Lecture 2: Orbital dynamics

2.1. Orbits in an axisymmetric potential

Consider the motion, according to Newtonian dynamics, of a test particle in the gravitational potential Φ of a star, planet, black hole, galaxy, etc. Use cylindrical polar coordinates (r, ϕ, z) (called *radial*, *azimuthal* and *vertical*).

Assume that Φ is *axisymmetric* and *reflectionally symmetric*:

$$\Phi = \Phi(r, z), \qquad \Phi(r, -z) = \Phi(r, z).$$

An important special case is the potential of a point mass M,

$$\Phi = -\frac{GM}{\sqrt{r^2 + z^2}}.$$

In this case the test particle follows a Keplerian orbit.

Lagrange's equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i},$$

where q_i are the generalized coordinates of the particle. The Lagrangian for a particle of unit mass is

$$L = T - V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z).$$

Two conserved quantities are the *specific angular momentum*,

$$h = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi},$$

and the *specific energy*,

$$\varepsilon = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = T + V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \Phi(r, z).$$

The radial and vertical equations of motion are

$$\ddot{r} = r\dot{\phi}^2 - \Phi_r, \qquad \ddot{z} = -\Phi_z,$$

where the subscripts on Φ denote partial derivatives. These are equivalent to

$$\ddot{r} = -\Phi^{\text{eff}}_r, \qquad \ddot{z} = -\Phi^{\text{eff}}_z,$$

with the *effective potential*

$$\Phi^{\text{eff}} = \frac{h^2}{2r^2} + \Phi.$$

Note that

$$\varepsilon = \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \Phi^{\text{eff}}.$$

Consider the family of circular orbits ($r = \text{constant}$) in the midplane ($z = 0$). These must satisfy

$$\begin{aligned} 0 &= -\Phi^{\text{eff}}_r(r,0) = \frac{h^2}{r^3} - \Phi_r(r,0), \\ 0 &= -\Phi^{\text{eff}}_z(r,0) \qquad (\checkmark \text{ by reflectional symmetry}). \end{aligned}$$

The specific angular momentum $h_\text{c}(r)$, angular velocity $\Omega_\text{c}(r)$ and specific energy $\varepsilon_\text{c}(r)$ of the circular orbits are therefore given by (assuming $\Phi_r(r,0) > 0$ and considering prograde orbits)

$$h_\text{c} = \sqrt{r^3 \Phi_r(r,0)}, \qquad \Omega_\text{c} = \frac{h_\text{c}}{r^2}, \qquad \varepsilon_\text{c} = \frac{h_\text{c}^2}{2r^2} + \Phi(r,0).$$

They satisfy the relation

$$\frac{d\varepsilon_\text{c}}{dh_\text{c}} = \Omega_\text{c}.$$

Proof:

$$\frac{d\varepsilon_\text{c}}{dr} = \frac{h_\text{c}}{r^2}\frac{dh_\text{c}}{dr} - \frac{h_\text{c}^2}{r^3} + \Phi_r(r,0) = \Omega_\text{c}\frac{dh_\text{c}}{dr}.$$

The *orbital shear rate* $S(r)$ and the dimensionless *orbital shear parameter* $q(r)$ are defined by

$$S = -r\frac{d\Omega_\text{c}}{dr}, \qquad q = -\frac{d\ln\Omega_\text{c}}{d\ln r} = \frac{S}{\Omega_\text{c}}.$$

In the case of a point-mass potential, the circular Keplerian orbits satisfy

$$\Phi(r,0) = -\frac{GM}{r}, \qquad h_\text{c} = \sqrt{GMr}, \qquad \Omega_\text{c} = \sqrt{\frac{GM}{r^3}}, \qquad \varepsilon_\text{c} = -\frac{GM}{2r}, \qquad S = \frac{3}{2}\Omega_\text{c}, \qquad q = \frac{3}{2}.$$

(See Example 1.1 for a revision of Keplerian orbits.)

2.2. Oscillations and precession

Small departures from a circular orbit of radius r in the midplane satisfy

$$\ddot{\delta r} = -\Omega_r^2\,\delta r, \qquad \ddot{\delta z} = -\Omega_z^2\,\delta z,$$

with

$$\Omega_r^2 = \Phi^{\text{eff}}_{rr}(r,0), \qquad \Omega_z^2 = \Phi^{\text{eff}}_{zz}(r,0),$$

defining the *radial frequency* $\Omega_r(r)$ and the *vertical frequency* $\Omega_z(r)$. (The radial frequency is more often called the *epicyclic frequency* and denoted κ. The vertical frequency is sometimes denoted ν. Note that $\Phi^{\text{eff}}_{rz}(r,0) = 0$ by reflectional symmetry.)

The circular orbit is stable if $\Omega_r^2 > 0$ and $\Omega_z^2 > 0$, i.e. if the orbit minimizes ε for a given h.

We have

$$\begin{aligned} \Omega_r^2 &= \frac{3h_\text{c}^2}{r^4} + \Phi_{rr}(r,0) \\ &= \frac{3h_\text{c}^2}{r^4} + \frac{d}{dr}\left(\frac{h_\text{c}^2}{r^3}\right) \\ &= \frac{1}{r^3}\frac{dh_\text{c}^2}{dr} \\ &= 4\Omega_\text{c}^2 + 2r\Omega_\text{c}\frac{d\Omega_\text{c}}{dr} \\ &= 2\Omega_\text{c}(2\Omega_\text{c} - S) \\ &= 2(2-q)\Omega_\text{c}^2, \\ \Omega_z^2 &= \Phi_{zz}(r,0). \end{aligned}$$

Keplerian orbits satisfy

$$\Omega_r = \Omega_z = \Omega,$$

meaning that (slightly) eccentric or inclined orbits close after one turn.

[FIGURE]

If $\Omega_r \approx \Omega$, an eccentric orbit precesses slowly. The minimum radius (*periapsis*) occurs at time intervals $\Delta t = 2\pi/\Omega_r$, corresponding to

$$\begin{aligned}\Delta\phi &= \frac{2\pi\Omega}{\Omega_r} \\ &= 2\pi\left(\frac{\Omega}{\Omega_r} - 1\right) + 2\pi \\ &= 2\pi\left(\frac{\Omega}{\Omega_r} - 1\right) \bmod 2\pi.\end{aligned}$$

The *apsidal precession rate* is therefore

$$\frac{\Delta\phi}{\Delta t} = \Omega - \Omega_r.$$

Similarly, if $\Omega_z \approx \Omega$, an inclined orbit precesses slowly with *nodal precession rate*

$$\Omega - \Omega_z.$$

(See Example 1.2 for precession of orbits in binary stars and around black holes.)

2.3. Mechanics of accretion

Consider two particles in circular orbits in the midplane. Can energy be released by a conservative exchange of angular momentum between the particles?

The total angular momentum and energy are

$$\begin{aligned}H &= H_1 + H_2 = m_1 h_1 + m_2 h_2, \\ E &= E_1 + E_2 = m_1 \varepsilon_1 + m_2 \varepsilon_2.\end{aligned}$$

In an infinitesimal exchange:

$$\begin{aligned}dH &= dH_1 + dH_2 = m_1\, dh_1 + m_2\, dh_2, \\ dE &= dE_1 + dE_2 = m_1\Omega_1\, dh_1 + m_2\Omega_2\, dh_2,\end{aligned}$$

If $dH = 0$ then

$$dE = (\Omega_1 - \Omega_2)\, dH_1.$$

So energy is released by transferring angular momentum from higher to lower angular velocity. In practice $d\Omega/dr < 0$, so this means an *outward transfer of angular momentum.*

Now generalize the argument to allow for an exchange of mass:

$$\begin{aligned}dM &= dm_1 + dm_2 = 0, \\ dH &= dH_1 + dH_2 = 0, \qquad dH_i = m_i\, dh_i + h_i\, dm_i, \\ dE_i &= m_i\Omega_i\, dh_i + \varepsilon_i\, dm_i \\ &= \Omega_i\, dH_i + (\varepsilon_i - h_i\Omega_i)\, dm_i, \\ dE &= (\Omega_1 - \Omega_2)\, dH_1 + \left[(\varepsilon_1 - h_1\Omega_1) - (\varepsilon_2 - h_2\Omega_2)\right]\, dm_1.\end{aligned}$$

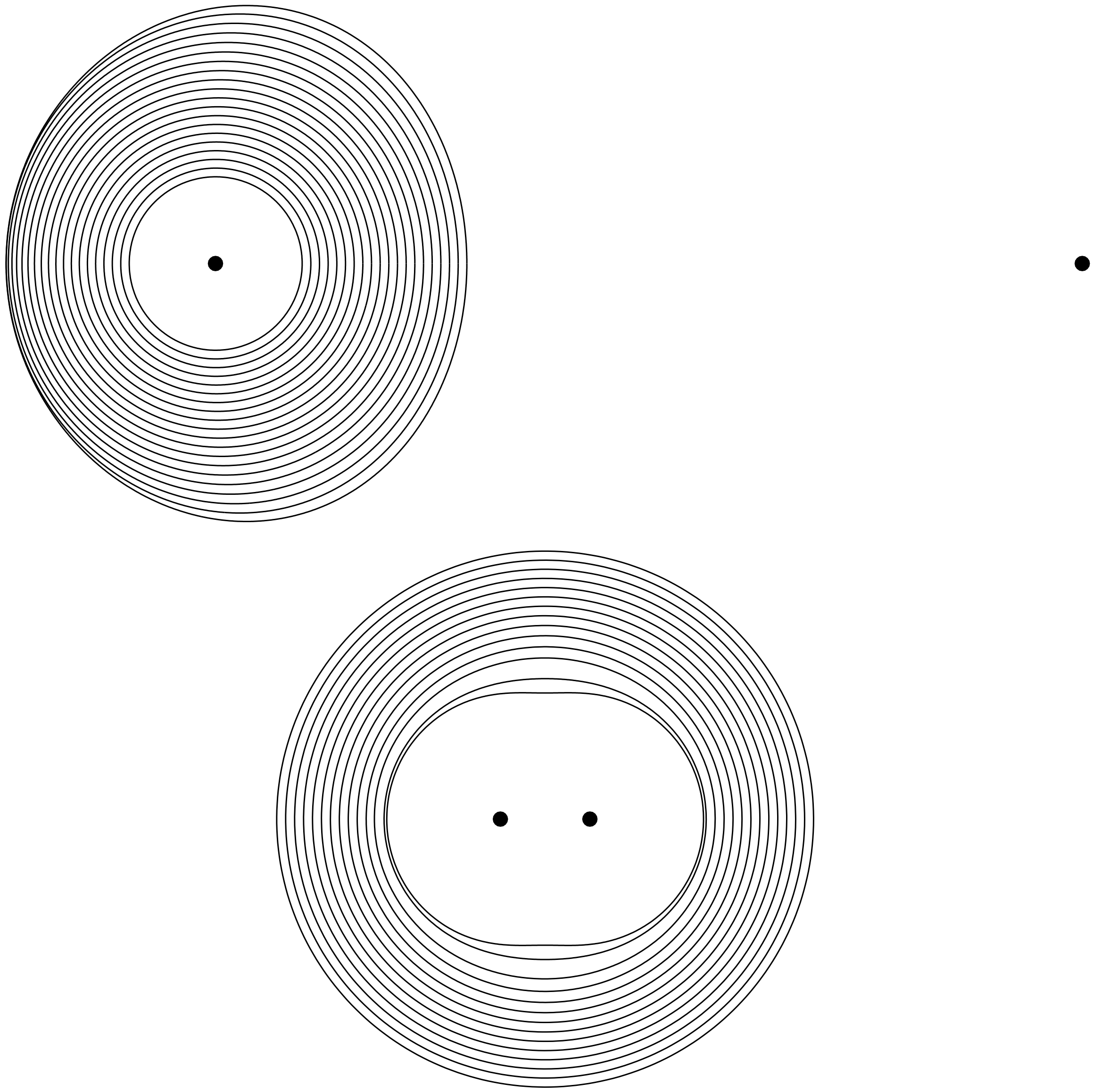
In practice

$$\frac{d}{dr}(\varepsilon - h\Omega) = -h\,\frac{d\Omega}{dr} > 0,$$

so energy is released by an *outward transfer of angular momentum* and an *inward transfer of mass.* This is the physical basis of an accretion disc.

[FIGURE]

2.4. Departures from Keplerian rotation



Families of prograde circumstellar (top) and circumbinary (bottom) periodic orbits of the restricted three-body problem for an equal-mass, circular binary. Orbits that are too large (top) or small (bottom) depart sufficiently from circular Keplerian orbits that they intersect their neighbours.

Exercise: Accretion on to a non-rotating black hole can be modelled using the potential $\Phi = -GM/(R - r_\mathrm{h})$, where $R = \sqrt{r^2 + z^2}$ is the spherical radius and $r_\mathrm{h} = 2GM/c^2$ is the (Schwarzschild) radius of the event horizon of the black hole. Calculate $\Omega_\mathrm{c}(r)$ and compare with the Keplerian angular velocity. Show that $h_\mathrm{c}(r)$ has a minimum at $r = 3r_\mathrm{h}$ and deduce that circular orbits in this potential are unstable for $r < 3r_\mathrm{h}$.