Lecture 3: Global and local views

In the *global view* of an astrophysical disc we consider the full disc, usually in cylindrical polar coordinates (r, ϕ, z).

In the *local view* (known as the local approximation, local model, shearing sheet, shearing box, Hill's approximation, etc.) we consider a small volume of the disc in the neighbourhood of an orbiting reference point, using local Cartesian coordinates (x, y, z) in the radial, azimuthal and vertical directions.

3.1. Local view of orbital dynamics

Select a reference point in a circular orbit of radius r_0 in the midplane. The orbit has angular velocity $\Omega_0 = \Omega(r_0)$, etc. (We drop the subscript 'c' on properties of circular orbits, but use a subscript '0' for the time being to indicate evaluation on the reference orbit.)

Introduce local coordinates (x, y, z) through

$$r = r_0 + x, \qquad \phi = \Omega_0 t + \frac{y}{r_0}, \qquad z = z.$$

Expand the Lagrangian for a particle of unit mass,

$$\begin{aligned} L &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z) \\ &= \frac{1}{2}\left(\dot{x}^2 + (r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right)^2 + \dot{z}^2\right) - \Phi(r_0 + x, z), \end{aligned}$$

to second order in (x, y, z) to obtain

$$L = L_0 + L_1 + L_2 + \dots,$$

where

$$\begin{aligned} L_0 &= \frac{1}{2}r_0^2\Omega_0^2 - \Phi_0 = \text{constant}, \\ L_1 &= r_0\Omega_0\dot{y} + (r_0\Omega_0^2 - \Phi_{r0})x = r_0\Omega_0\dot{y} = \frac{d}{dt}(r_0\Omega_0 y), \\ L_2 &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 2\Omega_0 x\dot{y} + \frac{1}{2}\Omega_0^2 x^2 - \frac{1}{2}\Phi_{rr0}x^2 - \frac{1}{2}\Phi_{zz0}z^2. \end{aligned}$$

The terms L_0 (a constant) and L_1 (a total time-derivative) make no contribution to Lagrange's equations and generate no motion. For small (x, y, z), the motion is dominated by L_2.

L_2 is separable into horizontal and vertical parts:

$$L_2 = L_\mathrm{h} + L_\mathrm{v} = \left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + 2\Omega_0 x\dot{y} - \frac{1}{2}(\Phi_{rr0} - \Omega_0^2)x^2\right] + \left[\frac{1}{2}\dot{z}^2 - \frac{1}{2}\Phi_{zz0}z^2\right].$$

The motion due to L_2 is

$$\begin{aligned} \ddot{x} &= 2\Omega_0\dot{y} - (\Phi_{rr0} - \Omega_0^2)x, \\ \ddot{y} + 2\Omega_0\dot{x} &= 0, \\ \ddot{z} &= -\Phi_{zz0}z. \end{aligned}$$

At this level of approximation, (x, y, z) may be interpreted as Cartesian coordinates in a frame rotating with the orbital angular velocity Ω_0. The equation of motion includes the Coriolis force and the *tidal potential*

$$\Phi_\mathrm{t} = \frac{1}{2}(\Phi_{rr0} - \Omega_0^2)x^2 + \frac{1}{2}\Phi_{zz0}z^2,$$

which comes from the expansion of the sum of the gravitational and centrifugal potentials, $\Phi - \frac{1}{2}\Omega_0^2 r^2$, to second order in (x, z).

From the radial force balance,

$$\Phi_r(r, 0) = r\Omega^2 \qquad \Rightarrow \qquad \Phi_{rr}(r, 0) = \Omega^2 - 2\Omega S,$$

so the tidal potential is

$$\Phi_\mathrm{t} = -\Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2.$$

In practice the orbital shear $S > 0$, so Φ_t has a saddle point at the origin.

Rewrite the equations of motion in the local view as

$$\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} &= 2\Omega_0 S_0 x, \\
\ddot{y} + 2\Omega_0\dot{x} &= 0, \\
\ddot{z} &= -\Omega_{z0}^2 z.
\end{aligned}$$

Three conserved quantities are

$$\begin{aligned}
p_y &= \frac{\partial L_2}{\partial \dot{y}} = \dot{y} + 2\Omega_0 x, \\
\varepsilon_\mathrm{h} &= \sum_i \dot{q}_i \frac{\partial L_\mathrm{h}}{\partial \dot{q}_i} - L_\mathrm{h} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \Omega_0 S_0 x^2, \\
\varepsilon_\mathrm{v} &= \sum_i \dot{q}_i \frac{\partial L_\mathrm{v}}{\partial \dot{q}_i} - L_\mathrm{v} = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\Omega_{z0}^2 z^2.
\end{aligned}$$

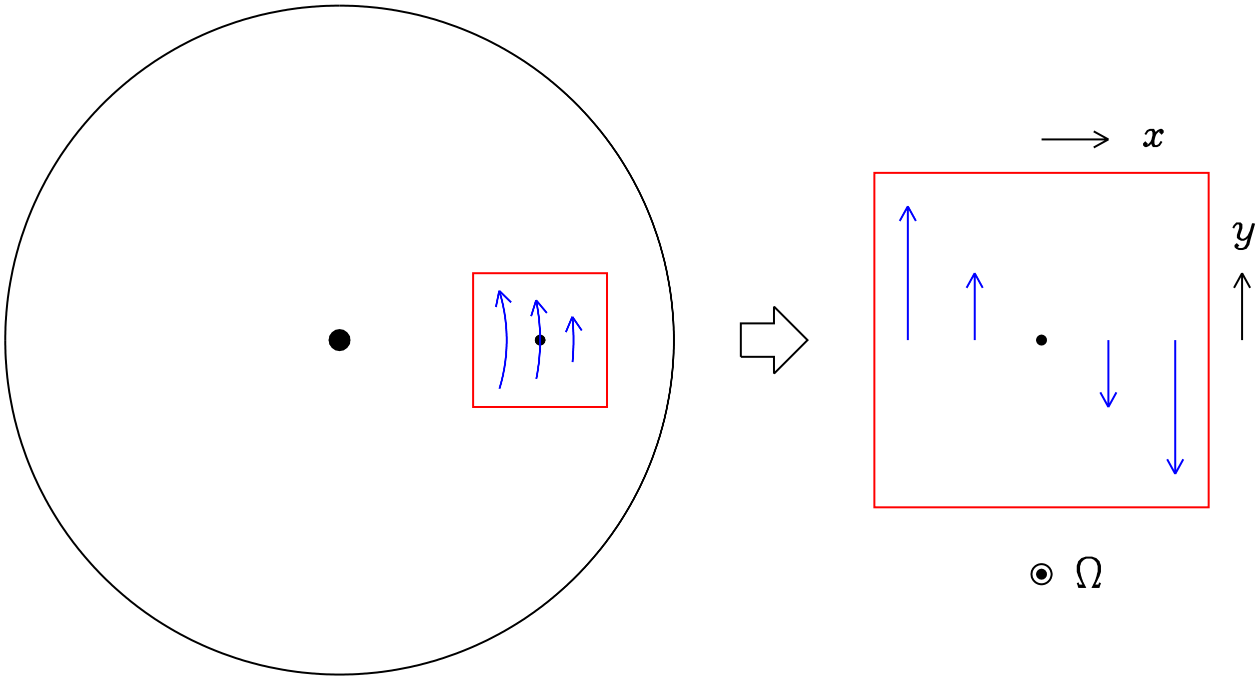
These can be related to the expansions of h and ε in the local view. Consider the conserved quantities

$$\begin{aligned}
\frac{h}{r_0} &= \frac{r^2\dot{\phi}}{r_0} = \frac{(r_0 + x)^2}{r_0}\left(\Omega_0 + \frac{\dot{y}}{r_0}\right) = \text{constant} + (\dot{y} + 2\Omega_0 x) + \cdots, \\
\varepsilon - \Omega_0 h &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \Phi(r, z) - \Omega_0 r^2\dot{\phi} \\
&= \frac{1}{2}\left(\dot{x}^2 + (r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right)^2 + \dot{z}^2\right) + \Phi(r_0 + x, z) - \Omega_0(r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right) \\
&= \text{constant} + \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2 + \cdots \qquad (\textbf{exercise}).
\end{aligned}$$

The local representation of the family of circular orbits in the midplane is

$$x = \text{constant}, \qquad \dot{y} = -S_0 x, \qquad z = 0,$$

which can be interpreted as an *orbital shear flow* with shear rate $S_0 = q_0\Omega_0$. Note that, in the rotating frame of the local view, the Coriolis force balances the tidal force for these orbits.



To obtain the general solution of the local equations of motion, note that

$$\begin{aligned}
&\ddot{x} - 2\Omega_0(p_y - 2\Omega_0 x) = 2\Omega_0 S_0 x \\
&\ddot{x} + 2\Omega_0(2\Omega_0 - S_0)x = 2\Omega_0 p_y \\
&\ddot{x} + \Omega_{r0}^2 x = 2\Omega_0 p_y,
\end{aligned}$$

so

$$\begin{aligned}
&x = x_0 + \mathrm{Re}\left(A\, e^{-i\Omega_{r0}t}\right), \\
&y = y_0 - S_0 x_0 t + \mathrm{Re}\left(\frac{2\Omega_0 A}{i\Omega_{r0}}\, e^{-i\Omega_{r0}t}\right), \\
&z = \mathrm{Re}\left(B\, e^{-i\Omega_{z0}t}\right),
\end{aligned}$$

for some real constant x_0 and complex oscillation amplitudes A and B. These are the local representation of (slightly) eccentric and inclined orbits. The three conserved quantities evaluate to (**exercise**)

$$\begin{aligned}
&p_y = (2\Omega_0 - S_0)x_0, \\
&\varepsilon_{\mathrm{h}} = \frac{1}{2}\Omega_{r0}^2\left(|A|^2 - \frac{S_0}{2\Omega_0}x_0^2\right), \\
&\varepsilon_{\mathrm{v}} = \frac{1}{2}\Omega_{z0}^2|B|^2.
\end{aligned}$$

Having derived the local model, we usually omit the subscript zero on Ω, S, Ω_r, Ω_z, etc. These quantities are regarded as constants, evaluated at r_0.

3.2. Symmetries of the local model

The local view has some symmetries inherited from the global view:

- Translational symmetry in y: $y \mapsto y + c$
- Reflectional symmetry in z: $z \mapsto -z$

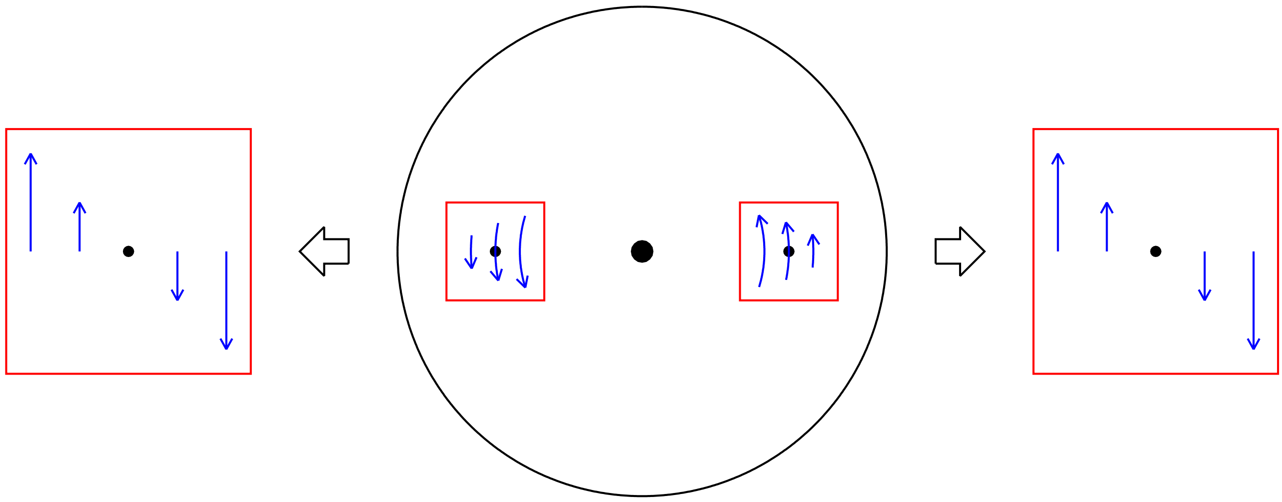
It has additional symmetries not present in the global view:

- Translational symmetry in x (when combined with a Galilean boost in y):

$$x \mapsto x + c, \qquad y \mapsto y - S_0 ct$$

- Rotational symmetry by π about the z axis (we cannot tell the inside from the outside):

$$x \mapsto -x, \qquad y \mapsto -y$$



- Scale-invariance (no characteristic length-scale, because we zoomed in to scales $\ll r_0$):

$$\mathbf{x} \mapsto c\,\mathbf{x}$$

- Separability of horizontal and vertical dimensions

The combination of translational symmetries in x and y means that the local model is *horizontally homogeneous.*

Exercise: Interpret the motion of a test particle in the local approximation for a Keplerian disc ($\Omega_r = \Omega_z = \Omega$). Starting with the case in which $x_0 = 0$ and $B = 0$ but $A \neq 0$, show that the epicyclic motion consists of a retrograde ellipse with an axis ratio of 2. Show that including $B \neq 0$ produces a tilted ellipse, and that including $x_0 \neq 0$ makes the centre of the ellipse (the *guiding centre* of the epicycle) drift in the azimuthal direction at the orbital shear rate.