Lecture 8: Radiative models

8.1. Equations of vertical structure

A *radiative model* describes the vertical structure of a disc in which the energy dissipated by viscosity is carried away by radiation from the surfaces of the disc.

The energy flux due to radiative diffusion, for an optically thick disc, is

$$\mathbf{F} = -\frac{16\sigma T^3}{3\kappa\rho}\boldsymbol{\nabla}T,$$

where κ is the (Rosseland mean) *opacity.* The dominant balance in the thermal energy equation, for a thin disc, is

$$0 = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2 - \frac{\partial F_z}{\partial z}.$$

Contributions from F_r and from radial advection are smaller by $O(H/r)^2$, e.g.

$$\left(\frac{1}{\gamma-1}\right)\bar{u}_r\frac{\partial p}{\partial r}\Big/\rho\nu\left(r\frac{d\Omega}{dr}\right)^2 \sim \frac{\bar{u}_r\rho c_\mathrm{s}^2}{r}\Big/\rho\bar{\nu}\Omega^2 \sim \frac{c_\mathrm{s}^2}{r^2\Omega^2} \sim \frac{H^2}{r^2}.$$

The equations of vertical structure for a radiative, Keplerian disc are then

$$\begin{aligned}\frac{\partial p}{\partial z} &= -\rho\Omega^2 z,\\ \frac{\partial F_z}{\partial z} &= \frac{9}{4}\rho\nu\Omega^2,\\ F_z &= -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z},\end{aligned}$$

together with an *equation of state*, e.g.

$$p = \frac{\mathscr{R}\rho T}{\mu} + \frac{4\sigma T^4}{3c} \qquad \text{(ideal gas + radiation)}$$

(where $\mathscr{R}$ is the gas constant and μ the mean molecular weight), an opacity function $\kappa(\rho, T)$, a viscosity prescription for $\bar{\nu}$ and boundary conditions, e.g. the 'zero boundary conditions' $\rho = p = T = 0$ at $z = \pm z_\mathrm{s}$ (or, more realistically, matching to an atmospheric model at the photosphere).

The problem is analogous to the radial structure of a star. In the local approximation, these are exact ODEs for equilibrium solutions independent of (x, y, t).

Opacity is often approximated by a power law, e.g. *Thomson opacity* (due to electron scattering: hotter regions of ionized discs)

$$\kappa = \text{constant} \approx 0.33\,\mathrm{cm^2\,g^{-1}},$$

or *Kramers opacity* (due to free-free/bound-free transitions: cooler regions of ionized discs)

$$\kappa = C_\kappa\rho T^{-7/2}, \qquad C_\kappa \approx 4.5\times10^{24}\,\mathrm{cm^5\,g^{-2}\,K^{7/2}}.$$

In cooler discs, dust and molecules dominate the opacity.

8.2. Radiative, Keplerian disc with gas pressure, power-law opacity and alpha viscosity

We aim to solve

$$\frac{dp}{dz} = -\rho\Omega^2 z, \qquad \frac{dF_z}{dz} = \frac{9}{4}\rho\nu\Omega^2, \qquad F_z = -\frac{16\sigma T^3}{3\kappa\rho}\frac{dT}{dz},$$

with

$$p = \frac{\mathscr{R}\rho T}{\mu}, \qquad \kappa = C_\kappa \rho^x T^y, \qquad \rho\nu = \frac{\alpha p}{\Omega}.$$

The problem can be reduced to a dimensionless form by writing

$$\rho(z) = \hat{\rho}\cdot\tilde{\rho}(\tilde{z}), \qquad p(z) = \hat{p}\cdot\tilde{p}(\tilde{z}), \qquad T(z) = \hat{T}\cdot\tilde{T}(\tilde{z}), \qquad F_z(z) = \hat{F}\cdot\tilde{F}(\tilde{z}),$$

with dimensionless vertical coordinate $\tilde{z} = z/H$ and characteristic values

$$\hat{\rho} = \frac{\Sigma}{H}, \qquad \hat{p} = \frac{P}{H}, \qquad \hat{T} = \frac{\mu}{\mathscr{R}}\frac{\hat{p}}{\hat{\rho}}, \qquad \hat{F} = \frac{16\sigma\hat{T}^4}{3\hat{\tau}},$$

where, again, $P = \Sigma H^2\Omega^2$, and the characteristic optical thickness is

$$\hat{\tau} = \hat{\kappa}\Sigma = C_\kappa \hat{\rho}^x \hat{T}^y \Sigma.$$

We obtain the dimensionless equations of vertical structure,

$$\frac{d\tilde{p}}{d\tilde{z}} = -\tilde{\rho}\tilde{z}, \qquad \frac{d\tilde{F}}{d\tilde{z}} = \lambda\tilde{p}, \qquad \frac{d\tilde{T}}{d\tilde{z}} = -\tilde{\rho}^{x+1}\tilde{T}^{y-3}\tilde{F}, \qquad \tilde{p} = \tilde{\rho}\tilde{T},$$

subject to the normalization conditions

$$\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = \int \tilde{\rho}\tilde{z}^2\, d\tilde{z} = 1$$

and the zero boundary conditions $\tilde{\rho} = \tilde{p} = \tilde{T} = 0$ at the surfaces $\tilde{z} = \pm\tilde{z}_{\mathrm{s}}$.

The inclusion of thermal physics leads to a specific solution of the problem of hydrostatic equilibrium considered in the previous lecture.

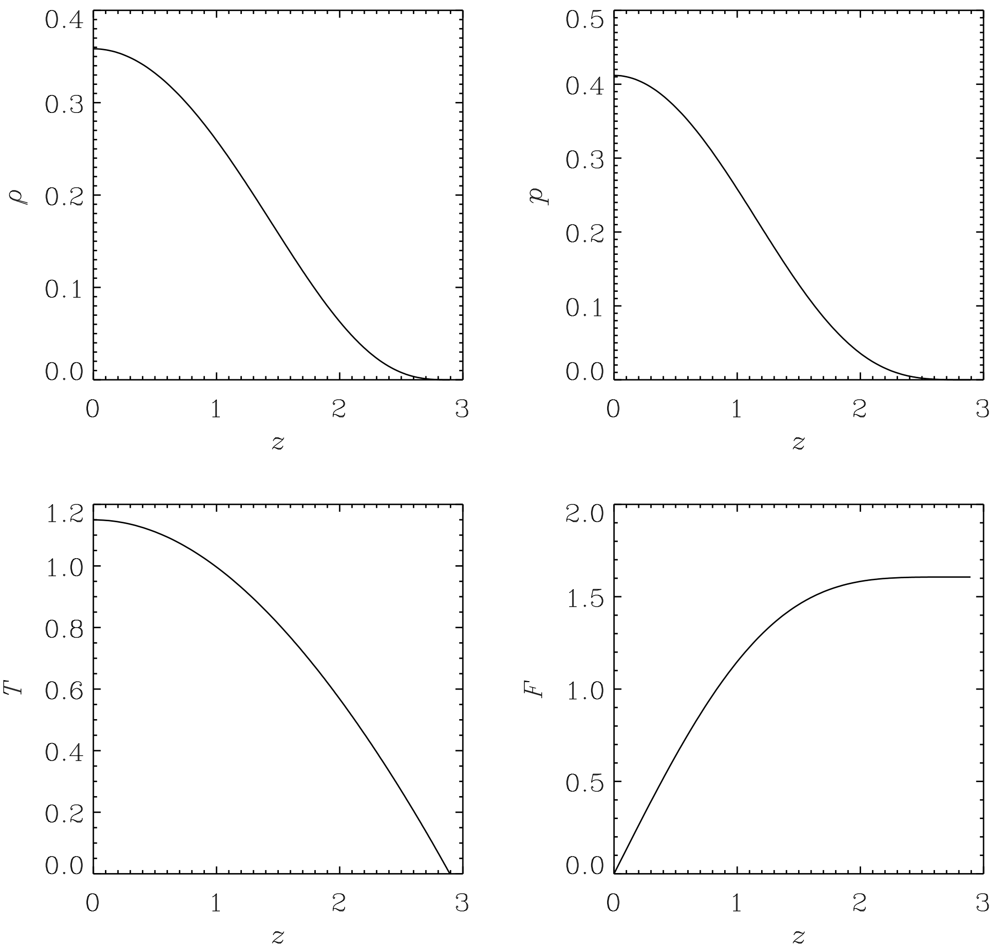
λ is an eigenvalue of the problem, which can be interpreted as the dimensionless cooling rate:

$$2\tilde{F}_{\mathrm{s}} = \int \frac{d\tilde{F}}{d\tilde{z}}\, d\tilde{z} = \lambda \int \tilde{p}\, d\tilde{z} = \lambda.$$

The numerical solution for Thomson opacity ($x = y = 0$) gives $\lambda = 3.213$ and $z_{\mathrm{s}} = 2.895$. This solution is similar to a polytropic model with $n \approx 2.4$.

The energy equation in a steady state gives the equilibrium condition for thermal balance:

$$\begin{aligned}
\frac{\hat{F}}{H}\lambda &= \frac{9}{4}\alpha\Omega\hat{p} \\
\lambda &= \frac{9}{4}\alpha\Omega\frac{P}{\hat{F}} \\
&= \frac{27}{64}\alpha\frac{\Sigma H^2\Omega^3\hat{\tau}}{\sigma\hat{T}^4} \\
&= \frac{27}{64}\alpha\frac{C_\kappa}{\sigma}\left(\frac{\mathscr{R}}{\mu}\right)^{4-y} H^{-x+2y-6}\Omega^{2y-5}\Sigma^{x+2}.
\end{aligned}$$



The vertically integrated viscosity is

$$\bar{\nu}\Sigma = \int \rho\nu\,dz = \int \frac{\alpha p}{\Omega}\,dz = \frac{\alpha P}{\Omega} = \alpha\Sigma H^2\Omega.$$

Thermal balance implies

$$H^{x-2y+6} \propto \Omega^{2y-5}\Sigma^{x+2},$$

so

$$\begin{aligned}\bar{\nu} &\propto H^2\Omega \\ &\propto \Omega^{(x+2y-4))/(x-2y+6)}\Sigma^{2(x+2)/(x-2y+6)} \\ &\propto r^{-3(x+2y-4)/2(x-2y+6)}\Sigma^{2(x+2)/(x-2y+6)}.\end{aligned}$$

e.g. for Thomson opacity ($x = y = 0$),

$$\bar{\nu} \propto r\Sigma^{2/3},$$

or for Kramers opacity ($x = 1$, $y = -7/2$),

$$\bar{\nu} \propto r^{15/14}\Sigma^{3/7}.$$

The constant of proportionality involves a numerical coefficient and various powers of C_κ/σ, $\mathscr{R}/\mu$ and GM.

The heating and cooling rates per unit area (equal in a thermal steady state) are

$$\mathscr{H} = \int \frac{9}{4}\alpha\Omega p\,dz = \frac{9}{4}\alpha\Omega P,$$

$$\mathscr{C} = 2\sigma T_{\mathrm{s}}^4 = 2F_z\Big|_{z=z_{\mathrm{s}}} = 2\hat{F}\tilde{F}_{\mathrm{s}} = \lambda\frac{16\sigma\hat{T}^4}{3\hat{\tau}}.$$

Note that

$$T_{\mathrm{s}}^4 = \frac{8\lambda}{3\hat{\tau}}\hat{T}^4,$$

so $\hat{T} \gg T_{\mathrm{s}}$ in a highly optically thick disc (as assumed when using zero boundary conditions).

8.3. Viscous instability

Consider the nonlinear diffusion equation for a Keplerian disc,

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right],$$

with $\bar{\nu} = \bar{\nu}(r,\Sigma)$. Linearize about any given solution $\Sigma_0(r,t)$:

$$\Sigma(r,t) = \Sigma_0(r,t) + \Sigma'(r,t), \qquad |\Sigma'| \ll \Sigma_0,$$

so that

$$(\bar{\nu}\Sigma)' = \frac{\partial(\bar{\nu}\Sigma)}{\partial\Sigma}\Sigma' = \beta\bar{\nu}\Sigma', \qquad \beta = \frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}.$$

We then obtain the linearized diffusion equation,

$$\frac{\partial\Sigma'}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\beta\bar{\nu}\Sigma'\right)\right].$$

The evolution is unstable (antidiffusive) for $\beta < 0$: perturbations grow rapidly on short length-scales, causing the disc to break into rings. Astrophysical applications may be complicated by thermal instability, which often coincides with it and dominates (see next lecture).

Exercise: Show that, when $\beta < 0$, short-wavelength perturbations Σ' with radial wavenumber k grow in time at the rate $3|\beta|\bar{\nu}k^2$, when $1/k$ is small (as in the WKB approximation) compared to the characteristic scale on which the solution Σ_0 varies. (The diffusion equation itself becomes inaccurate when $1/k$ is not large compared to H, so the fastest-growing modes typically have wavelengths of a few times H.)