## Lecture 7: Vertical structure

### 7.1. Hydrostatic equilibrium

The dominant force balance in the $z$ direction perpendicular to the plane of the disc is

$$0 = -\rho\frac{\partial\Phi}{\partial z} - \frac{\partial p}{\partial z}.$$

The Taylor expansion of $\Phi$ about the midplane is $\Phi(r,z) = \Phi(r,0) + \frac{1}{2}\Phi_{zz}(r,0)z^2 + \cdots$, so the vertical gravity in a thin disc is

$$g_z = -\frac{\partial\Phi}{\partial z} \approx -\Phi_{zz}(r,0)z = -\Omega_z^2 z,$$

where $\Omega_z(r)$ is the vertical frequency (recall that $\Omega_z = \Omega$ for a Keplerian disc).

The *equation of vertical hydrostatic equilibrium* is therefore

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z.$$

This is essentially an ordinary differential equation in $z$ at each $r$ (and $\phi$ and $t$). [In the local approximation the ODE $\frac{dp}{dz} = -\rho\Omega_z^2 z$ is exact for hydrostatic solutions independent of $(x,y,t)$.]

As in a star, pressure supports the disc against gravity in the vertical direction. But note that the disc is centrifugally supported in the radial direction.

This analysis is for a *non-self-gravitating* disc, using the vertical gravity due to the central object. In a *self-gravitating disc* the disc makes an additional contribution to $g_z$ and affects the hydrostatic structure.

If $p$ and $\rho$ are related in a known way, we can solve for the hydrostatic structure. e.g. for an isothermal ideal gas, $p = c_\mathrm{s}^2\rho$, where $c_\mathrm{s} = \text{constant}$ is the isothermal sound speed. Then the solution is a Gaussian:

$$p \propto \rho \propto \exp\left(-\frac{z^2}{2H^2}\right),$$

with scaleheight

$$H = \frac{c_\mathrm{s}}{\Omega_z} \qquad \left(= \frac{c_\mathrm{s}}{\Omega} \quad \text{for Keplerian}\right).$$

Formally, the disc extends to $z = \pm\infty$, and the thin-disc approximation breaks down once $z/r$ is no longer small, but there is essentially no mass at such heights.

## 7.2. Hydrostatic models

More generally, define the *surface density* $\Sigma$, *vertically integrated pressure* $P$ and *scaleheight* $H$ by

$$\Sigma = \int \rho\,dz, \qquad P = \int p\,dz, \qquad \Sigma H^2 = \int \rho z^2\,dz,$$

where the integrals are over the full vertical extent of disc. $H$ can be interpreted as the standard deviation of the mass distribution. Note that (assuming boundary conditions $zp \to 0$ as $z \to \pm\infty$)

$$P = \int 1 \cdot p\,dz = [zp] - \int z\frac{dp}{dz}\,dz = 0 + \int z\rho\Omega_z^2 z\,dz = \Sigma H^2 \Omega_z^2.$$

We can reduce the problem to a dimensionless form:

$$\rho(z) = \hat{\rho} \cdot \tilde{\rho}(\tilde{z}), \qquad p(z) = \hat{p} \cdot \tilde{p}(\tilde{z}),$$

where

$$\hat{\rho} = \frac{\Sigma}{H}, \qquad \hat{p} = \frac{P}{H}$$

are characteristic values of density and pressure, while $\tilde{\rho}$ and $\tilde{p}$ are dimensionless functions of the dimensionless coordinate

$$\tilde{z} = \frac{z}{H}.$$

These satisfy the dimensionless equation of hydrostatic equilibrium,

$$\frac{d\tilde{p}}{d\tilde{z}} = -\tilde{\rho}\tilde{z},$$

and the normalization conditions

$$\int \tilde{\rho}\,d\tilde{z} = \int \tilde{p}\,d\tilde{z} = \int \tilde{\rho}\tilde{z}^2\,d\tilde{z} = 1.$$

The *isothermal model* ($p \propto \rho$) is the solution

$$\tilde{\rho} = \tilde{p} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tilde{z}^2}{2}\right).$$

The *uniform model* ($\rho =$ constant) is the solution

$$\tilde{\rho} = \begin{cases} \frac{1}{2\sqrt{3}}, & \tilde{z}^2 < 3, \\ 0, & \tilde{z}^2 > 3, \end{cases} \qquad \tilde{p} = \begin{cases} \frac{1}{4\sqrt{3}}(3 - \tilde{z}^2), & \tilde{z}^2 < 3, \\ 0, & \tilde{z}^2 > 3. \end{cases}$$

The *polytropic model* of index $n$ ($p \propto \rho^{1+1/n}$, where $n > 0$ is not necessarily an integer) is the solution

$$\tilde{\rho} = C_\rho \left(1 - \frac{\tilde{z}^2}{2n+3}\right)^n, \qquad \tilde{p} = C_p \left(1 - \frac{\tilde{z}^2}{2n+3}\right)^{n+1},$$

(valid for $\tilde{z}^2 < 2n+3$ only, otherwise $\tilde{\rho} = \tilde{p} = 0$), with normalizing constants

$$C_\rho = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \frac{1}{\sqrt{(2n+3)\pi}}, \qquad C_p = \frac{(n+\frac{3}{2})}{(n+1)} C_\rho = \frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+2)} \frac{1}{\sqrt{(2n+3)\pi}}.$$

Here

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, dx, \qquad p > 0$$

is the Gamma function, equal to $(p-1)!$ for integers $p$. A useful integral here is

$$\int_{-1}^{1} (1-x^2)^p\, dx = \frac{\sqrt{\pi}\,\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}, \qquad p > -1.$$

It can be shown that the polytropic model reduces to the uniform model in the limit $n \to 0$ and reduces to the isothermal model in the limit $n \to \infty$.

While the isothermal model extends formally to $z = \pm\infty$, the other models have definite surfaces beyond which there is a vacuum.

## 7.3. Order-of-magnitude estimates and time-scales

Here we consider simple scaling relations ($\sim$), omitting numerical factors of order unity.

An important dimensionless parameter of a thin disc is the *aspect ratio*

$$\frac{H}{r} \ll 1.$$

From hydrostatic equilibrium,

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z \qquad \Rightarrow \qquad \frac{p}{H} \sim \rho\Omega^2 H \qquad \Rightarrow \qquad c_\mathrm{s} \sim \Omega H,$$

where $c_\mathrm{s} = \sqrt{p/\rho}$ is the isothermal sound speed.

The dimensions of dynamic viscosity

$$[\rho\nu] = ML^{-1}T^{-1}$$

are the same of those of $p/\Omega$. We write

$$\rho\nu = \frac{\alpha p}{\Omega},$$

where $\alpha$ is the *dimensionless viscosity parameter.* If $\alpha$ is regarded as a constant, this relation is known as the *alpha viscosity prescription.* Then

$$\nu = \frac{\alpha c_\mathrm{s}^2}{\Omega} \sim \alpha c_\mathrm{s} H.$$

In the kinetic theory of gases, the kinematic viscosity is $\nu \sim v\ell$, where $v$ is the mean speed of the molecules and $\ell$ is their mean free path. This molecular viscosity is negligible for astrophysical discs. But a similar estimate can be made for the effective 'eddy viscosity' of turbulence, if $v$ is a typical turbulent velocity and $\ell$ is the correlation length of the turbulence. For subsonic turbulence with $v \lesssim c_\mathrm{s}$ and $\ell \lesssim H$, we expect that $\alpha \lesssim 1$.

The stress is then

$$T_{r\phi} = \rho\nu r \frac{d\Omega}{dr} = -q\alpha p.$$

The idea behind $|T_{r\phi}| \sim \alpha p$ is that, whatever physical process gives rise to the stress, it should scale with the pressure. This assumption is probably correct for local processes such as small-scale turbulent motions resulting from instabilities (see later).

Three important characteristic time-scales in a disc can be defined:

*Dynamical time-scale* (time-scale of orbital motion and of vertical hydrostatic equilibrium):

$$t_{\rm dyn} \sim \frac{1}{\Omega} \sim \frac{H}{c_{\rm s}}.$$

*Viscous time-scale* (time-scale of radial motion and of evolution of the surface density):

$$t_{\rm visc} \sim \frac{r^2}{\bar{\nu}} \sim \alpha^{-1} \left(\frac{H}{r}\right)^{-2} t_{\rm dyn}.$$

*Thermal time-scale* (time-scale of vertical thermal balance):

$$t_{\rm th} \sim \frac{\text{internal energy/area}}{\text{dissipation rate/area}} \sim \frac{P}{\bar{\nu}\Sigma\Omega^2} \sim \frac{c_{\rm s}^2}{\bar{\nu}\Omega^2} \sim \frac{H^2}{\bar{\nu}} \sim \alpha^{-1} t_{\rm dyn}.$$

For a thin disc with $\alpha < 1$, we have the hierarchy

$$t_{\rm dyn} < t_{\rm th} \ll t_{\rm visc}.$$

Furthermore, all three time-scales usually increase with $r$.

The Mach number of the orbital motion is

$$\mathrm{Ma} \sim \frac{r\Omega}{c_{\rm s}} \sim \left(\frac{H}{r}\right)^{-1}.$$

The typical accretion velocity is

$$|\bar{u}_r| \sim \frac{\bar{\nu}}{r} \sim \alpha \left(\frac{H}{r}\right) c_{\rm s}.$$

For a thin disc, we have the hierarchy

$$|\bar{u}_r| \ll c_{\rm s} \ll r\Omega,$$

so the orbital motion is highly supersonic while the accretion flow is highly subsonic.

The relative contribution of the radial pressure gradient to the radial component of the equation of motion is

$$\frac{\partial p}{\partial r} \Big/ \rho r \Omega^2 \sim \frac{\rho c_{\rm s}^2}{r} \Big/ \rho r \Omega^2 \sim \frac{c_{\rm s}^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2.$$

Vertical variations of the radial gravitational acceleration are also of this order. Other terms in the radial equation of motion, such as inertial terms associated with the radial motion, are smaller still. Therefore

$$u_\phi = r\Omega \left[1 + O\left(\frac{H}{r}\right)^2\right],$$

so treating the azimuthal fluid velocity as equal to the orbital velocity of a test particle is an excellent approximation for a thin disc. In general, the thin-disc approximations involve fractional errors of $O(H/r)^2$. A formal asymptotic treatment of thin discs is possible, using as small parameter a characteristic value of $(H/r)^2$.

**Exercise:** If $\Phi = -GM/R$ and $p/\rho = c_{\rm s}^2 = \epsilon^2 GM/r$ ('locally isothermal'), where $R = \sqrt{r^2 + z^2}$ and $\epsilon =$ constant, show that *exact* force balances are achieved in all directions if

$$\rho = f(r) \exp\left(\frac{r - R}{\epsilon^2 R}\right), \qquad \Omega^2 = \frac{GM}{r^3}\left[\frac{r}{R} - \epsilon^2\left(1 - \frac{d\ln f}{d\ln r}\right)\right].$$