Lecture 13: Magnetic fields in discs

13.1. Equations of magnetohydrodynamics (MHD)

To examine the role of magnetic fields in astrophysical discs, we simplify the problem by considering a homogeneous, incompressible, inviscid fluid of uniform density ρ and electrical conductivity σ. We work in rationalized units. To convert the magnetic field $\mathbf{B}$ from rationalized to Gaussian units, multiply by $\sqrt{\mu_0/4\pi}$.

The *induction equation*,

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{B},$$

describes the advection of the magnetic field $\mathbf{B}$ by the fluid flow, together with its diffusion due to resistivity. The *magnetic diffusivity* is

$$\eta = \frac{1}{\mu_0\sigma}.$$

The induction equation comes from Maxwell's equations without the displacement current,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J},$$

together with Ohm's Law for a moving medium,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}).$$

The *Lorentz force* per unit volume is the divergence of the *Maxwell stress tensor*

$$\mathsf{M} = \frac{\mathbf{B}\mathbf{B}}{\mu_0} - \frac{|\mathbf{B}|^2}{2\mu_0}\mathtt{I},$$

of which the first term represents a *magnetic tension* in the field lines and the second term represents an isotropic *magnetic pressure*. Since $\nabla\cdot\mathbf{B} = 0$, we have

$$\nabla\cdot\mathsf{M} = \frac{1}{\mu_0}\mathbf{B}\cdot\nabla\mathbf{B} - \nabla\left(\frac{|\mathbf{B}|^2}{2\mu_0}\right).$$

The equation of motion including the Lorentz force is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\Phi - \frac{1}{\rho}\nabla\Pi + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\nabla\mathbf{B},$$

where

$$\Pi = p + \frac{|\mathbf{B}|^2}{2\mu_0}$$

is the total (gas plus magnetic) pressure.

The incompressibility and solenoidal conditions are

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0.$$

In the context of the local model of astrophysical discs, we add the Coriolis and centrifugal forces to obtain the equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} \Phi_{\mathrm{t}} - \frac{1}{\rho} \boldsymbol{\nabla} \Pi + \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B}.$$

Rotation does not affect the induction equation.

13.2. Horizontally invariant solutions

Write $\mathbf{u} = -Sx\,\mathbf{e}_y + \mathbf{v}$, where $\mathbf{v}$ is the departure from orbital motion. Look for horizontally invariant solutions in which $\mathbf{v}$, $\mathbf{B}$ and Π are independent of x and y. Then $\partial v_z/\partial z = \partial B_z/\partial z = 0$ and

$$\begin{aligned}
\frac{Dv_x}{Dt} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_x}{\partial z}, \\
\frac{Dv_y}{Dt} + (2\Omega - S) v_x &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_y}{\partial z}, \\
\frac{\partial v_z}{\partial t} &= -\Omega_z^2 z - \frac{1}{\rho} \frac{\partial \Pi}{\partial z}, \\
\frac{DB_x}{Dt} &= B_z \frac{\partial v_x}{\partial z} + \eta \frac{\partial^2 B_x}{\partial z^2}, \\
\frac{DB_y}{Dt} &= -SB_x + B_z \frac{\partial v_y}{\partial z} + \eta \frac{\partial^2 B_y}{\partial z^2}, \\
\frac{\partial B_z}{\partial t} &= 0,
\end{aligned}$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}.$$

In this model, $B_z =$ constant is a conserved uniform vertical magnetic flux passing through the disc.

Assume that the disc has upper and lower surfaces $z = z_\pm(t)$, with full thickness $z^+ - z^- = \Sigma/\rho =$ constant. Above and below the disc, assume that $\rho = 0$ and magnetic field is force-free: $B_x = B_x^\pm =$ constant and $B_y = B_y^\pm$, respectively. The inclinations $B_{x,y}^\pm/B_z$ are determined by global considerations.

Exercise: Integrate the vertical equation of motion over the disc, with the boundary condition $p = 0$ at $z = z^\pm$, to obtain

$$\ddot{Z} = -\Omega_z^2 Z - \frac{(\Pi^+ - \Pi^-)}{\Sigma},$$

where $Z = (z^+ + z^-)/2$ is the height of the centre of mass and $\Pi^\pm = |\mathbf{B}^+|^2/2\mu_0$. This equation allows a free oscillation about an equilibrium position (which is $Z = 0$ if $\Pi^+ = \Pi^-$).

Assume that the vertical oscillation is absent so that $v_z = 0$. Then the equations of the model reduce to a linear problem:

$$\begin{aligned}
\frac{\partial v_x}{\partial t} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_x}{\partial z},\\
\frac{\partial v_y}{\partial t} + (2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_y}{\partial z},\\
\frac{\partial B_x}{\partial t} &= B_z\frac{\partial v_x}{\partial z} + \eta\frac{\partial^2 B_x}{\partial z^2},\\
\frac{\partial B_y}{\partial t} + SB_x &= B_z\frac{\partial v_y}{\partial z} + \eta\frac{\partial^2 B_y}{\partial z^2}.
\end{aligned}$$

13.3. Equilibrium solutions

In a steady state

$$\begin{aligned}
-2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{dB_x}{dz},\\
(2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{dB_y}{dz},\\
0 &= B_z\frac{dv_x}{dz} + \eta\frac{d^2 B_x}{dz^2},\\
SB_x &= B_z\frac{dv_y}{dz} + \eta\frac{d^2 B_y}{dz^2}.
\end{aligned}$$

Eliminate variables in favour of B_x to obtain (**exercise**)

$$\frac{d^2 B_x}{dz^2} + K^2 B_x = 0, \qquad K^2 = \frac{2\Omega S v_{\mathrm{az}}^2}{v_{\mathrm{az}}^4 + \eta^2\Omega_r^2},$$

where

$$v_{\mathrm{az}} = \frac{B_z}{\sqrt{\mu_0 \rho}}$$

is the vertical *Alfvén velocity*. The solution is a combination of $\sin(Kz)$ and $\cos(Kz)$.

The situation that is usually considered involves symmetrical boundary conditions: $B_x = \pm B_x^+$ and $B_y = \pm B_y^+$ at $z = \pm z^+$. The relevant solution is

$$\begin{aligned}
B_x &= B_x^+ \frac{\sin(Kz)}{\sin(Kz^+)},\\
B_y &= B_y^+ \frac{z}{z^+} - \frac{(2\Omega - S)\eta}{v_{\mathrm{az}}^2} B_x^+ \left[\frac{\sin(Kz)}{\sin(Kz^+)} - \frac{z}{z^+}\right],\\
v_x &= \frac{v_{\mathrm{az}}^2}{(2\Omega - S)z^+}\frac{B_y^+}{B_z} - \eta\frac{B_x^+}{B_z}\left[\frac{K\cos(Kz)}{\sin(Kz^+)} - \frac{1}{z^+}\right],\\
v_y &= -\frac{v_{\mathrm{az}}^2}{2\Omega}\frac{B_x^+}{B_z}\frac{K\cos(Kz)}{\sin(Kz^+)}.
\end{aligned}$$

The meridional (x and z) components of $\mathbf{B}$ are called the *poloidal magnetic field*, while the azimuthal (y) component is called the *toroidal magnetic field*.

The poloidal magnetic field bends in the xz plane to match the boundary conditions. The shape is close to a parabola if $Kz^+ \ll 1$, i.e. in the limit of a strong field and/or a high resistivity; otherwise the shape is more 'bendy'.

There is a z-dependent departure from the orbital motion (v_y) with the Coriolis and Lorentz forces balancing. We have *isorotation* (i.e. constant angular velocity) along field lines if $\eta = 0$. More generally, the ratio of the two terms on the RHS of the y-component of the induction equation is $\eta^2\Omega_r^2/v_{\mathrm{a}z}^4$.

There is a mean radial velocity (i.e. an accretion flow) if a non-zero magnetic torque $\propto B_y^+ B_z$ acts on the disc to remove (or add) angular momentum. (This could result either from a magnetized outflow or from a magnetic connection to an external object rotating at a different rate.)

Above the disc, assuming very low density, we have

$$B_x = B_x^+, \qquad B_y = B_y^+, \qquad v_x = \text{constant}, \qquad v_y = \frac{B_x^+}{B_z} Sz + \text{constant}.$$

The uniform, force-free magnetic field acts as a rigid channel for the gas. The net acceleration parallel to $\mathbf{B}$ due to inertial forces and gravity is proportional to

$$\frac{B_x^+}{B_z} 2\Omega v_y - \frac{B_y^+}{B_z}(2\Omega - S)v_x - \Omega_z^2 z = \left(\frac{B_x^+}{B_z}\right)^2 2\Omega S z - \Omega_z^2 z + \text{constant}.$$

If the field is sufficiently inclined to the vertical, i.e. if

$$\left(\frac{B_x^+}{B_z}\right)^2 > \frac{\Omega_z^2}{2\Omega S},$$

then this net acceleration increases with z and will become positive at some height above the disc. A hydrostatic solution is then impossible and an outflow (jet or wind) is launched along the field lines. This is known as *magnetocentrifugal acceleration*. For a Keplerian disc ($\Omega_z = \Omega$, $S/\Omega = 3/2$), it requires $i > 30°$, where $i = \arctan|B_x^+/B_z|$ is the inclination.