/3$ and $B > 0$. Then $f(\mathcal{M})$ has a minimum at $\mathcal{M} = 1$. $g(r)$ has a minimum at

$$r = \frac{(5 - 3\gamma)}{4(\gamma - 1)} \frac{GM}{B}$$

This is the sonic radius $r_{\rm s}$ identified previously. A smooth passage through the sonic point is possible only if $|\dot{M}|$ has a special value, so that the minima of $f$ and $g$ are equal. If $|\dot{M}|$ is too large the solution does not work for all $r$. If it is too small the solution remains subsonic (or supersonic) for all $r$, which may not agree with the boundary conditions.



The $(r, \mathcal{M})$ plane shows an X-type critical point at $(r_{\rm s}, 1)$.




Solution curves for the case $\gamma = 4/3$.

For $r \ll r_{\rm s}$ the subsonic solution is close to a hydrostatic atmosphere. The supersonic solution is close to free fall.

For $r \gg r_{\rm s}$ the subsonic solution is close to $p =$ constant. The supersonic solution is close to $u =$ constant (so $\rho \propto r^{-2}$).

## 7.4 Stellar wind

For a stellar wind the appropriate solution is subsonic (hydrostatic) at small $r$ and supersonic (coasting) at large $r$. Parker (1958) first presented this simplified model for the solar wind.

### 7.5 Accretion

In spherical or Bondi (1952) accretion we consider a gas that is uniform and at rest at infinity (pressure $p_0$ and density $\rho_0$). Then $B = v_{s0}^2/(\gamma-1)$. The appropriate solution is subsonic (uniform) at large $r$ and supersonic (free fall) at small $r$. If the accreting object has a dense surface (a star rather than a black hole) then the accretion flow will be arrested by a shock above the surface.

The accretion rate of the critical solution is

$$\begin{aligned}\dot{M} &= 4\pi r_s^2 \rho_s v_{ss} = 4\pi r_s^2 \rho_0 v_{s0} \left(\frac{v_{ss}}{v_{s0}}\right)^{(\gamma+1)/(\gamma-1)} \\ &= f(\gamma)\dot{M}_B\end{aligned}$$

where

$$\dot{M}_B = \frac{\pi G^2 M^2 \rho_0}{v_{s0}^3} = 4\pi r_a^2 \rho_0 v_{s0}$$

$$f(\gamma) = \left(\frac{2}{5-3\gamma}\right)^{(5-3\gamma)/2(\gamma-1)}$$

Here

$$r_a = \frac{GM}{2v_{s0}^2}$$

is the nominal *accretion radius*, roughly the radius within which the mass $M$ captures the surrounding medium into a supersonic inflow.

Exercise: show that

$$\lim_{\gamma\to 1} f(\gamma) = e^{3/2}, \qquad \lim_{\gamma\to 5/3} f(\gamma) = 1$$

However, there is no sonic point for $\gamma = 5/3$.



# 8 Axisymmetric rotating magnetized flows: astrophysical jets

Stellar winds and jets from accretion discs are examples of outflows in which rotation and magnetic fields have important or essential roles. Using cylindrical polar coordinates $(R, \phi, z)$, we examine *steady* $(\partial/\partial t = 0)$, *axisymmetric* $(\partial/\partial\phi = 0)$ models based on the equations of ideal MHD.

## 8.1 Representation of an axisymmetric magnetic field

The solenoidal condition for an axisymmetric magnetic field is

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0.$$

We may write

$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z}, \qquad B_z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

where $\psi(R, z)$ is the *magnetic flux function*. This is related to the magnetic vector potential by $\psi = RA_\phi$. The magnetic flux contained inside the circle ($R =$ constant, $z =$ constant) is

$$\int_0^R B_z(R', z)\, 2\pi R'\, dR' = 2\pi\psi(R, z) \qquad (+\text{constant}).$$

Since $\boldsymbol{B}\cdot\boldsymbol{\nabla}\psi = 0$, $\psi$ labels magnetic field lines or their surfaces of revolution, *magnetic surfaces*. The magnetic field may be written in the form

$$\boldsymbol{B} = \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi + B_\phi\,\boldsymbol{e}_\phi = \left[-\frac{1}{R}\boldsymbol{e}_\phi\times\boldsymbol{\nabla}\psi\right] + \left[B_\phi\,\boldsymbol{e}_\phi\right].$$

The two square brackets represent the *poloidal* (meridional) and *toroidal* (azimuthal) parts of the magnetic field:

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi\,\boldsymbol{e}_\phi.$$

Note that

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{B}_\mathrm{p} = 0.$$

Similarly, one can write the velocity in the form

$$\boldsymbol{u} = \boldsymbol{u}_\mathrm{p} + u_\phi\, \boldsymbol{e}_\phi.$$

## 8.2 Mass loading and angular velocity

The steady induction equation in ideal MHD,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0},$$

implies

$$\boldsymbol{u} \times \boldsymbol{B} = -\boldsymbol{E} = \nabla \Phi_\mathrm{e},$$

where $\Phi_\mathrm{e}$ is the electrostatic potential. Now

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{B} &= (\boldsymbol{u}_\mathrm{p} + u_\phi\, \boldsymbol{e}_\phi) \times (\boldsymbol{B}_\mathrm{p} + B_\phi\, \boldsymbol{e}_\phi) \\ &= \Big[\boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_\mathrm{p} - B_\phi \boldsymbol{u}_\mathrm{p})\Big] + \Big[\boldsymbol{u}_\mathrm{p} \times \boldsymbol{B}_\mathrm{p}\Big]. \end{aligned}$$

For an axisymmetric solution with $\partial\Phi_\mathrm{e}/\partial\phi = 0$, we have

$$\boldsymbol{u}_\mathrm{p} \times \boldsymbol{B}_\mathrm{p} = \boldsymbol{0},$$

i.e. the poloidal velocity is parallel to the poloidal magnetic field. Let

$$\rho \boldsymbol{u}_\mathrm{p} = k \boldsymbol{B}_\mathrm{p},$$

where $k$ is the *mass loading*, i.e. the ratio of mass flux to magnetic flux.

The steady equation of mass conservation is

$$0 = \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot (\rho \boldsymbol{u}_\mathrm{p}) = \nabla \cdot (k \boldsymbol{B}_\mathrm{p}) = \boldsymbol{B}_\mathrm{p} \cdot \nabla k.$$



Therefore

$$k = k(\psi),$$

i.e. $k$ is a *surface function*, constant on each magnetic surface.

We now have

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_\mathrm{p} - B_\phi \boldsymbol{u}_\mathrm{p}) = \left(\frac{u_\phi}{R} - \frac{k B_\phi}{R\rho}\right) \nabla \psi.$$

Taking the curl of this, we find

$$\boldsymbol{0} = \nabla \left(\frac{u_\phi}{R} - \frac{k B_\phi}{R\rho}\right) \times \nabla \psi.$$

Therefore

$$\frac{u_\phi}{R} - \frac{k B_\phi}{R\rho} = \omega,$$

where $\omega(\psi)$ is another surface function, known as the *angular velocity of the magnetic surface*.

The complete velocity field may be written in the form

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + R\omega\, \boldsymbol{e}_\phi,$$

i.e. the total velocity is parallel to the total magnetic field in a frame of reference rotating with angular velocity $\omega$. It is useful to think of the fluid being constrained to move along the field line like a *bead on a rotating wire*.

## 8.3 Entropy

The steady thermal energy equation,

$$\boldsymbol{u} \cdot \nabla s = 0,$$

implies that $\boldsymbol{B}_\mathrm{p} \cdot \nabla s = 0$ and so

$$s = s(\psi)$$

is another surface function.

## 8.4 Angular momentum

The azimuthal component of the equation of motion is

$$\rho\left(\boldsymbol{u}_\mathrm{p}\cdot\boldsymbol{\nabla}u_\phi+\frac{u_Ru_\phi}{R}\right)=\frac{1}{\mu_0}\left(\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}B_\phi+\frac{B_RB_\phi}{R}\right)$$

$$\frac{1}{R}\rho\boldsymbol{u}_\mathrm{p}\cdot\boldsymbol{\nabla}(Ru_\phi)-\frac{1}{\mu_0R}\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}(RB_\phi)=0$$

$$\frac{1}{R}\boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{\nabla}\left(kRu_\phi-\frac{RB_\phi}{\mu_0}\right)=0$$

and so

$$Ru_\phi=\frac{RB_\phi}{\mu_0k}+\ell,$$

where

$$\ell=\ell(\psi)$$

is another surface function, the *angular momentum invariant*. This is the angular momentum removed in the outflow per unit mass, but part of the torque is carried by the magnetic field.

## 8.5 The Alfvén surface

Define the *poloidal Alfvén number* (cf. the Mach number)

$$A=\frac{u_\mathrm{p}}{v_\mathrm{ap}}.$$

Then

$$A^2=\frac{\mu_0\rho u_\mathrm{p}^2}{B_\mathrm{p}^2}=\frac{\mu_0k^2}{\rho},$$

and so $A\propto\rho^{-1/2}$ on each magnetic surface.



Consider the two equations

$$\frac{u_\phi}{R}=\frac{kB_\phi}{R\rho}+\omega,$$

$$Ru_\phi=\frac{RB_\phi}{\mu_0k}+\ell.$$

Eliminate $B_\phi$ to obtain

$$u_\phi=\frac{R^2\omega-A^2\ell}{R(1-A^2)}$$
$$=\left(\frac{1}{1-A^2}\right)R\omega+\left(\frac{A^2}{A^2-1}\right)\frac{\ell}{R}.$$

For $A\ll 1$ we have

$$u_\phi\approx R\omega,$$

i.e. the fluid is in uniform rotation, corotating with the magnetic surface. For $A\gg 1$ we have

$$u_\phi\approx\frac{\ell}{R},$$

i.e. the fluid conserves its specific angular momentum. The point $R=R_\mathrm{a}(\psi)$ where $A=1$ is the *Alfvén point*. The locus of Alfvén points for different magnetic surfaces forms the *Alfvén surface*. To avoid a singularity there we require

$$\ell=R_\mathrm{a}^2\omega.$$

Typically the outflow will start at low velocity in high-density material, where $A\ll 1$. We can therefore identify $\omega$ as the angular velocity $u_\phi/R=\Omega_0$ of the footpoint $R=R_0$ of the magnetic field line at the source of the outflow. It will then accelerate smoothly through an Alfvén surface and become super-Alfvénic ($A>1$). If mass is lost at a rate $\dot{M}$ in the outflow, angular momentum is lost at a rate

$\dot{M}\ell = \dot{M}R_{\mathrm{a}}^2\Omega_0$. In contrast, in a hydrodynamic outflow, angular momentum is conserved by fluid elements and is therefore lost at a rate $\dot{M}R_0^2\Omega_0$. A highly efficient removal of angular momentum occurs if the Alfvén radius is large compared to the footpoint radius. This effect is the *magnetic lever arm*. The loss of angular momentum through a stellar wind is called *magnetic braking*.

## 8.6 The Bernoulli equation

The total energy equation for a steady flow is

$$\boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]=0.$$

Now since

$$\boldsymbol{u}=\frac{k\boldsymbol{B}}{\rho}+R\omega\,\boldsymbol{e}_\phi,$$

we have

$$\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}=-R\omega\,\boldsymbol{e}_\phi\times\boldsymbol{B}=-R\omega\,\boldsymbol{e}_\phi\times\boldsymbol{B}_{\mathrm{p}},$$

which is purely poloidal. Thus

$$(\boldsymbol{E}\times\boldsymbol{B})_{\mathrm{p}}=\boldsymbol{E}\times(B_\phi\,\boldsymbol{e}_\phi)=-R\omega B_\phi\boldsymbol{B}_{\mathrm{p}}.$$

The total energy equation is then

$$\boldsymbol{\nabla}\cdot\left[k\boldsymbol{B}_{\mathrm{p}}(\tfrac{1}{2}u^2+\Phi+w)-\frac{R\omega B_\phi}{\mu_0}\boldsymbol{B}_{\mathrm{p}}\right]=0$$

$$\boldsymbol{B}_{\mathrm{p}}\cdot\boldsymbol{\nabla}\left[k\left(\tfrac{1}{2}u^2+\Phi+w-\frac{R\omega B_\phi}{\mu_0 k}\right)\right]=0$$

$$\tfrac{1}{2}u^2+\Phi+w-\frac{R\omega B_\phi}{\mu_0 k}=\varepsilon,$$

where

$$\varepsilon=\varepsilon(\psi)$$



is another surface function, the *energy invariant*.

An alternative invariant is

$$\begin{aligned}\varepsilon' &= \varepsilon-\ell\omega\\ &= \tfrac{1}{2}u^2+\Phi+w-\frac{R\omega B_\phi}{\mu_0 k}-\left(Ru_\phi-\frac{RB_\phi}{\mu_0 k}\right)\omega\\ &= \tfrac{1}{2}u^2+\Phi+w-Ru_\phi\omega\\ &= \tfrac{1}{2}u_{\mathrm{p}}^2+\tfrac{1}{2}(u_\phi-R\omega)^2+\Phi^{\mathrm{cg}}+w,\end{aligned}$$

where

$$\Phi^{\mathrm{cg}}=\Phi-\tfrac{1}{2}\omega^2R^2$$

is the centrifugal–gravitational potential associated with the magnetic surface. One can then see that $\varepsilon'$ is the Bernoulli function of the flow in the frame rotating with angular velocity $\omega$. In this frame the flow is strictly parallel to the field and the field therefore does no work because $\boldsymbol{J}\times\boldsymbol{B}\perp\boldsymbol{B}$ and so $\boldsymbol{J}\times\boldsymbol{B}\perp(\boldsymbol{u}-R\omega\,\boldsymbol{e}_\phi)$.

## 8.7 Summary

We have been able to integrate almost all of the MHD equations, reducing them to a set of algebraic relations on each magnetic surface. If the poloidal magnetic field $\boldsymbol{B}_{\mathrm{p}}$ (or, equivalently, the flux function $\psi$) is specified in advance, these algebraic equations are sufficient to determine the complete solution on each magnetic surface separately, although we must also (i) specify the initial conditions at the source of the outflow and (ii) ensure that the solution passes smoothly through critical points where the flow speed matches the speeds of slow and fast magnetoacoustic waves.

The component of the equation of motion perpendicular to the magnetic surfaces is the only piece of information not yet used. This 'transfield' or 'Grad–Shafranov' equation ultimately determines the equilibrium shape of the magnetic surfaces. It is a very complicated nonlinear partial differential equation for $\psi(R,z)$ and cannot be reduced to simple terms. We do not consider it here.

## 8.8 Acceleration from the surface of an accretion disc

We consider the launching of an outflow from a thin accretion disc. The angular velocity $\Omega(R)$ of the disc corresponds to circular Keplerian orbital motion around a central mass $M$:

$$\Omega = \left(\frac{GM}{R^3}\right)^{1/2}$$

If the flow starts essentially from rest in high-density material ($A \ll 1$), we have

$$\omega \approx \Omega,$$

i.e. the angular velocity of the magnetic surface is the angular velocity of the disc at the foot-point of the field line. In the sub-Alfvénic region we have

$$\varepsilon' \approx \tfrac{1}{2}u_{\rm p}^2 + \Phi^{\rm cg} + w.$$

As in the case of stellar winds, if the gas is hot (comparable to the escape temperature) an outflow can be driven by thermal pressure. Of more interest here is the possibility of a dynamically driven outflow. For a 'cold' wind the enthalpy makes a negligible contribution in this equation. Whether the flow accelerates or not above the disc then depends on the variation of the centrifugal–gravitational potential along the field line.

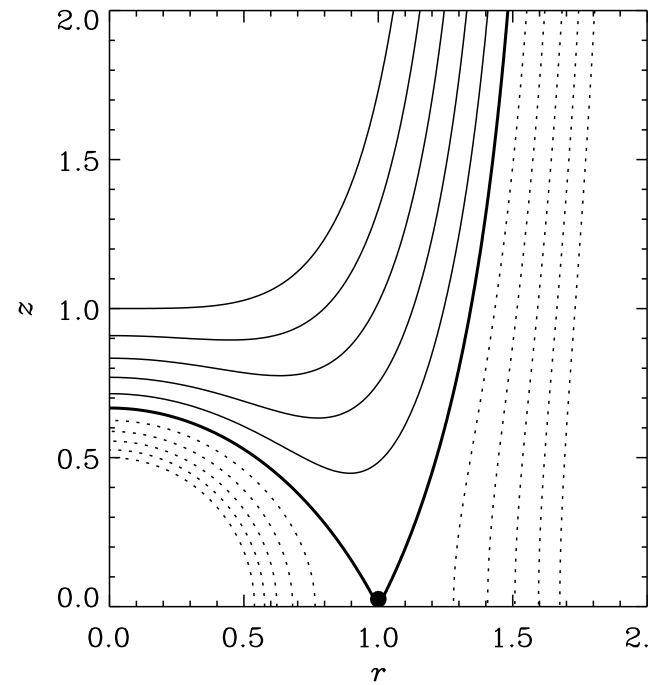
Consider a Keplerian disc in a point-mass potential. Let the foot-point of the field line be at $R = R_0$, and let the angular velocity of the field line be

$$\omega = \Omega_0 = \left(\frac{GM}{R_0^3}\right)^{1/2},$$

as argued above. Then

$$\Phi^{\rm cg} = -GM(R^2+z^2)^{-1/2} - \frac{1}{2}\frac{GM}{R_0^3}R^2.$$






Contours of $\Phi^{\rm cg}$, in units such that $R_0 = 1$.
The downhill directions are indicated by dashed contours.

In units such that $R_0 = 1$, the equation of the equipotential passing through the foot-point $(R_0, z)$ is

$$(R^2+z^2)^{-1/2} + \frac{R^2}{2} = \frac{3}{2}.$$

This can be rearranged into the form

$$z^2 = \frac{(2-R)(R-1)^2(R+1)^2(R+2)}{(3-R^2)^2}.$$

Close to the foot-point $(1, 0)$ we have

$$z^2 \approx 3(R-1)^2$$

and so

$$z \approx \pm\sqrt{3}(R-1).$$

The foot-point lies at a saddle point of $\Phi^{\rm cg}$. If the inclination of the field line to the vertical, $i$, at the surface of the disc exceeds $30°$, the flow is accelerated without thermal assistance. This is *magnetocentrifugal acceleration.*

The critical equipotential has an asymptote at $r = r_0\sqrt{3}$. The field line must continue to expand sufficiently in the radial direction in order to sustain the magnetocentrifugal acceleration.

## 8.9 Magnetically driven accretion

To allow a quantity of mass $\Delta M_{\rm acc}$ to be accreted from radius $R_0$, the angular momentum that must be removed is $R_0^2\Omega_0\,\Delta M_{\rm acc}$. The angular momentum removed by a quantity of mass $\Delta M_{\rm jet}$ flowing out in a magnetized jet from radius $R_0$ is $\ell\,\Delta M_{\rm jet} = R_{\rm a}^2\Omega_0\,\Delta M_{\rm jet}$. Therefore accretion can in principle be driven by an outflow, with

$$\frac{\dot{M}_{\rm acc}}{\dot{M}_{\rm jet}} \sim \frac{R_{\rm a}^2}{R_0^2}.$$

The magnetic lever arm allows an efficient removal of angular momentum if the Alfvén radius is large compared to the foot-point radius.



# 9 Waves and instabilities in stratified rotating astrophysical bodies

## 9.1 Eulerian and Lagrangian perturbations

We have used the symbol $\delta$ to denote an *Eulerian perturbation*, i.e. the perturbation of a quantity at a fixed point in space. Let $\Delta$ denote the *Lagrangian perturbation*, i.e. the perturbation of a quantity as experienced by a fluid element, taking into account its displacement by a distance $\boldsymbol{\xi}$. For infinitesimal perturbations these are related by

$$\Delta = \delta + \boldsymbol{\xi}\cdot\boldsymbol{\nabla},$$

and so

$$\Delta\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi},$$

$$\Delta p = -\gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi},$$

$$\Delta\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\boldsymbol{B}.$$

These relations hold even if the basic state is not static, if $\boldsymbol{\xi}$ is the *Lagrangian displacement*, i.e. the displacement of a fluid element the between the unperturbed and perturbed flows. Thus

$$\Delta\boldsymbol{u} = \frac{{\rm D}\boldsymbol{\xi}}{{\rm D}t}$$

If the basic state is static ($\boldsymbol{u} = \boldsymbol{0}$) this reduces to $\delta\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$ as before.

## 9.2 The energy principle

In Section 4 we derived the linearized equation

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\delta\rho\boldsymbol{\nabla}\Phi - \rho\boldsymbol{\nabla}\delta\Phi - \boldsymbol{\nabla}\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\boldsymbol{B}) \quad (1)$$

governing the displacement $\boldsymbol{\xi}(\boldsymbol{x},t)$ of the fluid from an arbitrary magnetostatic equilibrium, where

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}),$$

$$\nabla^2\delta\Phi = 4\pi G\delta\rho,$$

$$\delta\Pi = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}),$$

$$\delta\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\boldsymbol{B} - (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\boldsymbol{B}.$$

Using the equation of magnetostatic equilibrium,

$$\mathbf{0} = -\rho\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B},$$

equation (1) can be put into the equivalent form (**exercise**)

$$\rho\frac{\partial^2\xi_i}{\partial t^2} = -\rho\frac{\partial\delta\Phi}{\partial x_i} - \rho\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial\xi_k}{\partial x_l}\right), \qquad (2)$$

where

$$\begin{aligned} V_{ijkl} &= \left(\gamma p + \frac{B^2}{\mu_0}\right)\delta_{ij}\delta_{kl} + \Pi(\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) \\ &\quad + \frac{1}{\mu_0}(B_jB_l\delta_{ik} - B_iB_j\delta_{kl} - B_kB_l\delta_{ij}) \\ &= V_{klij}. \end{aligned}$$

In this form (but with $\rho\,\mathrm{D}^2\boldsymbol{\xi}/\mathrm{D}t^2$ on the left-hand side) this equation can be shown to hold for perturbations from an arbitrary flow $\boldsymbol{u}(\boldsymbol{x},t)$, if $\boldsymbol{\xi}$ is the Lagrangian displacement.

We may write the equation in the form

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = \mathcal{F}\boldsymbol{\xi},$$



where $\mathcal{F}$ is a linear differential operator (or integro-differential if self-gravitation is taken into account). The force operator $\mathcal{F}$ can be shown to be self-adjoint with respect to the inner product

$$\langle\boldsymbol{\eta},\boldsymbol{\xi}\rangle = \int \rho\boldsymbol{\eta}^*\cdot\boldsymbol{\xi}\,\mathrm{d}V$$

if appropriate boundary conditions apply. Let $\delta\Psi$ be the gravitational potential perturbation associated with the displacement $\boldsymbol{\eta}$, so $\nabla^2\delta\Psi = -4\pi G\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\eta})$. Then

$$\begin{aligned} \langle\boldsymbol{\eta},\mathcal{F}\boldsymbol{\xi}\rangle &= \int\left[-\rho\eta_i^*\frac{\partial\delta\Phi}{\partial x_i} - \rho\eta_i^*\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \eta_i^*\frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial\xi_k}{\partial x_l}\right)\right]\mathrm{d}V \\ &= \int\left[-\delta\Phi\frac{\nabla^2\delta\Psi^*}{4\pi G} - \rho\xi_i\eta_j^*\frac{\partial^2\Phi}{\partial x_i\partial x_j} - V_{ijkl}\frac{\partial\xi_k}{\partial x_l}\frac{\partial\eta_i^*}{\partial x_j}\right]\mathrm{d}V \\ &= \int\left[\frac{\boldsymbol{\nabla}(\delta\Phi)\cdot\boldsymbol{\nabla}(\delta\Psi^*)}{4\pi G} - \rho\xi_i\eta_j^*\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \xi_k\frac{\partial}{\partial x_l}\left(V_{ijkl}\frac{\partial\eta_i^*}{\partial x_j}\right)\right]\mathrm{d}V \\ &= \int\left[-\delta\Psi^*\frac{\nabla^2\delta\Phi}{4\pi G} - \rho\xi_i\eta_j^*\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \xi_i\frac{\partial}{\partial x_j}\left(V_{klij}\frac{\partial\eta_k^*}{\partial x_l}\right)\right]\mathrm{d}V \\ &= \int\left[-\rho\xi_i\frac{\partial\delta\Psi^*}{\partial x_i} - \rho\xi_i\eta_j^*\frac{\partial^2\Phi}{\partial x_i\partial x_j} + \xi_i\frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial\eta_k^*}{\partial x_l}\right)\right]\mathrm{d}V \\ &= \langle\mathcal{F}\boldsymbol{\eta},\boldsymbol{\xi}\rangle. \end{aligned}$$

Here the integrals are over all space. We assume that the exterior of the body is a medium of zero density in which the force-free limit of MHD holds and $\boldsymbol{B}$ decays sufficiently fast as $|\boldsymbol{r}|\to\infty$ that we may integrate freely by parts and ignore surface terms. Also note that $\delta\Phi = O(r^{-1})$, or in fact $O(r^{-2})$ if $\delta M = 0$.

The functional

$$\begin{aligned} W[\boldsymbol{\xi}] &= -\frac{1}{2}\langle\boldsymbol{\xi},\mathcal{F}\boldsymbol{\xi}\rangle \\ &= \frac{1}{2}\int\left[-\frac{|\boldsymbol{\nabla}\delta\Phi|^2}{4\pi G} + \rho\xi_i^*\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + V_{ijkl}\frac{\partial\xi_i^*}{\partial x_j}\frac{\partial\xi_k}{\partial x_l}\right]\mathrm{d}V \end{aligned}$$

is therefore real and represents the change in potential energy associated with the displacement $\boldsymbol{\xi}$.

If the basic state is static, we may consider solutions of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}(\boldsymbol{x})\exp(-\mathrm{i}\omega t)\right],$$

so we obtain

$$-\omega^2\tilde{\boldsymbol{\xi}} = \mathcal{F}\tilde{\boldsymbol{\xi}}$$

and

$$\omega^2 = -\frac{\langle\tilde{\boldsymbol{\xi}},\mathcal{F}\tilde{\boldsymbol{\xi}}\rangle}{\langle\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}\rangle} = \frac{2W[\tilde{\boldsymbol{\xi}}]}{\langle\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}\rangle}.$$

Therefore $\omega^2$ is real and we have either oscillations ($\omega^2 > 0$) or instability ($\omega^2 < 0$).

The above expression for $\omega^2$ satisfies the usual Rayleigh–Ritz variational principle for self-adjoint eigenvalue problems. The eigenvalues $\omega^2$ are the stationary values of $2W[\boldsymbol{\xi}]/\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle$ among trial displacements $\boldsymbol{\xi}$ satisfying the boundary conditions. In particular, the lowest eigenvalue is the global minimum value of $2W[\boldsymbol{\xi}]/\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle$. Therefore the equilibrium is unstable if and only if $W[\boldsymbol{\xi}]$ can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions. This is called the *energy principle*.

This discussion is incomplete because it assumes that the eigenfunctions form a complete set. In general a continuous spectrum of non-square-integrable modes is also present. However, it can be shown that a necessary and sufficient condition for instability is that $W[\boldsymbol{\xi}]$ can be made negative as described above. Consider the equation for twice the energy of the perturbation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}]\right) &= \langle\ddot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}},\ddot{\boldsymbol{\xi}}\rangle - \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\boldsymbol{\xi},\mathcal{F}\dot{\boldsymbol{\xi}}\rangle\\ &= \langle\mathcal{F}\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\mathcal{F}\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle\\ &= 0.\end{aligned}$$

Therefore

$$\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = \text{constant}.$$



where $E$ is determined by the initial conditions $\boldsymbol{\xi}_0, \dot{\boldsymbol{\xi}}_0$. If $W$ is positive definite then the equilibrium is stable because $\boldsymbol{\xi}$ is limited by the constraint $W[\boldsymbol{\xi}] \leqslant E$.

Suppose that a (real) trial displacement $\boldsymbol{\eta}$ can be found for which

$$\frac{2W[\boldsymbol{\eta}]}{\langle\boldsymbol{\eta},\boldsymbol{\eta}\rangle} = -\gamma^2, \qquad \gamma > 0.$$

Then let the initial conditions be $\boldsymbol{\xi}_0 = \boldsymbol{\eta}$, $\dot{\boldsymbol{\xi}}_0 = \gamma\boldsymbol{\eta}$ so that

$$\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = 0.$$

Now let

$$a(t) = \ln\left(\frac{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{\langle\boldsymbol{\eta},\boldsymbol{\eta}\rangle}\right)$$

so that

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle}\\ \frac{\mathrm{d}^2a}{\mathrm{d}t^2} &= \frac{2(\langle\boldsymbol{\xi},\mathcal{F}\boldsymbol{\xi}\rangle + \langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle)\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - 4\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2}\\ &= \frac{2(-2W[\boldsymbol{\xi}] + \langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle)\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - 4\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2}\\ &= \frac{4(\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - \langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2)}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2}\\ &\geqslant 0\end{aligned}$$

by the Cauchy–Schwarz inequality. Thus

$$\frac{\mathrm{d}a}{\mathrm{d}t} \geqslant \dot{a}_0 = 2\gamma$$

$$a \geqslant 2\gamma t + a_0 = 2\gamma t.$$

Therefore the disturbance with these initial conditions grows at least as fast as $\exp(\gamma t)$ and the equilibrium is unstable.

## 9.3 Spherically symmetric star

The simplest model of a star neglects rotation and magnetic fields and assumes a spherically symmetric hydrostatic equilibrium in which $\rho(r)$ and $p(r)$ satisfy

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g$$

with inward radial gravitational acceleration

$$g(r) = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2}\int_0^r \rho(r')\,4\pi r'^2\,\mathrm{d}r'$$

The stratification induced by gravity provides a non-uniform background for wave propagation.

In this case

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\delta\rho\boldsymbol{\nabla}\Phi - \rho\boldsymbol{\nabla}\delta\Phi - \boldsymbol{\nabla}\delta p$$

$$\omega^2\int_V \rho|\boldsymbol{\xi}|^2\,\mathrm{d}V = \int_V \boldsymbol{\xi}^*\cdot(\delta\rho\boldsymbol{\nabla}\Phi + \rho\boldsymbol{\nabla}\delta\Phi + \boldsymbol{\nabla}\delta p)\,\mathrm{d}V$$

At the surface $S$ of the star, we assume that $\rho$ and $p$ vanish. Then $\delta p$ also vanishes on $S$ (assuming that $\boldsymbol{\xi}$ and its derivatives are bounded). Now

$$\begin{aligned}
\int_V \boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}\delta p\,\mathrm{d}V &= -\int_V(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^*\delta p\,\mathrm{d}V \\
&= \int_V \frac{1}{\gamma p}(\delta p + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}p)^*\delta p\,\mathrm{d}V \\
&= \int_V\left[\frac{|\delta p|^2}{\gamma p} + \frac{1}{\gamma p}(\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}p)(-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p - \gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\right]\mathrm{d}V \\
&= \int_V\left[\frac{|\delta p|^2}{\gamma p} - \frac{|\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p|^2}{\gamma p} - (\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}p)\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right]\mathrm{d}V \\
&= \int_V\left[\frac{|\delta p|^2}{\gamma p} - \frac{|\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p|^2}{\gamma p} - (\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}\Phi)(\delta\rho + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho)\right]\mathrm{d}V
\end{aligned}$$



Thus

$$\begin{aligned}
\omega^2\int_V\rho|\boldsymbol{\xi}|^2\,\mathrm{d}V = &-\frac{1}{4\pi G}\int_\infty|\boldsymbol{\nabla}\delta\Phi|^2\,\mathrm{d}V \\
&+\int_V\left[\frac{|\delta p|^2}{\gamma p} - (\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}p)\cdot\left(\frac{1}{\gamma}\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\ln p - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\ln\rho\right)\right]\mathrm{d}V
\end{aligned}$$

$$\omega^2\int_V\rho|\boldsymbol{\xi}|^2\,\mathrm{d}V = -\frac{1}{4\pi G}\int_\infty|\boldsymbol{\nabla}\delta\Phi|^2\,\mathrm{d}V + \int_V\left(\frac{|\delta p|^2}{\gamma p} + \rho N^2|\xi_r|^2\right)\mathrm{d}V$$

where $N(r)$ is the *Brunt–Väisälä frequency* (or *buoyancy frequency*) given by

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) \propto g\frac{\mathrm{d}s}{\mathrm{d}r}$$

$N$ is the frequency of oscillation of a fluid element that is displaced vertically in a stably stratified atmosphere if it maintains pressure equilibrium with its surroundings.

There are three contributions to $\omega^2$: the self-gravitational term (destabilizing), the acoustic term (stabilizing) and the buoyancy term (stabilizing if $N^2 > 0$).

If $N^2 < 0$ for any interval of $r$, a trial displacement can always be found such that $\omega^2 < 0$. This is done by localizing $\xi_r$ in that interval and arranging the other components of $\boldsymbol{\xi}$ such that $\delta p = 0$. Therefore the star is unstable if $\partial s/\partial r < 0$ anywhere. This is *Schwarzschild's criterion* for convective instability.

## 9.4 Modes of an incompressible sphere

Analytical solutions can be obtained in the case of a homogeneous incompressible 'star' of mass $M$ and radius $R$ which has

$$\rho = \left(\frac{3M}{4\pi R^3}\right)H(R-r),$$

where $H$ is the Heaviside step function. For $r \leqslant R$ we have

$$g = \frac{GMr}{R^3},$$

$$p = \frac{3GM^2(R^2 - r^2)}{8\pi R^6}.$$

For an incompressible fluid

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0,$$

$$\delta\rho = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\rho = \xi_r \left(\frac{3M}{4\pi R^3}\right) \delta(r - R),$$

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho = \xi_r \left(\frac{3GM}{R^3}\right) \delta(r - R). \qquad (1)$$

$\delta p$ is indeterminate and is a variable independent of $\boldsymbol{\xi}$. The linearized equation of motion is

$$-\rho\omega^2 \boldsymbol{\xi} = -\rho \boldsymbol{\nabla} \delta\Phi - \boldsymbol{\nabla} \delta p.$$

Thus $\boldsymbol{\xi} = \boldsymbol{\nabla} U$ with $\nabla^2 U = 0$ and $-\rho\omega^2 U = -\rho\delta\Phi - \delta p$ in $r \leqslant R$. Appropriate solutions of Laplace's equation regular at $r = 0$ are the solid spherical harmonics (with arbitrary normalization)

$$U = r^\ell Y_\ell^m(\theta, \phi),$$

where $\ell$ and $m$ are integers with $\ell \geqslant |m|$. Equation (1) also implies

$$\delta\Phi = \begin{cases} A r^\ell Y_\ell^m, & r < R \\ B r^{-\ell-1} Y_\ell^m, & r > R \end{cases}$$

The matching conditions from equation (1) at $r = R$ are

$$[\delta\Phi] = 0$$

$$\left[\frac{\partial \delta\Phi}{\partial r}\right] = \xi_r \left(\frac{3GM}{R^3}\right)$$



Thus

$$BR^{-\ell-1} - AR^\ell = 0$$

$$-(\ell+1) BR^{-\ell-2} - \ell A R^{\ell-1} = \ell R^{\ell-1} \left(\frac{3GM}{R^3}\right)$$

with solution

$$A = -\frac{\ell}{2\ell+1} \left(\frac{3GM}{R^3}\right), \qquad B = AR^{2\ell+1}$$

At $r = R$ the Lagrangian pressure perturbation should vanish:

$$\Delta p = \delta p + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p = 0$$

$$\left(\frac{3M}{4\pi R^3}\right) \left[\omega^2 R^\ell + \left(\frac{\ell}{2\ell+1}\right) \left(\frac{3GM}{R^3}\right) R^\ell\right] - \frac{3GM^2}{4\pi R^5} \ell R^{\ell-1} = 0$$

$$\omega^2 = \frac{2\ell(\ell-1)}{2\ell+1} \frac{GM}{R^3}$$

Since $\omega^2 \geqslant 0$ the star is stable. Note that $\ell = 0$ corresponds to $\boldsymbol{\xi} = \mathbf{0}$ and $\ell = 1$ corresponds to $\boldsymbol{\xi} = $ constant. The remaining modes are non-trivial and are called *f modes* (fundamental modes). These can be thought of as surface gravity waves, related to ocean waves for which $\omega^2 = gk$.

## 9.5 The plane-parallel atmosphere

The local dynamics of a stellar atmosphere can be studied in a Cartesian ('plane-parallel') approximation. The gravitational acceleration is taken to be constant (appropriate to an atmosphere) and in the $-z$ direction. For hydrostatic equilibrium,

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$$

A simple example is an *isothermal atmosphere* in which $p = c_\mathrm{s}^2 \rho$ with $c_\mathrm{s} = $ constant:

$$\rho = \rho_0 \, \mathrm{e}^{-z/H}, \qquad p = p_0 \, \mathrm{e}^{-z/H}$$

$H = c_s^2/g$ is the *isothermal scale-height*. The Brunt–Väisälä frequency in an isothermal atmosphere is given by

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}z} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}z}\right) = \left(1 - \frac{1}{\gamma}\right)\frac{g}{H}$$

which is constant and is positive for $\gamma > 1$. An isothermal atmosphere is *stably (subadiabatically) stratified* if $\gamma > 1$ and *neutrally (adiabatically) stratified* if $\gamma = 1$.

A further example is a *polytropic atmosphere* in which $p \propto \rho^{1+1/m}$ in the undisturbed state, where $m$ is a positive constant. In general $1+1/m$ differs from the adiabatic exponent $\gamma = 1+1/n$ of the gas. For hydrostatic equilibrium,

$$\rho^{1/m}\frac{\mathrm{d}\rho}{\mathrm{d}z} \propto -\rho g$$

$$\rho^{1/m} \propto -z$$

if $z = 0$ is the surface of the atmosphere. Let

$$\rho = \rho_0\left(-\frac{z}{H}\right)^m$$

where $\rho_0$ and $H$ are constants. Then

$$p = p_0\left(-\frac{z}{H}\right)^{m+1}$$

where

$$p_0 = \frac{\rho_0 g H}{m+1}$$

to satisfy $\mathrm{d}p/\mathrm{d}z = -\rho g$. In this case

$$N^2 = \left(\frac{m+1}{\gamma} - m\right)\frac{g}{z} = \left(\frac{m-n}{n+1}\right)\frac{g}{-z}.$$

We return to the linearized equations looking for solutions of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}(z)\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}_\mathrm{h}\cdot\boldsymbol{x})\right], \qquad \text{etc.}$$



where 'h' stands for horizontal ($x$ and $y$ components). Then

$$-\rho\omega^2\boldsymbol{\xi}_\mathrm{h} = -\mathrm{i}\boldsymbol{k}_\mathrm{h}\delta p,$$

$$-\rho\omega^2\xi_z = -g\,\delta\rho - \frac{\mathrm{d}\delta p}{\mathrm{d}z},$$

$$\delta\rho = -\xi_z\frac{\mathrm{d}\rho}{\mathrm{d}z} - \rho\Delta,$$

$$\delta p = -\xi_z\frac{\mathrm{d}p}{\mathrm{d}z} - \gamma p\Delta,$$

$$\Delta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \mathrm{i}\boldsymbol{k}_\mathrm{h}\cdot\boldsymbol{\xi}_\mathrm{h} + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}$$

The self-gravitation of the perturbation (i.e. $\delta\Phi$) is neglected in the atmosphere. This is known as *Cowling's approximation*.

Eliminate variables in favour of $\xi_z$ and $\Delta$:

$$\omega^2\frac{\mathrm{d}\xi_z}{\mathrm{d}z} - gk_\mathrm{h}^2\xi_z = (\omega^2 - v_\mathrm{s}^2k_\mathrm{h}^2)\Delta,$$

$$g\frac{\mathrm{d}\xi_z}{\mathrm{d}z} - \omega^2\xi_z = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}(\gamma p\Delta) + g\Delta,$$

where $k_\mathrm{h} = |\boldsymbol{k}_\mathrm{h}|$. A general property of these equations is that they admit an incompressible mode in which $\Delta = 0$. For compatibility of these equations,

$$\frac{\omega^2}{gk_\mathrm{h}^2} = \frac{g}{\omega^2}$$

$$\omega^2 = \pm gk_\mathrm{h}.$$

The acceptable solution with $\xi_z$ decaying with depth is

$$\omega^2 = gk_\mathrm{h}, \qquad \xi_z \propto \exp(k_\mathrm{h}z)$$

This is a *surface gravity wave* known in stellar oscillations as the *f mode* (fundamental mode). It is vertically evanescent.

The other wave solutions can be found analytically in the case of a polytropic atmosphere. Eliminate variables in favour of $\Delta$ (algebra omitted):

$$z\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2}+(m+2)\frac{\mathrm{d}\Delta}{\mathrm{d}z}-(A+k_\mathrm{h}z)k_\mathrm{h}\Delta=0$$

where

$$A=\frac{n(m+1)}{n+1}\frac{\omega^2}{gk_\mathrm{h}}+\left(\frac{m-n}{n+1}\right)\frac{gk_\mathrm{h}}{\omega^2}$$

is a constant. Let $\Delta=w(z)\,\mathrm{e}^{k_\mathrm{h}z}$:

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(m+2+2k_\mathrm{h}z)\frac{\mathrm{d}w}{\mathrm{d}z}-(A-m-2)k_\mathrm{h}w=0$$

This is related to the *confluent hypergeometric equation* and has a regular singular point at $z=0$. Using the method of Frobenius, we seek power-series solutions

$$w=\sum_{r=0}^{\infty}a_r z^{\sigma+r},\qquad a_0\neq 0$$

The indicial equation is

$$\sigma(\sigma+m+1)=0$$

and the regular solution has $\sigma=0$. The recurrence relation is then

$$\frac{a_{r+1}}{a_r}=\frac{(A-m-2-2r)k_\mathrm{h}}{(r+1)(r+m+2)}$$

In the case of an infinite series, $a_{r+1}/a_r\sim -2k_\mathrm{h}/r$ as $r\to\infty$, so $w$ behaves like $\mathrm{e}^{-2k_\mathrm{h}z}$ and $\Delta$ diverges like $\mathrm{e}^{-k_\mathrm{h}z}$ as $z\to-\infty$. Solutions in which $\Delta$ decays with depth are those for which the series terminates and $w$ is a polynomial. For a polynomial of degree $\mathcal{N}-1$ ($\mathcal{N}\geqslant 1$),

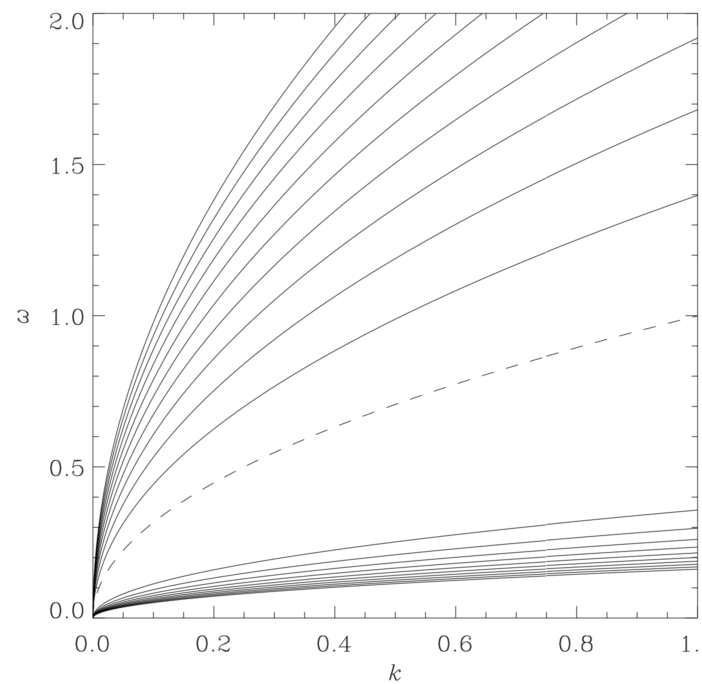
$$A=2\mathcal{N}+m$$



Rearrange:

$$n(m+1)\left(\frac{\omega^2}{gk_\mathrm{h}}\right)^2-(n+1)(2\mathcal{N}+m)\left(\frac{\omega^2}{gk_\mathrm{h}}\right)+(m-n)=0$$

A negative root for $\omega^2$ exists if and only if $m-n<0$, i.e. $N^2<0$, as expected from Schwarzschild's criterion for stability.




Dispersion relation, in arbitrary units, for a stably stratified plane-parallel polytropic atmosphere with $m=3$ and $n=3/2$. The dashed line is the f mode. Above it are the first ten p modes and below it are the first ten g modes.

For $\mathcal{N} \gg 1$, the large root is

$$\frac{\omega^2}{gk_\mathrm{h}} \sim \frac{(n+1)2\mathcal{N}}{n(m+1)} \qquad \text{(p modes, } \omega^2 \propto v_\mathrm{s}^2\text{)}$$

and the small root is

$$\frac{\omega^2}{gk_\mathrm{h}} \sim \frac{m-n}{(n+1)2\mathcal{N}} \qquad \text{(g modes, } \omega^2 \propto N^2\text{)}$$

The f mode is the 'trivial' solution $\Delta = 0$. p modes ('p' for pressure) are *acoustic waves*, which rely on compressibility. g modes are *gravity waves*, which rely on buoyancy.

In solar-type stars the inner part (radiative zone) is convectively stable ($N^2 > 0$) and the outer part (convective zone) is unstable ($N^2 < 0$). However, the convection is so efficient that only a very small entropy gradient is required to sustain the convective heat flux, so $N^2$ is very small and negative in the convective zone. Although g modes propagate in the radiative zone at frequencies smaller than $N$, they cannot reach the surface. Only f and p modes are observed at the solar surface.

In more massive stars the situation is reversed. Then f, p and g modes can be observed, in principle, at the surface. g modes are particularly well observed in certain classes of white dwarf.

## 9.6 Rotating fluid bodies

### 9.6.1 Equilibrium

The equations of ideal gas dynamics in cylindrical polar coordinates are

$$\frac{\mathrm{D}u_R}{\mathrm{D}t} - \frac{u_\phi^2}{R} = -\frac{\partial\Phi}{\partial R} - \frac{1}{\rho}\frac{\partial p}{\partial R}$$

$$\frac{\mathrm{D}u_\phi}{\mathrm{D}t} + \frac{u_R u_\phi}{R} = -\frac{1}{R}\frac{\partial\Phi}{\partial\phi} - \frac{1}{\rho R}\frac{\partial p}{\partial\phi}$$



$$\frac{\mathrm{D}u_z}{\mathrm{D}t} = -\frac{\partial\Phi}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\left[\frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\right]$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p\left[\frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}\right]$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_R\frac{\partial}{\partial R} + \frac{u_\phi}{R}\frac{\partial}{\partial\phi} + u_z\frac{\partial}{\partial z}$$

Consider a steady, axisymmetric basic state with density $\rho(R,z)$, pressure $p(R,z)$, gravitational potential $\Phi(R,z)$ and with differential rotation

$$\boldsymbol{u} = R\Omega(R,z)\,\boldsymbol{e}_\phi$$

For equilibrium we require

$$-R\Omega^2\,\boldsymbol{e}_R = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}p$$

Take the curl to obtain

$$-R\frac{\partial\Omega^2}{\partial z}\,\boldsymbol{e}_\phi = \boldsymbol{\nabla}p\times\boldsymbol{\nabla}\left(\frac{1}{\rho}\right) = \boldsymbol{\nabla}T\times\boldsymbol{\nabla}s$$

This is just the vorticity equation in a steady state. It is sometimes called the *thermal wind equation*. The equilibrium is called *barotropic* if $\boldsymbol{\nabla}p$ is parallel to $\boldsymbol{\nabla}\rho$, otherwise it is called *baroclinic*. In a barotropic state the angular velocity is independent of $z$: $\Omega = \Omega(R)$. This is a version of the *Taylor–Proudman theorem* which states that under certain conditions the velocity in a rotating fluid is independent of height.

We can also write

$$\frac{1}{\rho}\boldsymbol{\nabla}p = \boldsymbol{g} = -\boldsymbol{\nabla}\Phi + R\Omega^2\,\boldsymbol{e}_R$$

where $\boldsymbol{g}$ is the *effective gravitational acceleration*, including the centrifugal force associated with the (non-uniform) rotation.

In a barotropic state with $\Omega(R)$ we can write

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi^{\rm cg}, \qquad \Phi^{\rm cg} = \Phi(R,z) + \Psi(R), \qquad \Psi = -\int R\Omega^2\,{\rm d}R$$

Also, since $p = p(\rho)$ in the equilibrium state, we can define the *pseudo-enthalpy* $\tilde{w}(\rho)$ such that ${\rm d}\tilde{w} = {\rm d}p/\rho$. An example is a polytropic model for which

$$p = K\rho^{1+1/m}, \qquad \tilde{w} = (m+1)K\rho^{1/m}$$

($\tilde{w}$ equals the true enthalpy only if the equilibrium is homentropic.) The equilibrium condition then reduces to

$$\boldsymbol{0} = -\boldsymbol{\nabla}\Phi^{\rm cg} - \boldsymbol{\nabla}\tilde{w}$$

or

$$\Phi + \Psi + \tilde{w} = C = \text{constant} \tag{1}$$

An example of a rapidly and differentially rotating equilibrium is an accretion disc around a central mass $M$. For a non-self-gravitating disc $\Phi = -GM(R^2+z^2)^{-1/2}$. Assume the disc is barotropic and let the arbitrary additive constant in $\tilde{w}$ be defined (as in the polytropic example above) such that $\tilde{w} = 0$ at the surfaces $z = \pm H(R)$ of the disc where $\rho = p = 0$. Then

$$-GM(R^2+H^2)^{-1/2} + \Psi(R) = C$$

from which

$$R\Omega^2 = -\frac{\rm d}{{\rm d}R}\left[GM(R^2+H^2)^{-1/2}\right]$$

For example, if $H = \epsilon R$ with $\epsilon = \text{constant}$,

$$\Omega^2 = (1+\epsilon^2)^{-1/2}\frac{GM}{R^3}$$

The thinner the disc is, the closer it is to Keplerian rotation. Having determined the relation between $\Omega(R)$ and $H(R)$, equation (1) then determines the spatial distribution of $\tilde{w}$ (and therefore of $\rho$ and $p$) within the disc.



### 9.6.2 Linear perturbations

The basic state is independent of $t$ and $\phi$, allowing us to consider linear perturbations of the form

$$\mathrm{Re}\left[\delta u_R(R,z)\exp(-{\rm i}\omega t + {\rm i}m\phi)\right], \qquad \text{etc.}$$

where $m$ is the azimuthal wavenumber (an integer). The linearized equations in the Cowling approximation are

$$-{\rm i}\hat{\omega}\delta u_R - 2\Omega\delta u_\phi = -\frac{1}{\rho}\frac{\partial\delta p}{\partial R} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial R}$$

$$-{\rm i}\hat{\omega}\delta u_\phi + \frac{1}{R}\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}(R^2\Omega) = -\frac{{\rm i}m\delta p}{\rho R}$$

$$-{\rm i}\hat{\omega}\delta u_z = -\frac{1}{\rho}\frac{\partial\delta p}{\partial z} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial z}$$

$$-{\rm i}\hat{\omega}\delta\rho + \delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = -\rho\left[\frac{1}{R}\frac{\partial}{\partial R}(R\delta u_R) + \frac{{\rm i}m\delta u_\phi}{R} + \frac{\partial\delta u_z}{\partial z}\right]$$

$$-{\rm i}\hat{\omega}\delta p + \delta\boldsymbol{u}\cdot\boldsymbol{\nabla}p = -\gamma p\left[\frac{1}{R}\frac{\partial}{\partial R}(R\delta u_R) + \frac{{\rm i}m\delta u_\phi}{R} + \frac{\partial\delta u_z}{\partial z}\right]$$

where

$$\hat{\omega} = \omega - m\Omega$$

is the Doppler-shifted frequency, i.e. the frequency measured in a frame of reference that rotates with the local angular velocity of the fluid.

Eliminate $\delta u_\phi$ and $\delta\rho$ to obtain

$$(\hat{\omega}^2 - A)\delta u_R - B\delta u_z = -\frac{{\rm i}\hat{\omega}}{\rho}\left(\frac{\partial\delta p}{\partial R} - \frac{\partial p}{\partial R}\frac{\delta p}{\gamma p}\right) + 2\Omega\frac{{\rm i}m\delta p}{\rho R}$$

$$-C\delta u_R + (\hat{\omega}^2 - D)\delta u_z = -\frac{{\rm i}\hat{\omega}}{\rho}\left(\frac{\partial\delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right)$$

where

$$A = \frac{2\Omega}{R}\frac{\partial}{\partial R}(R^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial R}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial R} - \frac{1}{\rho}\frac{\partial \rho}{\partial R}\right)$$

$$B = \frac{2\Omega}{R}\frac{\partial}{\partial z}(R^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial R}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}\right)$$

$$C = -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial R} - \frac{1}{\rho}\frac{\partial \rho}{\partial R}\right)$$

$$D = -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}\right)$$

Note that $A$, $B$, $C$ and $D$ involve radial and vertical deriavtives of the specific angular momentum and the specific entropy. The thermal wind equation implies

$$B = C$$

so the matrix

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

is symmetric.

### 9.6.3 The Solberg–Høiland criteria

It can be useful to introduce the Lagrangian displacement $\boldsymbol{\xi}$ such that

$$\Delta \boldsymbol{u} = \delta \boldsymbol{u} + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t},$$

i.e.

$$\delta u_R = -\mathrm{i}\hat{\omega}\xi_R,$$

$$\delta u_\phi = -\mathrm{i}\hat{\omega}\xi_\phi - R\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Omega,$$

$$\delta u_z = -\mathrm{i}\hat{\omega}\xi_z.$$



Note that

$$\frac{1}{R}\frac{\partial}{\partial R}(R\delta u_R) + \frac{im\delta u_\phi}{R} + \frac{\partial \delta u_z}{\partial z} = -\mathrm{i}\hat{\omega}\left[\frac{1}{R}\frac{\partial}{\partial R}(R\xi_R) + \frac{im\xi_\phi}{R} + \frac{\partial \xi_z}{\partial z}\right]$$

The linearized equations constitute an eigenvalue problem for $\omega$ but it is not self-adjoint except when $m = 0$.

We specialize to the case $m = 0$ (axisymmetric perturbations). Then

$$(\omega^2 - A)\xi_R - B\xi_z = \frac{1}{\rho}\left(\frac{\partial \delta p}{\partial R} - \frac{\partial p}{\partial R}\frac{\delta p}{\gamma p}\right)$$

$$-B\xi_R + (\omega^2 - D)\xi_z = \frac{1}{\rho}\left(\frac{\partial \delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right)$$

with

$$\delta p = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p - \gamma p \boldsymbol{\nabla} \cdot \boldsymbol{\xi}$$

Multiply the first equation by $\rho\xi_R^*$ and the second by $\rho\xi_z^*$ and integrate over the volume $V$ of the fluid (using the boundary condition $\delta p = 0$) to obtain

$$\begin{aligned}\omega^2 \int_V \rho(|\xi_R|^2 + |\xi_z|^2)\,\mathrm{d}V &= \int_V \left[\rho Q(\boldsymbol{\xi}) + \boldsymbol{\xi}^* \cdot \boldsymbol{\nabla}\delta p - \frac{\delta p}{\gamma p}\boldsymbol{\xi}^* \cdot \boldsymbol{\nabla} p\right] \mathrm{d}V \\ &= \int_V \left[\rho Q(\boldsymbol{\xi}) - \frac{\delta p}{\gamma p}(\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{\xi}^* + \boldsymbol{\xi}^* \cdot \boldsymbol{\nabla} p)\right] \mathrm{d}V \\ &= \int_V \left(\rho Q(\boldsymbol{\xi}) + \frac{|\delta p|^2}{\gamma p}\right) \mathrm{d}V\end{aligned}$$

where

$$Q(\boldsymbol{\xi}) = A|\xi_R|^2 + B(\xi_R^*\xi_z + \xi_z^*\xi_R) + D|\xi_z|^2 = \begin{bmatrix} \xi_R^* & \xi_z^* \end{bmatrix} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_z \end{bmatrix}$$

is the (real) Hermitian form associated with the matrix $\mathbf{M}$.

Note that this integral involves only the meridional components of the displacement. If we had not made the Cowling approximation there would be the usual negative definite contribution to $\omega^2$ from self-gravitation.

The above integral relation therefore shows that $\omega^2$ is real and a variational property ensures that instability *to axisymmetric perturbations* occurs if and only if the integral on the right-hand side can be made negative by a suitable trial displacement. If $Q$ is positive definite then $\omega^2 > 0$ and we have stability. Now the characteristic equation of the matrix $\mathbf{M}$ is

$$\lambda^2 - (A+D)\lambda + AD - B^2 = 0$$

The eigenvalues $\lambda_\pm$ are both positive if and only if

$$A + D > 0 \qquad \text{and} \qquad AD - B^2 > 0$$

If these conditions are satisfied throughout the fluid then $Q > 0$, which implies $\omega^2 > 0$, so the fluid is stable to axisymmetric perturbations (neglecting self-gravitation). These conditions are also necessary for stability. If one of the eigenvalues is negative in some region in the meridional plane, a trial displacement can be found which is localized in that region, has $\delta p = 0$ and $Q < 0$, implying instability. (By choosing $\boldsymbol{\xi}$ in the correct direction and tuning $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ appropriately, it is possible to arrange for $\delta p$ to vanish.)

Using $\ell = R^2\Omega$ (specific angular momentum) and $s = c_p(\gamma^{-1}\ln p - \ln\rho) +$ constant (specific entropy) for a polytropic ideal gas, we have

$$A = \frac{1}{R^3}\frac{\partial \ell^2}{\partial R} - \frac{g_R}{c_p}\frac{\partial s}{\partial R}$$

$$B = \frac{1}{R^3}\frac{\partial \ell^2}{\partial z} - \frac{g_R}{c_p}\frac{\partial s}{\partial z} = -\frac{g_z}{c_p}\frac{\partial s}{\partial R}$$

$$D = -\frac{g_z}{c_p}\frac{\partial s}{\partial z}$$

so the two conditions become

$$\frac{1}{R^3}\frac{\partial \ell^2}{\partial R} - \frac{1}{c_p}\boldsymbol{g}\cdot\boldsymbol{\nabla}s > 0$$



and

$$-g_z\left(\frac{\partial \ell^2}{\partial R}\frac{\partial s}{\partial z} - \frac{\partial \ell^2}{\partial z}\frac{\partial s}{\partial R}\right) > 0$$

These are the *Solberg–Høiland stability criteria.*

(If the criteria are marginally satisfied a further investigation may be required.)

Consider first the non-rotating case $\ell = 0$. The first criterion reduces to the *Schwarzschild criterion* for convective stability,

$$-\frac{1}{c_p}\boldsymbol{g}\cdot\boldsymbol{\nabla}s \equiv N^2 > 0$$

In the homentropic case $s =$ constant (which is a barotropic model) they reduce to the *Rayleigh criterion* for centrifugal (inertial) stability,

$$\frac{\mathrm{d}\ell^2}{\mathrm{d}R} > 0$$

which states that the specific angular momentum should increase with $R$ for stability.

The second Solberg-Høiland criterion is equivalent to

$$(\boldsymbol{e}_R \times (-\boldsymbol{g}))\cdot(\boldsymbol{\nabla}\ell^2 \times \boldsymbol{\nabla}s) > 0$$

In other words the vectors $\boldsymbol{e}_R \times (-\boldsymbol{g})$ and $\boldsymbol{\nabla}\ell^2 \times \boldsymbol{\nabla}s$ should be parallel (rather than antiparallel). In a rotating star, for stability we require that the specific angular momentum should increase with $R$ on each surface of constant entropy.