Lecture 14: The magnetorotational instability

14.1. Stability analysis

We return to the equations

$$\frac{\partial v_x}{\partial t} - 2\Omega v_y = \frac{B_z}{\mu_0\rho}\frac{\partial B_x}{\partial z},$$
$$\frac{\partial v_y}{\partial t} + (2\Omega - S)v_x = \frac{B_z}{\mu_0\rho}\frac{\partial B_y}{\partial z},$$
$$\frac{\partial B_x}{\partial t} = B_z\frac{\partial v_x}{\partial z} + \eta\frac{\partial^2 B_x}{\partial z^2},$$
$$\frac{\partial B_y}{\partial t} + SB_x = B_z\frac{\partial v_y}{\partial z} + \eta\frac{\partial^2 B_y}{\partial z^2}$$

describing the time-dependent vertical structure of a magnetized disc.

Consider perturbations to an equilibrium state, of the form

$$\delta v_x = \mathrm{Re}\left(\tilde{v}_x\, e^{\lambda t + ikz}\right),$$

etc., where λ is the (possibly complex) growth rate and k is the (real) vertical wavenumber. The equations require

$$\lambda\tilde{v}_x - 2\Omega\tilde{v}_y = \frac{ikB_z}{\mu_0\rho}\tilde{B}_x,$$
$$\lambda\tilde{v}_y + (2\Omega - S)\tilde{v}_x = \frac{ikB_z}{\mu_0\rho}\tilde{B}_y,$$
$$(\lambda + \eta k^2)\tilde{B}_x = ikB_z\tilde{v}_x,$$
$$(\lambda + \eta k^2)\tilde{B}_y + S\tilde{B}_x = ikB_z\tilde{v}_y.$$

Multiply the first two equations by ikB_z and use the last two to substitute for $\tilde{v}_x$ and $\tilde{v}_y$:

$$\lambda\lambda_\eta\tilde{B}_x - 2\Omega\left(\lambda_\eta\tilde{B}_y + S\tilde{B}_x\right) = -\omega_\mathrm{a}^2\tilde{B}_x,$$
$$\lambda\left(\lambda_\eta\tilde{B}_y + S\tilde{B}_x\right) + (2\Omega - S)\lambda_\eta\tilde{B}_x = -\omega_\mathrm{a}^2\tilde{B}_y,$$

where $\lambda_\eta = \lambda + \eta k^2$ and the *Alfvén frequency* is

$$\omega_\mathrm{a} = \mathbf{k}\cdot\mathbf{v}_\mathrm{a} = \frac{kB_z}{\sqrt{\mu_0\rho}}.$$

Algebraic elimination leads to the *magnetorotational dispersion relation*

$$(\lambda\lambda_\eta + \omega_\mathrm{a}^2)^2 + \Omega_r^2\lambda_\eta^2 - 2\Omega S\omega_\mathrm{a}^2 = 0.$$

This gives marginal stability ($\lambda = 0$) for $k^2 = K^2$ (K being the equilibrium wavenumber) and instability for $k^2 < K^2$. To prove this we use the *Routh–Hurwitz stability criteria*: the roots of the real quartic polynomial

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0$$

all have $\mathrm{Re}(\lambda) < 0$ if and only if

$$a, b, c, d > 0 \qquad \text{and} \qquad abc - a^2 d - c^2 > 0.$$

In our case we have

$$a = 2\eta k^2, \qquad b = 2\omega_\mathrm{a}^2 + (\eta k^2)^2 + \Omega_r^2, \qquad c = 2(\omega_\mathrm{a}^2 + \Omega_r^2)\eta k^2, \qquad d = \omega_\mathrm{a}^4 + \Omega_r^2(\eta k^2)^2 - 2\Omega S \omega_\mathrm{a}^2,$$

giving (**exercise**)

$$abc - a^2 d - c^2 = (2\eta k^2)^2 \omega_\mathrm{a}^2 \left[(\eta k^2)^2 + 4\Omega^2\right] > 0.$$

Assume that $\Omega_r^2 > 0$; otherwise we already have orbital and hydrodynamic instability. Then all of the Routh–Hurwitz stability criteria are satisfied, except possibly the criterion $d > 0$. But

$$\begin{aligned} d &= \left(v_{\mathrm{a}z}^4 + \eta^2 \Omega_r^2\right) k^4 - 2\Omega S v_{\mathrm{a}z}^2 k^2 \\ &= \left(v_{\mathrm{a}z}^4 + \eta^2 \Omega_r^2\right) k^2 (k^2 - K^2), \end{aligned}$$

and we have *magnetorotational instability* (MRI) for $d < 0$.

The usual boundary conditions for normal modes are that $\delta B_x = \delta B_y = 0$ at $z = \pm z^+$. This gives solutions involving $\sin(kz)$ or $\cos(kz)$ (note that the dispersion relation is even in k, so $e^{\pm ikz}$ can be combined), with the quantization

$$k = \frac{n\pi}{2z^+}, \qquad n = 1, 2, 3, \dots.$$

The $n = 1$ mode has the lowest value of k and therefore determines the overall stability of the equilibrium, although it may not be the fastest-growing mode.

14.2. The ideal MRI

For ideal MHD (the perfectly conducting limit of zero resistivity) we set $\eta = 0$, in which case $\lambda_\eta = \lambda$. The dispersion relation is then a quadratic for λ^2:

$$\lambda^4 + (2\omega_\mathrm{a}^2 + \Omega_r^2)\lambda^2 + \omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2\Omega S) = 0,$$

with discriminant

$$\begin{aligned} (2\omega_\mathrm{a}^2 + \Omega_r^2)^2 - 4\omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2\Omega S) &= \Omega_r^4 + 4\omega_\mathrm{a}^2(\Omega_r^2 + 2\Omega S) \\ &= \Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2 \end{aligned}$$

and roots

$$\lambda^2 = -\omega_\mathrm{a}^2 + \frac{1}{2}\left(-\Omega_r^2 \pm \sqrt{\Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2}\right).$$

Assume again that $\Omega_r^2 > 0$. The + root is maximized wrt ω_a^2 when

$$0 = \frac{\partial \lambda^2}{\partial \omega_\mathrm{a}^2} = -1 + \frac{4\Omega^2}{\sqrt{\cdots}}$$

$$\sqrt{\cdots} = 4\Omega^2$$

$$\Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2 = 16\Omega^4$$

$$\omega_\mathrm{a}^2 = \Omega^2 - \frac{\Omega_r^4}{16\Omega^2},$$

giving

$$
\begin{aligned}
(\lambda^2)_{\max} &= -\Omega^2 + \frac{\Omega_r^4}{16\Omega^2} + \frac{1}{2}(-\Omega_r^2 + 4\Omega^2) \\
&= \Omega^2 \left(1 - \frac{\Omega_r^2}{4\Omega^2}\right)^2 \\
&= \Omega^2 \left(\frac{2\Omega S}{4\Omega^2}\right)^2 \\
&= \left(\frac{S}{2}\right)^2 .
\end{aligned}
$$

So the maximum growth rate is half the orbital shear rate, *independent of the magnetic field*. The weaker the field is, the shorter the wavelength of the fastest-growing mode, to achieve $\omega_{\mathrm{a}} \sim \Omega$.

Exercise: Show that the fastest-growing mode has $\delta v_x = \delta v_y$ and $\delta B_x = -\delta B_y$, which maximizes the correlations leading to outward angular momentum transport:

$$
-T_{xy} = \rho\, \delta v_x\, \delta v_y - \frac{\delta B_x\, \delta B_y}{\mu_0}.
$$

This shear stress also extracts energy from the orbital shear, allowing the perturbation to grow.

For ideal instability we require

$$
\omega_{\mathrm{a}}^2(\omega_{\mathrm{a}}^2 - 2\Omega S) < 0.
$$

Crucially, this condition is satisfied for a Keplerian disc, provided that the field is not too strong, whereas the Rayleigh criterion for hydrodynamic instability, $4\Omega^2 - 2\Omega S < 0$, is not.

The $n = 1$ mode, with wavenumber $k = \pi/2z^+$, is the last to be stabilized as B_z is increased. The disc is unstable for

$$
0 < \frac{v_{\mathrm{a}z}}{\Omega z^+} < \frac{2\sqrt{2q}}{\pi} \qquad (\approx 1.1 \text{ for a Keplerian disc}).
$$

14.3. Nonlinear outcome

The MRI typically develops into sustained MHD turbulence in discs that are sufficiently ionized (η small enough) and not very strongly magnetized. It leads to outward angular-momentum transport with typically $\alpha \lesssim 0.1$, depending on the field strength and the degree of ionization.

In the absence of a large-scale, imposed magnetic field, it is thought that the MRI can act as a *dynamo*, in which the turbulent motions due to the instability sustain the magnetic field against Ohmic dissipation. Whether this dynamo can operate at the very low viscosities found in astrophysical discs is an open question.