Lecture 11: Density waves and gravitational instability

11.1. The compressible shearing sheet

Now consider, in the local model, a 2D compressible sheet that is self-gravitating but inviscid. (See Example 2.4 for the effects of viscosity.) The sheet has velocity $\mathbf{u}(x,y,t)$, surface density $\Sigma(x,y,t) = \int \rho\,dz$ and 2D pressure $P(x,y,t) = \int p\,dz$, satisfying the equation of mass conservation,

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla}\cdot(\Sigma\mathbf{u}) = 0,$$

and the equation of motion,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}\Phi_{\mathrm{t,m}} - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}P,$$

where the tidal potential in the midplane is $\Phi_{\mathrm{t,m}} = -\Omega S x^2$, the disc potential $\Phi_{\mathrm{d}}(x,y,z,t)$ satisfies Poisson's equation

$$\nabla^2\Phi_{\mathrm{d}} = 4\pi G\Sigma\,\delta(z),$$

and its value in the midplane is $\Phi_{\mathrm{d,m}}(x,y,t) = \Phi_{\mathrm{d}}(x,y,0,t)$.

To avoid the complications of thermal physics and focus on the dynamics, we assume a barotropic relation $P = P(\Sigma)$.

(These equations are only a model; they cannot be derived exactly from the true 3D equations.)

Poisson's equation can be solved conveniently in the Fourier domain. Let

$$\tilde{\Sigma}(k_x,k_y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Sigma(x,y,t)\,e^{-ik_xx}\,e^{-ik_yy}\,dx\,dy,$$

etc., so that

$$\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{\Phi}_{\mathrm{d}} = 4\pi G\tilde{\Sigma}\,\delta(z),$$

where

$$k = \sqrt{k_x^2 + k_y^2}$$

is the horizontal wavenumber. The relevant solution (decaying as $|z|\to\infty$) for $k\neq 0$ is

$$\tilde{\Phi}_{\mathrm{d}} = -\frac{2\pi G\tilde{\Sigma}}{k}\,e^{-k|z|},$$

so that

$$\left[\frac{\partial\tilde{\Phi}_{\mathrm{d}}}{\partial z}\right]_{0-}^{0+} = 4\pi G\tilde{\Sigma},$$

as required. So

$$\tilde{\Phi}_{\mathrm{d,m}} = -\frac{2\pi G\tilde{\Sigma}}{k}.$$

(The $k=0$ component of the potential gives no horizontal force anyway.)

11.2. Conservation of potential vorticity

Use the vector identity

$$(\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}=\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\left(\tfrac{1}{2}|\mathbf{u}|^2\right)$$

to rewrite the equation of motion as

$$\frac{\partial\mathbf{u}}{\partial t}+[(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}]=-\boldsymbol{\nabla}(\cdots),$$

since $P=P(\Sigma)$. Take the curl:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{u})+\boldsymbol{\nabla}\times[(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}]=\mathbf{0}.$$

Now use the vector identity

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A}-\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}+\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})$$

to obtain (since the problem is 2D)

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})&=-(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})(\boldsymbol{\nabla}\cdot\mathbf{u})\\&=(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\frac{1}{\Sigma}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\Sigma.\end{aligned}$$

Thus

$$\frac{Df}{Dt}=0,$$

where

$$f=\frac{2\Omega+(\boldsymbol{\nabla}\times\mathbf{u})_z}{\Sigma}$$

is the potential vorticity or 'vortensity' (vorticity divided by surface density).

Vortices and zonal flows correspond to coherent structures in this conserved quantity. Unlike the incompressible 2D case, though, vortex dynamics is not the whole story. Vortical disturbances are coupled to acoustic ones, so a vortex can excite waves.

11.3. Linear stability of a uniform 2D self-gravitating sheet

The uniform basic state of the sheet is the solution $\mathbf{u}=-Sx\,\mathbf{e}_y$, $\Sigma=$ constant, $P=$ constant.

The linearized equations for small perturbations $\mathbf{v}$, Σ', etc., are

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-Sx\frac{\partial}{\partial y}\right)\Sigma'+\Sigma\,\boldsymbol{\nabla}\cdot\mathbf{v}=0,\\&\left(\frac{\partial}{\partial t}-Sx\frac{\partial}{\partial y}\right)\mathbf{v}-Sv_x\,\mathbf{e}_y+2\boldsymbol{\Omega}\times\mathbf{v}=-\boldsymbol{\nabla}\Phi'_{\mathrm{d,m}}-\frac{1}{\Sigma}\boldsymbol{\nabla}P',\\&\nabla^2\Phi'_{\mathrm{d}}=4\pi G\Sigma'\,\delta(z),\end{aligned}$$

with

$$P'=\frac{dP}{d\Sigma}\Sigma'=v_{\mathrm{s}}^2\Sigma',$$

where $v_{\mathrm{s}}=$ constant is the (adiabatic) sound speed of the basic state.

The solutions are shearing waves:

$$\Sigma'(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\Sigma}'(t)\,\exp\left[i\mathbf{k}(t)\cdot\mathbf{x}\right]\right\},$$

etc., satisfying the amplitude equations

$$\begin{aligned}
&\frac{d\tilde{\Sigma}'}{dt} + \Sigma\, i\mathbf{k}\cdot\tilde{\mathbf{v}} = 0,\\
&\frac{d\tilde{v}_x}{dt} - 2\Omega\tilde{v}_y = -ik_x\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right),\\
&\frac{d\tilde{v}_y}{dt} + (2\Omega - S)\tilde{v}_x = -ik_y\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_{\mathrm{s}}^2\frac{\tilde{\Sigma}'}{\Sigma}\right),\\
&\tilde{\Phi}'_{\mathrm{d,m}} = -\frac{2\pi G\tilde{\Sigma}'}{k}.
\end{aligned}$$

The potential vorticity perturbation

$$\tilde{f}' = \frac{ik_x\tilde{v}_y - ik_y\tilde{v}_x}{\Sigma} - \frac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2}$$

satisfies $d\tilde{f}'/dt$, as expected (**exercise**).

Consider axisymmetric waves: $k_y = 0$, $k_x =$ constant, $k = |k_x|$. Then the equations have constant coefficients, so we can assume the amplitudes are $\propto e^{-i\omega t}$:

$$\begin{aligned}
&-i\omega\tilde{\Sigma}' + \Sigma\, ik_x\tilde{v}_x = 0,\\
&-i\omega\tilde{v}_x - 2\Omega\tilde{v}_y = -ik_x\left(v_{\mathrm{s}}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\frac{\tilde{\Sigma}'}{\Sigma},\\
&-i\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x = 0.
\end{aligned}$$

Multiply the second equation by $i\omega$ and eliminate $\tilde{\Sigma}'$ and $\tilde{v}_y$:

$$\omega^2\tilde{v}_x - 2\Omega(2\Omega - S)\tilde{v}_x = k_x^2\left(v_{\mathrm{s}}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\tilde{v}_x.$$

We deduce the dispersion relation for *density waves*:

$$\omega^2 = \Omega_r^2 - 2\pi G\Sigma|k_x| + v_{\mathrm{s}}^2 k_x^2. \qquad (1)$$

There is also a time-independent ($\omega = 0$) vortical solution ($\tilde{f}' \neq 0$) with $\tilde{v}_x = 0$. This involves a sinusoidal azimuthal velocity perturbation $v_y(x)$ giving rise to a Coriolis force that is balanced by pressure and self-gravity: an example of a zonal flow.

The dispersion relation (1) has positive, stabilizing contributions from inertial forces (Ω_r^2) and acoustic forces ($v_{\mathrm{s}}^2 k_x^2$), and a negative, destabilizing contribution from self-gravity. It describes 'acoustic–inertial waves' that can potentially be destabilized by self-gravity.

The disc is unstable to axisymmetric disturbances if $\omega^2 < 0$ for some k_x. ω^2 is minimized wrt $|k_x|$ when

$$0 = -2\pi G\Sigma + 2v_\mathrm{s}^2|k_x| \qquad \Rightarrow \qquad |k_x| = \frac{\pi G\Sigma}{v_\mathrm{s}^2},$$

so

$$(\omega^2)_\mathrm{min} = \Omega_r^2 - \frac{(\pi G\Sigma)^2}{v_\mathrm{s}^2} = \Omega_r^2\left(1 - \frac{1}{Q^2}\right),$$

where

$$Q = \frac{v_\mathrm{s}\Omega_r}{\pi G\Sigma}$$

is the *Toomre stability parameter*. We have *gravitational instability* (GI) if $Q < 1$.

The definition of Q involves the product of the stabilizing influences (acoustic and inertial restoring forces) divided by a measure of the destabilizing influence (self-gravity).

Dispersion relations for density waves: