**Lecture 10: Vortices in discs**

## 10.1. The vorticity equation

To study the behaviour of vortices in discs we consider a 2D incompressible sheet. Velocity perturbations in the plane of the disc satisfy the nonlinear equations

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \nu\nabla^2 v_x,$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \nu\nabla^2 v_y,$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$

Introduce the *streamfunction* $\chi(x, y, t)$ such that

$$v_x = \frac{\partial\chi}{\partial y}, \qquad v_y = -\frac{\partial\chi}{\partial x}.$$

The instantaneous streamlines are the curves $\chi = \text{constant}$. The vorticity perturbation is

$$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z = (-\nabla^2\chi)\,\mathbf{e}_z = \zeta\,\mathbf{e}_z.$$

Take the curl of the equation of motion to eliminate $\psi$: many terms cancel, leaving the *vorticity equation*

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta, \tag{1}$$

a nonlinear advection–diffusion equation to be solved in conjunction with Poisson's equation

$$\nabla^2\chi = -\zeta.$$

The total *absolute vorticity* is $(2\Omega - S + \zeta)\,\mathbf{e}_z$, with contributions from background rotation, background shear and the velocity perturbation.

Note that the Coriolis force drops out of the 2D incompressible dynamics, so the fact that the sheet is rotating is irrelevant. This model is too constrained to allow epicyclic/inertial oscillations; it involves pure vortex dynamics with background shear.

Multiply equation (1) by $\zeta$ to obtain an equation for the *enstrophy* $\frac{1}{2}\zeta^2$:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{2}\zeta^2\right) &= \nu\zeta\nabla^2\zeta \\ &= \boldsymbol{\nabla}\cdot(\nu\zeta\boldsymbol{\nabla}\zeta) - \nu|\boldsymbol{\nabla}\zeta|^2.\end{aligned}$$

Integrated over an area $A$, assuming suitable boundary conditions, this equation implies that the enstrophy decays:

$$\frac{d}{dt}\int\frac{1}{2}\zeta^2\,dA = -\int\nu|\boldsymbol{\nabla}\zeta|^2\,dA.$$

To maintain vorticity perturbations in the presence of viscosity requires baroclinic or 3D effects, or other source terms.

### 10.2. Zonal flows

Axisymmetric structures in the vorticity correspond to $y$-independent solutions of equation (1). These have $v_x = 0$ and satisfy the diffusion equation

$$\frac{\partial\zeta}{\partial t} = \nu\frac{\partial^2\zeta}{\partial x^2}.$$

They involve a purely *zonal flow* $v_y(x,t)$ ('zonal' = 'azimuthal') and are unaffected by background shear. To the extent that viscosity is negligible, they are equilibrium solutions. They involve a 'geostrophic' balance between the Coriolis force and a radial pressure gradient.

### 10.3. Shearing vortices

Shearing-wave solutions of equation (1),

$$\zeta(\mathbf{x},t) = \mathrm{Re}\left\{\tilde\zeta(t)\,\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$

satisfy the amplitude equation

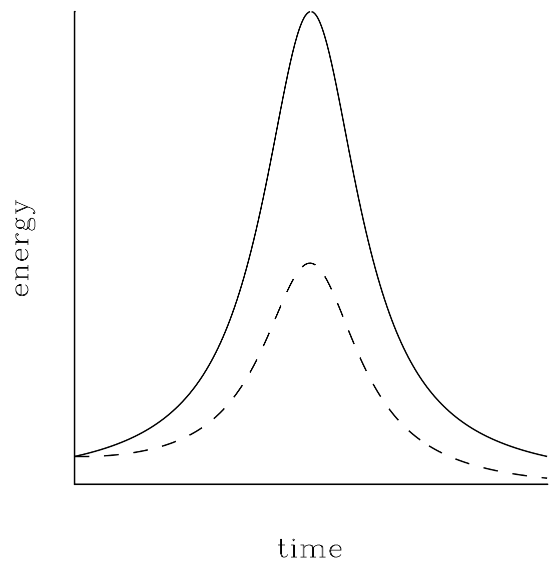
$$\frac{d\tilde\zeta}{dt} = -\nu k^2\tilde\zeta.$$

The nonlinear term $\mathbf{v}\cdot\boldsymbol{\nabla}\zeta$ vanishes because $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ implies $i\mathbf{k}\cdot\tilde{\mathbf{v}} = 0$. So the vorticity amplitude decays viscously:

$$\tilde\zeta \propto E_\nu(t).$$

The kinetic energy can undergo *transient growth*:

$$|\tilde{\mathbf{v}}|^2 \propto k^2|\tilde\chi|^2 \propto k^2\left|\frac{\tilde\zeta}{k^2}\right|^2 \propto \frac{E_\nu^2}{k^2}.$$



**10.4. Elliptical vortex patches**

In the absence of viscosity, the vorticity equation (1) reduces to

$$\frac{D\zeta}{Dt} = 0.$$

Consider a uniform vortex patch defined by a closed contour, inside which $\zeta = \zeta_0$, a non-zero constant, and outside which $\zeta = 0$.

The vorticity perturbation $\zeta$ generates a velocity field $\mathbf{v}$ that, together with the background shear $-Sx\,\mathbf{e}_y$, advects the contour. Do steady solutions exist in which the flow induced by the vortex resists the shear?

Consider an elliptical vortex patch (centred on the origin WLOG), with semi-axes $a$ and $b$, inclined at an angle $\theta$ with respect to $y$ and $x$ axes).

As shown in Example 2.3, the velocity $\mathbf{v}$ induced by $\zeta_0$ causes the ellipse to rotate with angular velocity

$$\dot{\theta} = \frac{ab\zeta_0}{(a+b)^2},$$

while the background shear $-Sx\,\mathbf{e}_y$ deforms the ellipse according to

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta, \qquad \dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2-b^2}.$$

Combine these effects to obtain

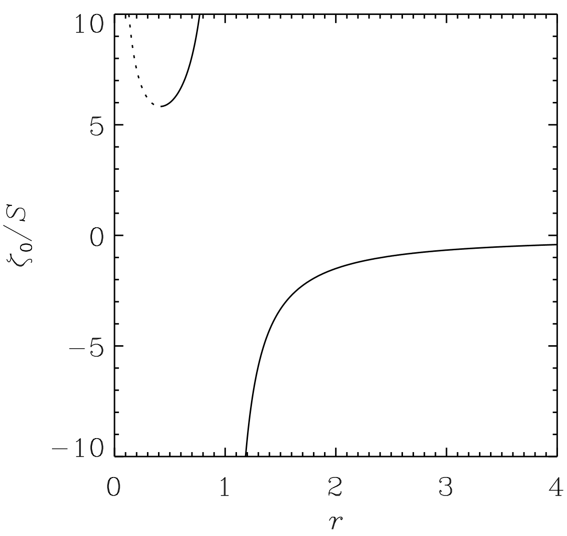
$$\begin{aligned}\frac{\dot{a}}{a} &= -\frac{\dot{b}}{b} = S\sin\theta\cos\theta,\\ \dot{\theta} &= \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2-b^2} + \frac{ab\,\zeta_0}{(a+b)^2}.\end{aligned}$$

Note that the area $\pi ab$ is conserved, as expected. Rewrite in terms of the aspect ratio $r = a/b$:

$$\begin{aligned}\frac{\dot{r}}{r} &= 2S\sin\theta\cos\theta,\\ \dot{\theta} &= \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2}.\end{aligned}$$

Equilibrium solutions representing steady vortices have $\theta = 0$ WLOG (let $r < 1$ if necessary) and

$$\frac{S}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2} = 0 \qquad \Rightarrow \qquad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)} = f(r).$$




In the context of a rotating disc (and assuming $S/\Omega > 0$), vortices are called *cyclonic* if $\zeta_0/S > 0$ (vorticity in the same sense as rotation) and *anticyclonic* if $\zeta_0/S < 0$.

The linearized equations governing the stability of an equilibrium vortex ($\theta = 0$) are

$$\dot{\delta r} = 2Sr\,\delta\theta, \qquad \dot{\delta\theta} = S\frac{\partial g}{\partial r}\,\delta r,$$

where

$$g\left(r, \frac{\zeta_0}{S}\right) = \frac{1}{r^2 - 1} + \frac{r}{(r+1)^2}\frac{\zeta_0}{S}$$

vanishes at equilibrium, where its derivative is (**exercise**)

$$\frac{\partial g}{\partial r} = \frac{2 - (r+1)^2}{r(r^2-1)^2} = -\frac{r}{(r+1)^2}\frac{df}{dr}.$$

So

$$\ddot{\delta r} = 2S^2 r\frac{\partial g}{\partial r}\,\delta r,$$

which implies instability for

$$\frac{\partial g}{\partial r} > 0, \qquad \text{i.e.} \qquad \frac{df}{dr} < 0, \qquad \text{i.e.} \qquad r < \sqrt{2} - 1.$$

Vortices have elliptical streamlines and are susceptible to the *elliptical instability* in 3D. In a Keplerian disc, sufficiently strong anticyclonic vortices with $r < 4$ are vigorously unstable (through violation of a Rayleigh-like criterion). Weaker anticyclonic vortices with $r > 4$ can exist in a Keplerian disc; these tend to have weaker forms of elliptical instability involving resonant destabilization of inertial waves, which may produce turbulent motion without destroying the vortex.

**Exercise:** The velocity field inside the vortex has a linear dependence on the Cartesian coordinates. Show that it has the form (including the contribution from background shear)

$$\mathbf{u} = \frac{S}{r-1}\left(\frac{y}{r}, -rx\right).$$