Lecture 5: Boundary conditions and steady accretion

5.1. Boundary conditions

At a *free boundary*, which can move radially as the disc spreads, the internal torque vanishes:

$$\mathscr{G} = 0.$$

A spreading disc will reach $r = 0$ in a finite time, whereas the outer edge might expand indefinitely. The *inner boundary condition* is then particularly important, and depends on the nature of the central object.

Black hole

Circular orbits are unstable sufficiently close to the event horizon (see Example 1.3).

e.g. for a non-rotating (Schwarzschild) black hole:

$$\Omega^2 = \frac{GM}{r^3}, \qquad \Omega_r^2 = \Omega^2\left(1 - \frac{6\,GM}{c^2 r}\right),$$

so orbits are unstable for

$$r < r_{\mathrm{ms}} = \frac{6\,GM}{c^2} = 3\,r_{\mathrm{h}}.$$

This critical radius is referred to as the *marginally stable circular orbit* or *innermost stable circular orbit* (ISCO). Actually this orbit is unstable. Given a radial displacement, the particle rapidly spirals into the black hole without needing to lose any angular momentum.

In a fluid disc, a rapid transition occurs near r_{ms}. The radial velocity $|\bar{u}_r|$ increases very rapidly as r decreases near r_{ms}, so Σ decreases very rapidly to conserve the radial mass flux.

It is expected that the torque $\mathscr{G} \approx 0$ at $r = r_{\mathrm{in}} \approx r_{\mathrm{ms}}$.

Exercise: Using Newtonian dynamics in the model potential $\Phi = -GM/(R - r_{\mathrm{h}})$, calculate ε and h for a circular orbit of radius $r = r_{\mathrm{ms}} = 3r_{\mathrm{h}}$. Use these conserved quantities to work out how the radial and azimuthal velocities depend on r for a particle that is infinitesimally displaced inwards from this circular orbit.

Star with a negligible magnetic field

In this case the disc may extend to the stellar surface $r = R_*$. In contrast to the disc, the star (supported mainly by pressure) usually rotates at only a small fraction of the Keplerian rate:

$$\Omega_* \ll \sqrt{\frac{GM}{R_*^3}}.$$

Ω makes a rapid adjustment from the Keplerian value to the stellar value in a *boundary layer*:

The usual argument is that the 'viscous' torque $\mathscr{G} = 0$ at $r = r_{\text{in}} \approx R_*$ where $\dfrac{d\Omega}{dr} = 0$.

Star with a significant magnetic field

In this case the disc is disrupted within the *magnetospheric radius*, leading to the formation of a cavity and accretion along field lines towards the magnetic poles of the star. This *polar accretion* is observed in many young stars and neutron stars.

The star may exert a torque on the disc if they are linked by magnetic field lines. The outcome of this complicated process depends on the stellar rotation rate and magnetic field, the accretion rate, etc.

Treatment for $r \gg r_{\text{in}}$

If we are mainly interested in scales $r \gg r_{\text{in}}$, we may formally let $r_{\text{in}} \to 0$.

To allow a mass flux at the origin but no torque:

$$\mathscr{F} \to \text{constant}, \qquad \mathscr{G} \to 0 \qquad \text{as } r \to 0.$$

To allow a torque at the origin:

$$\mathscr{G} \to \text{constant} \qquad \text{as } r \to 0.$$

For a Keplerian disc:

$$\mathscr{F} \propto r\Sigma\bar{u}_r \propto r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right), \qquad \mathscr{G} \propto r^{1/2}\bar{\nu}\Sigma.$$

In the first case (no torque at the origin): $\bar{\nu}\Sigma \to$ constant as $r \to 0$.

In the second case (torque at the origin): $r^{1/2}\bar{\nu}\Sigma \to$ constant as $r \to 0$.

5.2. Steady accretion discs

In a steady state, mass conservation gives

$$\mathscr{F} = -\dot{M} = \text{constant},$$

where $\dot{M}$ is the *mass accretion rate*. (We neglect the slow variation of the potential due to the increase in M.) Angular momentum conservation gives

$$\mathscr{F}h + \mathscr{G} = \text{constant}.$$

If the inner boundary condition is $\mathscr{G} = 0$ at $r = r_{\text{in}}$, the solution is

$$\mathscr{G} = \dot{M}(h - h_{\text{in}}), \qquad h_{\text{in}} = h(r_{\text{in}}).$$

For a Keplerian disc, we have $\mathscr{G} = 3\pi\bar{\nu}\Sigma h$ and $h \propto r^{1/2}$, so

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_{\text{in}}}{r}}\right).$$

If we know the function $\bar{\nu}(r, \Sigma)$, this provides a complete solution for the disc.

From the expression for the surface temperature,

$$2\sigma T_{\text{s}}^4 = \bar{\nu}\Sigma r^2\left(\frac{d\Omega}{dr}\right)^2,$$

in the case of a steady Keplerian disc we have

$$\sigma T_{\text{s}}^4 = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \sqrt{\frac{r_{\text{in}}}{r}}\right).$$

The total luminosity of a disc extending to $r_{\rm out} = \infty$ is

$$L_{\rm disc} = \int_{r_{\rm in}}^{\infty} \frac{3GM\dot{M}}{4\pi r^3} \left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right) 2\pi r \, dr = \frac{1}{2} \frac{GM\dot{M}}{r_{\rm in}}.$$

This is exactly equal to the rate at which orbital binding energy is transferred to the gas as it passes from to $r_{\rm in}$. But it is only half the potential energy released. In the case of accretion on to a stellar surface, the remaining energy is released in the boundary layer.

The spectrum of a disc emitting blackbody radiation is a stretched version of the blackbody spectrum, with the high-energy radiation coming mainly from the inner disc and the low-energy radiation coming mainly from the outer disc.

Exercise: Show that, at radius r in a steady Keplerian accretion disc, the ratio of the rate of energy dissipation to the rate at which orbital energy is given up by the inflowing matter is

$$3\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right).$$

(The fact that this ratio is not equal to unity everywhere is explained by the fact that the internal torque provides an outward flux of energy as well as angular momentum.)