Lecture 9: Thermal instability / Hydrodynamics of the shearing sheet

9.1. Thermal instability

So far, we have assumed a balance between heating and cooling: \( \frac{9}{4} \dot{\nu} \Sigma \Omega^2 = \mathcal{H} = \mathcal{C} = 2F_s \).

Now relax this assumption, but assume that \( \alpha \ll 1 \) so that \( t_{\text{dyn}} \ll t_{\text{th}} \ll t_{\text{visc}} \). Consider behaviour on the timescale \( t_{\text{th}} \); we can then assume that the disc is hydrostatic and that the surface density does not evolve.

By solving the equations of vertical structure except thermal balance, we can calculate \( \mathcal{H} \) and \( \mathcal{C} \) as functions of \( (\Sigma, \dot{\nu} \Sigma) \). In fact \( \mathcal{H} \) depends only on \( \dot{\nu} \Sigma \). The equation of thermal balance \( \mathcal{H} = \mathcal{C} \) defines a curve in the \( (\Sigma, \dot{\nu} \Sigma) \) plane.

Along the equilibrium curve, \( d\mathcal{H} = d\mathcal{C} \) and \( d(\dot{\nu} \Sigma) = \beta \dot{\nu} d\Sigma \), where \( \beta = \left( \frac{\partial \ln(\dot{\nu} \Sigma)}{\partial \ln \Sigma} \right)_r \):

\[
\frac{d\mathcal{H}}{d(\dot{\nu} \Sigma)} = \frac{\partial \mathcal{H}}{\partial \Sigma} d\Sigma + \frac{\partial \mathcal{H}}{\partial (\dot{\nu} \Sigma)} d(\dot{\nu} \Sigma)
\]

\[
\frac{d\mathcal{H}}{d(\dot{\nu} \Sigma)} = \frac{1}{\beta \dot{\nu}} \frac{\partial \mathcal{C}}{\partial \Sigma} + \frac{\partial \mathcal{C}}{\partial (\dot{\nu} \Sigma)}.
\]

The internal energy content of disc per unit area is \( \sim P \sim (\Omega/\alpha)\dot{\nu} \Sigma \). If some heat is added, \( \dot{\nu} \Sigma \) increases but \( \Sigma \) is fixed on the timescale \( t_{\text{th}} \). The system is thermally unstable if the excess heating outweighs the excess cooling, i.e. if

\[
\frac{d\mathcal{H}}{d(\dot{\nu} \Sigma)} > \frac{\partial \mathcal{C}}{\partial (\dot{\nu} \Sigma)}, \quad \text{i.e. if} \quad \frac{1}{\beta \dot{\nu}} \frac{\partial \mathcal{C}}{\partial \Sigma} > 0.
\]

In practice \( \partial \mathcal{C}/\partial \Sigma < 0 \) (because, at fixed \( \dot{\nu} \Sigma \), \( \Sigma \propto 1/\dot{\nu} \propto 1/(\alpha T) \), and \( \mathcal{C} \) generally increases with \( T \)), so thermal instability occurs (like viscous instability) when \( \beta < 0 \). Thermal instability then dominates (as its timescale is shorter).

9.2. Outbursts

We have seen that a radiative disc with gas pressure and Thomson opacity has \( \dot{\nu} \Sigma \propto \nu \Sigma^{5/3} \) and is viscously and thermally stable. For cooler discs undergoing H ionization, the graph of \( \dot{\nu} \Sigma \) versus \( \Sigma \) can involve an ‘S curve’, leading to instability and limit-cycle behaviour, which explains the outbursts in many cataclysmic variables, X-ray binaries and other systems.
9.3. Hydrodynamics of the shearing sheet

Recall the local view of an astrophysical disc: a linear shear flow $\mathbf{u}_0 = -Sx \mathbf{e}_y$ in a frame rotating with $\Omega_0 = \Omega_0 \mathbf{e}_z$. Here $\Omega_0$ and $S = -r \Omega_0/dr$ are evaluated at the reference radius $r_0$.

The model is either horizontally unbounded or equipped with (modified) periodic boundary conditions (see later). Possible treatments of the vertical structure are:

- ignore $z$ completely (2D shearing sheet)
- neglect vertical gravity: homogeneous in $z$
- include vertical gravity: isothermal, uniform, polytropic, radiative, etc. models

9.4. Homogeneous incompressible fluid

Consider a 3D model, unbounded or periodic in $(x, y, z)$, with a uniform kinematic viscosity $\nu$. The equation of motion is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla \rho - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u},$$

subject to the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0.$$
In components:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + \mathbf{v} \cdot \mathbf{\nabla} \right) v_x - 2\Omega v_y &= -\frac{\partial \psi}{\partial x} + \nu \nabla^2 v_x, \\
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + \mathbf{v} \cdot \mathbf{\nabla} \right) v_y + (2\Omega - S)v_x &= -\frac{\partial \psi}{\partial y} + \nu \nabla^2 v_y, \\
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + \mathbf{v} \cdot \mathbf{\nabla} \right) v_z &= -\frac{\partial \psi}{\partial z} + \nu \nabla^2 v_z.
\end{align*}
\]

Consider a plane-wave solution in the form of a shearing wave:

\[
\begin{align*}
v(x, t) &= \text{Re} \left\{ \tilde{\psi}(t) \exp[i\mathbf{k}(t) \cdot \mathbf{x}] \right\}, \\
\psi(x, t) &= \text{Re} \left\{ \tilde{\psi}(t) \exp[i\mathbf{k}(t) \cdot \mathbf{x}] \right\},
\end{align*}
\]

with time-dependent wavevector \( \mathbf{k}(t) \). Then

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} \right) \mathbf{v} &= \text{Re} \left\{ \left[ \frac{d\tilde{\mathbf{v}}}{dt} + \left( i \frac{dk}{dt} + Sx ik_y \right) \tilde{\mathbf{v}} \right] \exp[i\mathbf{k}(t) \cdot \mathbf{x}] \right\}.
\end{align*}
\]

If we choose

\[
\frac{dk}{dt} = S_k \mathbf{e}_x,
\]

then two terms cancel and we are left with \( \frac{d\tilde{\mathbf{v}}}{dt} \).

This means

\[
k_x = k_{x0} + S_y t, \quad k_y = \text{constant}, \quad k_z = \text{constant}.
\]

Tilting of the wavefronts by the shear flow, and

Dual shear flow in Fourier space:
Furthermore, the nonlinear term vanishes:
\[
\mathbf{v} \cdot \nabla \mathbf{v} = \text{Re} \left[ \mathbf{v} e^{i\mathbf{k} \cdot \mathbf{x}} \right] \cdot \nabla \text{Re} \left[ \mathbf{v} e^{i\mathbf{k} \cdot \mathbf{x}} \right] = \text{Re} \left[ \mathbf{k} \cdot \tilde{\mathbf{v}} e^{i\mathbf{k} \cdot \mathbf{x}} \right] \text{Re} \left[ i\tilde{\mathbf{v}} e^{i\mathbf{k} \cdot \mathbf{x}} \right] = 0,
\]
because \( \nabla \cdot \mathbf{v} = 0 \) implies \( i\mathbf{k} \cdot \tilde{\mathbf{v}} = 0 \). (This is a special result for an incompressible fluid. Note also that the nonlinear term does not vanish for a superposition of shearing waves.)

The amplitude equations for a shearing wave are
\[
\frac{d\tilde{v}_x}{dt} - 2\Omega \tilde{v}_y = -i\mathbf{k}_x \tilde{\psi} - \nu k^2 \tilde{v}_x,
\]
\[
\frac{d\tilde{v}_y}{dt} + (2\Omega - S)\tilde{v}_x = -i\mathbf{k}_y \tilde{\psi} - \nu k^2 \tilde{v}_y,
\]
\[
\frac{d\tilde{v}_z}{dt} = -i\mathbf{k}_z \tilde{\psi} - \nu k^2 \tilde{v}_z,
\]
\[i\mathbf{k} \cdot \tilde{\mathbf{v}} = 0,
\]
with \( k^2 = |\mathbf{k}|^2 \).

The viscous terms can be taken care of by a viscous decay factor
\[
E_\nu(t) = \exp \left( -\int \nu k^2 \, dt \right) = \exp \left\{ -\nu \left[ (k_x^2 + k_y^2 + k_z^2) t + S k_x k_y t^2 + \frac{1}{3} S^2 k_y^2 t^3 \right] \right\}.
\]

The decay is faster than exponential if \( k_y \neq 0 \).

Write \( \tilde{\mathbf{v}} = E_\nu(t) \tilde{\mathbf{v}}(t) \) and \( \tilde{\psi} = E_\nu(t) \tilde{\psi}(t) \) to eliminate the \( \nu \) terms in the amplitude equations. Then eliminate variables in favour of \( \tilde{v}_x \) to obtain (see Example 2.1)
\[
\frac{d^2}{dt^2} \left( k_x^2 \tilde{v}_x \right) + \Omega_x^2 k_y^2 \tilde{v}_x = 0,
\]
where \( \Omega_x^2 = 2\Omega(2\Omega - S) \) is the square of the epicyclic frequency in the local approximation.

Summary of outcomes (see Example 2.1):

- Stable if \( \Omega_x^2 > 0 \): \( |\tilde{v}|^2 \) oscillates if \( k_y = 0 \), or decays algebraically if \( k_y \neq 0 \).
- Unstable if \( \Omega_x^2 < 0 \): \( |\tilde{v}|^2 \) grows exponentially if \( k_y = 0 \), or grows algebraically if \( k_y \neq 0 \).

When \( \nu > 0 \), \( E_\nu \) kills off any algebraic growth for \( k_y \neq 0 \). But axisymmetric disturbances \( (k_y = 0) \) of sufficiently large scale grow exponentially.

We conclude that a rotating shear flow is linearly stable when \( \Omega_x^2 > 0 \), but unstable when \( \Omega_x^2 < 0 \).

This agrees with the stability of circular test-particle orbits. It also agrees with Rayleigh’s criterion for the linear stability of a cylindrical shear flow \( \mathbf{u} = r \Omega(r) \mathbf{e}_\phi \) to axisymmetric perturbations: the flow is unstable if the specific angular momentum \( |r^2 \Omega| \) decreases outwards.

The case \( \Omega_x^2 = 0 \) (either a non-rotating shear flow or one with uniform specific angular momentum) is marginally Rayleigh-stable and allows algebraic growth in the absence of viscosity.